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Research Article

Extended Laguerre Polynomials Associated with Hermite, Bernoulli, and Euler Numbers and Polynomials

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Let $\mathbf{P}_n = \{p(x) \in \mathbb{R}[x] \mid \deg p(x) \le n\}$ be an inner product space with the inner product $\langle p(x), q(x)\rangle = \int_0^\infty x^\alpha e^{-x} p(x)q(x)dx$, where $p(x), q(x) \in \mathbf{P}_n$ and $\alpha \in \mathbb{R}$ with $\alpha > -1$. In this paper we study the properties of the extended Laguerre polynomials which are an orthogonal basis for $\mathbf{P}_n$. From those properties, we derive some interesting relations and identities of the extended Laguerre polynomials associated with Hermite, Bernoulli, and Euler numbers and polynomials.

1. Introduction/Preliminaries

For $\alpha \in \mathbb{R}$ with $\alpha > -1$, the extended Laguerre polynomials are defined by the generating function as follows:

$$\frac{\exp(-xt/(1-t))}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^\alpha(x) t^n, \tag{1.1}$$

see [1–6].

From (1.1), we can derive the following:

$$L_n^\alpha(x) = \sum_{r=0}^{n} \frac{(-1)^r \binom{n+\alpha}{n-r}}{r!} x^r, \tag{1.2}$$

see [1–9].

As is well known, Rodrigues' formula for $L_n^{\alpha}(x)$ is given by

$$L_n^{\alpha}(x) = \frac{1}{n!}x^{-\alpha}e^{x}\left(\frac{d^n}{dx^n}e^{-x}x^{n+\alpha}\right), \tag{1.3}$$

see [1–6, 8, 9].

From (1.3), we note that

$$\int_0^{\infty} x^{\alpha}e^{-x}L_m^{\alpha}(x)L_n^{\alpha}(x)dx = \frac{1}{n!}\Gamma(\alpha+n+1)\delta_{m,n}, \quad (\alpha > -1), \tag{1.4}$$

where $\delta_{m,n}$ is the Kronecker symbol.

From (1.1), (1.2), and (1.3), we can derive the following identities:

$$(n+1)L_{n+1}^{\alpha}(x) + (x-\alpha-2n-1)L_n^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x) = 0, \quad (n \in \mathbb{N}), \tag{1.5}$$

$$\frac{d}{dx}L_n^{\alpha}(x) - \frac{d}{dx}L_{n-1}^{\alpha}(x) + L_{n-1}^{\alpha}(x) = 0, \quad \text{for } n \geq 1, \tag{1.6}$$

$$x\frac{d}{dx}L_n^{\alpha}(x) = nL_n^{\alpha}(x) - (n+\alpha)L_{n-1}^{\alpha}(x) = 0, \quad (n \geq 1), \tag{1.7}$$

and $L_n^{\alpha}(x)$ is a solution of $xy'' + (\alpha+1-x)y' + xy = 0$.

The derivatives of general Laguerre polynomials are given by

$$\begin{aligned} &\frac{d}{dx}L_n^{\alpha}(x) = -L_{n-1}^{\alpha+1}(x), && \frac{d}{dx}\left(x^{\alpha}L_n^{\alpha}(x)\right) = (n+\alpha)x^{\alpha-1}L_n^{\alpha-1}(x), \\ &\frac{d}{dx}\left(e^{-x}L_n^{\alpha}(x)\right) = -e^{-x}L_n^{\alpha+1}(x), && \frac{d}{dx}\left(x^{\alpha}e^{-x}L_n^{\alpha}(x)\right) = (n+1)x^{\alpha-1}e^{-x}L_{n+1}^{\alpha-1}(x). \end{aligned} \tag{1.8}$$

The nth Bernoulli polynomials, $B_n(x)$, are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.9}$$

see [10–17], with the usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x=0$, $B_n(0) = B_n$ are called the nth Bernoulli numbers.

It is well known that the nth Euler polynomials are also defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1.10}$$

see [18–22], with the usual convention about replacing $E^n(x)$ by $E_n(x)$.

The Hermite polynomials are given by

$$H_n(x) = (H + 2x)^n = \sum_{l=0}^{n} \binom{n}{l} 2^l x^l H_{n-l}, \tag{1.11}$$

see [23, 24], with the usual convention about replacing H^n by H_n. In the special case, $x = 0$, $H_n(0) = H_n$ are called the nth Hermite numbers.

From (1.11), we note that

$$\frac{d}{dx} H_n(x) = 2n(H + 2x)^{n-1} = 2nH_{n-1}(x), \tag{1.12}$$

see [23, 24], and $H_n(x)$ is a solution of Hermite differential equation which is given by

$$y'' - 2xy' + ny = 0, \tag{1.13}$$

(see [1–6, 23–32]).

Throughout this paper we assume that $\alpha \in \mathbb{R}$ with $\alpha > -1$. Let $\mathbf{P}_n = \{p(x) \in \mathbb{R}[x] \mid \deg p(x) \leq n\}$. Then $\mathbf{P}_n$ is an inner product space with the inner product $\langle p(x), q(x) \rangle = \int_0^\infty x^\alpha e^{-x} p(x) q(x) dx$, where $p(x), q(x) \in \mathbf{P}_n$. By (1.4) the set of the extended Laguerre polynomials $\{L_0^\alpha(x), L_1^\alpha(x), \ldots, L_n^\alpha(x)\}$ is an orthogonal basis for $\mathbf{P}_n$. In this paper we study the properties of the extended Laguerre polynomials which are an orthogonal basis for $\mathbf{P}_n$. From those properties, we derive some new and interesting relations and identities of the extended Laguerre polynomials associated with Hermite, Bernoulli and Euler numbers and polynomials.

2. On the Extended Laguerre Polynomials Associated with Hermite, Bernoulli, and Euler Polynomials

For $p(x) \in \mathbf{P}_n$, $p(x)$ is given by

$$p(x) = \sum_{k=0}^{n} C_k L_k^\alpha(x), \quad \text{for uniquely determined real numbers } C_k. \tag{2.1}$$

From (1.3), (1.4), and (2.1), we note that

$$\langle p(x), L_k^\alpha(x) \rangle = C_k \langle L_k^\alpha(x), L_k^\alpha(x) \rangle = C_k \int_0^\infty x^\alpha e^{-x} L_k^\alpha(x) L_k^\alpha(x) dx = C_k \frac{\Gamma(\alpha + k + 1)}{k!}. \tag{2.2}$$

Thus, by (2.2), we get

$$
\begin{aligned}
C_k &= \frac{k!}{\Gamma(\alpha+k+1)}\left\langle p(x), L_k^{\alpha}(x)\right\rangle \\
&= \frac{k!}{\Gamma(\alpha+k+1)}\frac{1}{k!}\int_0^{\infty}\left(\frac{d^k}{dx^k}x^{k+\alpha}e^{-x}\right)p(x)dx \\
&= \frac{1}{\Gamma(\alpha+k+1)}\int_0^{\infty}\left(\frac{d^k}{dx^k}x^{k+\alpha}e^{-x}\right)p(x)dx.
\end{aligned} \tag{2.3}
$$

Therefore, by (2.1) and (2.3), we obtain the following proposition.

Proposition 2.1. *For $p(x) \in \mathbf{P}_n$, let*

$$
p(x) = \sum_{k=0}^{n} C_k L_k^{\alpha}(x), \quad (\alpha > -1). \tag{2.4}
$$

Then one has the following:

$$
C_k = \frac{1}{\Gamma(\alpha+k+1)}\int_0^{\infty}\left(\frac{d^k}{dx^k}x^{k+\alpha}e^{-x}\right)p(x)dx. \tag{2.5}
$$

To derive inverse formula of (1.2), let take one $p(x) = x^n \in \mathbf{P}_n$. Then, by Proposition 2.1, one gets

$$
\begin{aligned}
C_k &= \frac{1}{\Gamma(\alpha+k+1)}\int_0^{\infty}\left(\frac{d^k}{dx^k}e^{-x}x^{k+\alpha}\right)x^n dx \\
&= (-1)^k\frac{n(n-1)\cdots(n-k+1)}{\Gamma(\alpha+k+1)}\int_0^{\infty} e^{-x}x^{\alpha+n}dx \\
&= (-1)^k\frac{n(n-1)\cdots(n-k+1)}{\Gamma(\alpha+k+1)}\Gamma(\alpha+n+1) \\
&= \frac{(-1)^k n!(\alpha+n)\cdots\alpha\Gamma(\alpha)}{(\alpha+k)\cdots\alpha\Gamma(\alpha)(n-k)!} \\
&= (-1)^k n!\frac{(\alpha+n)\cdots(\alpha+k+1)}{(n-k)!} = (-1)^k n!\binom{\alpha+n}{n-k}.
\end{aligned} \tag{2.6}
$$

Therefore, by (2.6), we obtain the following corollary.

Corollary 2.2 (Inverse formula of $L_n^{\alpha}(x)$). *For $n \in Z_+$, one has*

$$
x^n = n!\sum_{k=0}^{n}\binom{\alpha+n}{n-k}(-1)^k L_k^{\alpha}(x). \tag{2.7}
$$

Let one takes Bernoulli polynomials of degree n with $p(x) = B_n(x) \in \mathbf{P}_n$. Then $B_n(x)$ can be written as

$$B_n(x) = \sum_{k=0}^{n} C_k L_k^{\alpha}(x), \quad (\alpha \in \mathbb{R} \text{ with } \alpha > -1). \tag{2.8}$$

From Proposition 2.1, one has

$$\begin{aligned}
C_k &= \frac{1}{\Gamma(\alpha+k+1)} \int_0^{\infty} \left(\frac{d^k}{dx^k} e^{-x} x^{k+\alpha} \right) B_n(x) dx \\
&= \frac{(-1)^k n(n-1)\cdots(n-k+1)}{\Gamma(\alpha+k+1)} \int_0^{\infty} e^{-x} x^{k+\alpha} B_{n-k}(x) dx \\
&= \frac{(-1)^k n(n-1)\cdots(n-k+1)}{\Gamma(\alpha+k+1)} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_0^{\infty} e^{-x} x^{k+\alpha+l} dx \\
&= \frac{(-1)^k n(n-1)\cdots(n-k+1)}{\Gamma(\alpha+k+1)} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \Gamma(\alpha+k+l+1).
\end{aligned} \tag{2.9}$$

By the fundamental property of gamma function, one gets

$$\frac{\Gamma(\alpha+k+l+1)}{\Gamma(\alpha+k+1)(n-k)!} \binom{n-k}{l} = \frac{(\alpha+l+k)\cdots(\alpha+k+1)\Gamma(\alpha+k+1)(n-k)!}{\Gamma(\alpha+k+1)(n-k)!(n-k-l)!l!} = \frac{\binom{\alpha+k+l}{l}}{(n-k-l)!}. \tag{2.10}$$

Therefore, by (2.8), (2.9), and (2.10), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$ with $\alpha > -1$, one has*

$$B_n(x) = n! \sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k \binom{\alpha+k+l}{l} \frac{B_{n-k-l}}{(n-k-l)!} L_k^{\alpha}(x). \tag{2.11}$$

As is known, relationships between Hermite and Laguerre polynomials are given by

$$H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{-1/2}\left(x^2\right), \tag{2.12}$$

$$H_{2m+1}(x) = (-1)^m 2^{2m+1} m! L_m^{-1/2}\left(x^2\right), \tag{2.13}$$

see [1–6]. In the special case $\alpha = -1/2$, by (2.12) and (2.13), we obtain the following corollary.

Corollary 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$B_n\left(x^2\right) = n! \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{H_{2k}(x)}{2^{2k}k!} \binom{-\frac{1}{2}+k+l}{l} \frac{B_{n-k-l}}{(n-k-l)!}. \tag{2.14}$$

By the same method as Theorem 2.3, one gets

$$E_n(x) = n! \sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k \binom{\alpha+k+l}{l} \frac{E_{n-k-l}}{(n-k-l)!} L_k^{\alpha}(x), \tag{2.15}$$

where $E_n(x)$ are the nth Euler polynomials. In the special case, $x = 0$, $E_n(0) = E_n$ are called the nth Euler numbers.

Let one considers the nth Hermite polynomials with $p(x) = H_n(x) \in \mathbf{P}_n$. Then $H_n(x)$ can be written as

$$H_n(x) = \sum_{k=0}^{n} C_k L_k^{\alpha}(x), \quad (\alpha \in \mathbb{R} \text{ with } \alpha > -1). \tag{2.16}$$

From Proposition 2.1, one notes that

$$\begin{aligned} C_k &= \frac{1}{\Gamma(\alpha+k+1)} \int_0^{\infty} \left(\frac{d^k}{dx^k} e^{-x} x^{k+\alpha}\right) H_n(x)dx \\ &= \frac{(-2n)}{\Gamma(\alpha+k+1)} \int_0^{\infty} \left(\frac{d^{k-1}}{dx^{k-1}} e^{-x} x^{k+\alpha}\right) H_{n-1}(x)dx \\ &= \cdots \\ &= \frac{(-2n)(-2(n-1))\cdots(-2(n-k+1))}{\Gamma(\alpha+k+1)} \int_0^{\infty} e^{-x} x^{k+\alpha} H_{n-k}(x)dx \\ &= \frac{(-1)^k 2^k n!}{\Gamma(\alpha+k+1)(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l} 2^l \int_0^{\infty} e^{-x} x^{k+\alpha+l} dx \\ &= \frac{(-1)^k 2^k n!}{\Gamma(\alpha+k+1)(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l} 2^l \Gamma(\alpha+k+l+1). \end{aligned} \tag{2.17}$$

It is not difficult to show that

$$\frac{\binom{n-k}{l}\Gamma(\alpha+k+l+1)}{\Gamma(\alpha+k+1)(n-k)!} = \frac{(n-k)!(\alpha+k+l)\cdots(\alpha+k+1)\Gamma(\alpha+k+1)}{(n-k-l)!l!\Gamma(\alpha+k+1)(n-k)!} = \frac{\binom{\alpha+k+l}{l}}{(n-k-l)!}. \tag{2.18}$$

Therefore, by (2.16), (2.17), and (2.18), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$ with $\alpha > -1$, one has*

$$H_n(x) = n! \sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k 2^{k+l} \binom{\alpha+k+l}{l} \frac{H_{n-k-l}}{(n-k-l)!} L_k^{\alpha}(x). \tag{2.19}$$

In the special case, $\alpha = -1/2$, we obtain the following corollary.

Corollary 2.6. *For $n \in \mathbb{Z}_+$, one has*

$$H_n\left(x^2\right) = n! \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{2^{l-k} H_{2k}(x)}{k!} \binom{-\frac{1}{2}+k+l}{l} \frac{H_{n-k-l}}{(n-k-l)!}. \tag{2.20}$$

For $\beta \in \mathbb{R}$ with $\beta > -1$, let one takes

$$p(x) = L_n^{\beta}(x) \in \mathbf{P}_n. \tag{2.21}$$

Then $L_n^{\beta}(x)$ is also written as

$$L_n^{\beta}(x) = \sum_{k=0}^{n} C_k L_k^{\alpha}(x). \tag{2.22}$$

From Proposition 2.1, one can determine the coefficients of (2.22) as follows:

$$\begin{aligned}
C_k &= \frac{1}{\Gamma(\alpha+k+1)} \int_0^{\infty} \left(\frac{d^k}{dx^k} e^{-x} x^{k+\alpha} \right) L_n^{\beta}(x) dx \\
&= \frac{1}{\Gamma(\alpha+k+1)} \int_0^{\infty} \left(\frac{d^{k-1}}{dx^{k-1}} e^{-x} x^{k+\alpha} \right) L_{n-1}^{\beta+1}(x) dx \\
&= \cdots \\
&= \frac{1}{\Gamma(\alpha+k+1)} \int_0^{\infty} e^{-x} x^{k+\alpha} L_{n-k}^{\beta+k}(x) dx \\
&= \frac{1}{\Gamma(\alpha+k+1)} \sum_{r=0}^{n-k} \frac{(-1)^r \binom{n+\beta}{n-k-r}}{r!} \int_0^{\infty} e^{-x} x^{k+\alpha+r} dx \\
&= \frac{1}{\Gamma(\alpha+k+1)} \sum_{r=0}^{n-k} \frac{(-1)^r \binom{n+\beta}{n-k-r}}{r!} \Gamma(k+\alpha+r+1).
\end{aligned} \tag{2.23}$$

By the fundamental property of gamma function, one gets

$$\frac{\Gamma(k+\alpha+r+1)}{r!\Gamma(\alpha+k+1)} = \frac{(k+\alpha+r)\cdots(\alpha+k+1)\Gamma(\alpha+k+1)}{r!\Gamma(\alpha+k+1)} = \binom{k+\alpha+r}{r}. \tag{2.24}$$

Therefore, by (2.22), (2.23), and (2.24), we obtain the following theorem.

Theorem 2.7. *For $\beta \in \mathbb{R}$ with $\beta > -1$, and $n \in \mathbb{Z}_+$, one has*

$$L_n^{\beta}(x) = \sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^r\binom{n+\beta}{n-k-r}\binom{\alpha+k+r}{r}L_k^{\alpha}(x). \tag{2.25}$$

In the special case, $\alpha = \beta$, one has

$$\sum_{k=0}^{n-1}\left(\sum_{r=0}^{n-k}(-1)^r\binom{n+\alpha}{n-k-r}\binom{\alpha+k+r}{r}\right)L_k^{\alpha}(x) = 0. \tag{2.26}$$

Thus, by (2.26), we obtain the following corollary.

Corollary 2.8. *For $0 \le k \le n-1$, $\alpha \in \mathbb{R}$ with $\alpha > -1$, one has*

$$\sum_{r=0}^{n-k}(-1)^r\binom{n+\alpha}{n-k-r}\binom{\alpha+k+r}{r} = 0. \tag{2.27}$$

Let one assumes that

$$p(x) = \sum_{l=0}^{n} B_l(x)B_{n-l}(x) \in \mathbf{P}_n. \tag{2.28}$$

Then $p(x)$ can be rewritten as a linear combination of $L_0^{\alpha}(x), L_1^{\alpha}(x), \dots, L_n^{\alpha}(x)$ as follows:

$$p(x) = \sum_{l=0}^{n} B_l(x)B_{n-l}(x) = \sum_{k=0}^{n} C_k L_k^{\alpha}(x). \tag{2.29}$$

By Proposition 2.1, one can determine the coefficients of (2.29) as follows:

$$C_k = \frac{1}{\Gamma(\alpha+k+1)}\sum_{l=0}^{n}\int_0^{\infty}\left(\frac{d^k}{dx^k}e^{-x}x^{k+\alpha}\right)B_l(x)B_{n-l}(x)dx. \tag{2.30}$$

It is known that

$$\sum_{l=0}^{n} B_l(x)B_{n-l}(x) = \frac{2}{n+2}\sum_{l=0}^{n-2}\binom{n+2}{l}B_{n-l}B_l(x) + (n+1)B_n(x), \tag{2.31}$$

see [25].

From (2.30) and (2.31), one notes that

$$\begin{aligned}
C_n &= \frac{n+1}{\Gamma(\alpha+n+1)}\int_0^\infty\left(\frac{d^n}{dx^n}e^{-x}x^{n+\alpha}\right)B_n(x)dx\\
&= \frac{n+1}{\Gamma(\alpha+n+1)}(-1)^n n!\int_0^\infty e^{-x}x^{n+\alpha}dx = \frac{(n+1)!(-1)^n}{\Gamma(\alpha+n+1)}\Gamma(n+\alpha+1)\\
&= (n+1)!(-1)^n,\\
C_{n-1} &= \frac{n+1}{\Gamma(\alpha+n)}\int_0^\infty\left(\frac{d^{n-1}}{dx^{n-1}}e^{-x}x^{n-1+\alpha}\right)B_n(x)dx\\
&= \frac{n+1}{\Gamma(\alpha+n)}(-1)^{n-1}n!\int_0^\infty e^{-x}x^{n-1+\alpha}B_1(x)dx\\
&= \frac{n+1}{\Gamma(\alpha+n)}(-1)^{n-1}n!\left\{\Gamma(\alpha+n+1)-\frac{1}{2}\Gamma(\alpha+n)\right\}\\
&= (n+1)!(-1)^{n-1}\left(n+\alpha-\frac{1}{2}\right).
\end{aligned}\tag{2.32}$$

For $0 \le k \le n-2$, one has

$$\begin{aligned}
C_k &= \frac{1}{\Gamma(\alpha+k+1)}\left\{\frac{2}{n+2}\sum_{l=0}^{n-2}\binom{n+2}{l}B_{n-l}\int_0^\infty\left(\frac{d^k}{dx^k}e^{-x}x^{k+\alpha}\right)B_l(x)dx\right.\\
&\qquad\left.+(n+1)\int_0^\infty\left(\frac{d^k}{dx^k}e^{-x}x^{k+\alpha}\right)B_n(x)dx\right\}\\
&= \frac{1}{\Gamma(\alpha+k+1)}\left\{\frac{2}{n+2}\sum_{l=k}^{n-2}\binom{n+2}{l}B_{n-l}(-1)^k l(l-1)\cdots(l-k+1)\int_0^\infty e^{-x}x^{k+\alpha}B_{l-k}(x)dx\right.\\
&\qquad\left.+(n+1)(-1)^k n(n-1)\cdots(n-k+1)\int_0^\infty e^{-x}x^{k+\alpha}B_{n-k}(x)dx\right\}\\
&= \frac{1}{\Gamma(\alpha+k+1)}\left\{\frac{2}{n+2}\sum_{l=k}^{n-2}\binom{n+2}{l}B_{n-l}(-1)^k\frac{l!}{(l-k)!}\sum_{j=0}^{l-k}\binom{l-k}{j}B_{l-k-j}\int_0^\infty e^{-x}x^{k+\alpha+j}dx\right.\\
&\qquad\left.+(n+1)(-1)^k\frac{n!}{(n-k)!}\sum_{j=0}^{n-k}\binom{n-k}{j}B_{n-k-j}\int_0^\infty e^{-x}x^{k+\alpha+j}dx\right\}\\
&= \frac{1}{\Gamma(\alpha+k+1)}\left\{\frac{2}{n+2}\sum_{l=k}^{n-2}\sum_{j=0}^{l-k}\binom{n+2}{l}B_{n-l}(-1)^k\frac{l!}{(l-k)!}\binom{l-k}{j}B_{l-k-j}\Gamma(\alpha+k+j+1)\right.
\end{aligned}$$

$$
\begin{aligned}
&\left.+(n+1)(-1)^k\frac{n!}{(n-k)!}\sum_{j=0}^{n-k}\binom{n-k}{j}B_{n-k-j}\Gamma(\alpha+k+j+1)\right\}\\
&=\frac{2}{n+2}\sum_{l=k}^{n-2}\sum_{j=0}^{l-k}\binom{n+2}{l}B_{n-l}(-1)^k l!\frac{(\alpha+k+j)\cdots(\alpha+k+1)B_{l-k-j}}{j!(l-k-j)!}\\
&\quad+(n+1)(-1)^k n!\sum_{j=0}^{n-k}\frac{(\alpha+k+j)\cdots(\alpha+k+1)}{j!(n-k-j)!}B_{n-k-j}\\
&=\frac{2}{n+2}\sum_{l=k}^{n-2}\sum_{j=0}^{l-k}\binom{n+2}{l}B_{n-l}(-1)^k l!\binom{\alpha+k+j}{j}\frac{B_{l-k-j}}{(l-k-j)!}\\
&\quad+(n+1)(-1)^k n!\sum_{j=0}^{n-k}\binom{\alpha+k+j}{j}\frac{B_{n-k-j}}{(n-k-j)!}.
\end{aligned}
\tag{2.33}
$$

Therefore, by (2.29) and (2.32), we obtain the following theorem.

Theorem 2.9. *For $n\in\mathbb{Z}_+$, $\alpha\in\mathbb{R}$ with $\alpha>-1$, one has*

$$
\begin{aligned}
&\sum_{k=0}^{n}B_k(x)B_{n-k}(x)\\
&=\sum_{k=0}^{n-2}\left\{\frac{2}{n+2}\sum_{l=k}^{n-2}\sum_{0\le j\le n-k}(-1)^k\binom{n+2}{l}l!B_{n-l}\binom{\alpha+k+j}{j}\frac{B_{l-k-j}}{(l-k-j)!}\right.\\
&\qquad\left.+(n+1)!\sum_{0\le j\le n-k}(-1)^k\binom{\alpha+k+j}{j}\frac{B_{n-k-j}}{(n-k-j)!}\right\}L_k^{\alpha}(x)\\
&\quad+(-1)^n(n+1)!\left\{L_n^{\alpha}(x)-\left(n+\alpha-\frac{1}{2}\right)L_{n-1}^{\alpha}(x)\right\}.
\end{aligned}
\tag{2.34}
$$

Let one takes the polynomial $p(x)$ in $\mathbf{P}_n$ as follows:

$$
p(x)=\sum_{i_1+\cdots+i_r=n}B_{i_1}(x)B_{i_2}(x)\cdots B_{i_r}(x)\in\mathbf{P}_n.
\tag{2.35}
$$

From the orthogonality of $\{L_0^{\alpha}(x),\dots,L_n^{\alpha}(x)\}$, one notes that

$$
p(x)=\sum_{i_1+\cdots+i_r=n}B_{i_1}(x)B_{i_2}(x)\cdots B_{i_r}(x)=\sum_{k=0}^{n}C_kL_k^{\alpha}(x),
\tag{2.36}
$$

where

$$
C_k=\frac{1}{\Gamma(\alpha+k+1)}\int_0^{\infty}\left(\frac{d^k}{dx^k}e^{-x}x^{k+\alpha}\right)p(x)dx.
\tag{2.37}
$$

It is known in [25] that

$$\begin{aligned}
&\sum_{i_1+\cdots+i_r=n} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\quad= \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+r-1}{k}\left\{\sum_{\max\{0,k+r-n\}\le a\le r}\binom{r}{a}\sum_{i_1+\cdots+i_a=n+a-k-r} B_{i_1}B_{i_2}\cdots B_{i_a}\right.\\
&\qquad\left.+\sum_{i_1+\cdots+i_r=n-k} B_{i_1}\cdots B_{i_r}\right\}E_k(x)+\binom{n+r-1}{n}E_n(x).
\end{aligned}\tag{2.38}$$

From (2.35), (2.37), and (2.38), one notes that

$$\begin{aligned}
C_n &= \frac{\binom{n+r-1}{n}}{\Gamma(\alpha+n+1)}\int_0^{\infty}\left(\frac{d^n}{dx^n}e^{-x}x^{n+\alpha}\right)E_n(x)dx\\
&= \frac{\binom{n+r-1}{n}}{\Gamma(\alpha+n+1)}(-1)^n n!\int_0^{\infty} x^{n+\alpha}e^{-x}dx\\
&= \frac{\binom{n+r-1}{n}}{\Gamma(\alpha+n+1)}(-1)^n n!\Gamma(\alpha+n+1) = \binom{n+r-1}{n}(-1)^n n!,\\
C_{n-1} &= \frac{\binom{n+r-1}{n}}{\Gamma(\alpha+n)}\int_0^{\infty}\left(\frac{d^{n-1}}{dx^{n-1}}e^{-x}x^{n+\alpha-1}\right)E_n(x)dx\\
&= \frac{\binom{n+r-1}{n}}{\Gamma(\alpha+n)}(-1)^{n-1}n!\int_0^{\infty} e^{-x}x^{n+\alpha-1}E_1(x)dx\\
&= \frac{\binom{n+r-1}{n}}{\Gamma(\alpha+n)}(-1)^{n-1}n!\left\{\Gamma(\alpha+n+1)-\frac{1}{2}\Gamma(\alpha+n)\right\}\\
&= \binom{n+r-1}{n}(-1)^{n-1}n!\left(n+\alpha-\frac{1}{2}\right).
\end{aligned}\tag{2.39}$$

For $0 \le k \le n-2$, by (2.37) and (2.38), one gets

$$\begin{aligned}
C_k = \frac{1}{\Gamma(\alpha+k+1)}&\left\{\frac{1}{2}\sum_{l=0}^{n-2}\binom{n+r-1}{l}\left(\sum_{\max\{0,l+r-n\}\le a\le r}\binom{r}{a}\sum_{i_1+\cdots+i_a=n+a-l-r} B_{i_1}B_{i_2}\cdots B_{i_a}\right.\right.\\
&\qquad\left.+\sum_{i_1+\cdots+i_r=n-l} B_{i_1}\cdots B_{i_r}\right)\int_0^{\infty}\left(\frac{d^k}{dx^k}e^{-x}x^{n+\alpha}\right)E_l(x)dx\\
&\left.+\binom{n+r-1}{n}\int_0^{\infty}\left(\frac{d^k}{dx^k}e^{-x}x^{n+\alpha}\right)E_n(x)dx\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha+k+1)}\left\{\frac{1}{2}\sum_{l=k}^{n-2}\binom{n+r-1}{l}\left(\sum_{\max\{0,l+r-n\}\le a\le r}\binom{r}{a}\sum_{i_1+\cdots+i_a=n+a-l-r}B_{i_1}B_{i_2}\cdots B_{i_a}\right.\right.\\
&\qquad\left.+\sum_{i_1+\cdots+i_r=n-l}B_{i_1}\cdots B_{i_r}\right)(-1)^k l(l-1)\cdots(l-k+1)\\
&\qquad\times\int_0^\infty e^{-x}x^{k+\alpha}E_{l-k}(x)dx\\
&\qquad\left.+\binom{n+r-1}{n}(-1)^k n(n-1)\cdots(n-k+1)\int_0^\infty e^{-x}x^{k+\alpha}E_{n-k}(x)dx\right\}\\
&= \frac{1}{\Gamma(\alpha+k+1)}\left\{\frac{1}{2}\sum_{l=k}^{n-2}\binom{n+r-1}{l}\left(\sum_{\max\{0,l+r-n\}\le a\le r}\sum_{i_1+\cdots+i_a=n+a-l-r}B_{i_1}\cdots B_{i_a}\right.\right.\\
&\qquad\left.+\sum_{i_1+\cdots+i_r=n-l}B_{i_1}\cdots B_{i_r}\right)\frac{(-1)^k l!}{(l-k)!}\sum_{j=0}^{l-k}\binom{l-k}{j}E_{l-k-j}\\
&\qquad\times\int_0^\infty e^{-x}x^{k+\alpha+j}dx\\
&\qquad\left.+\binom{n+r-1}{n}\frac{(-1)^k n!}{(n-k)!}\sum_{j=0}^{n-k}\binom{n-k}{j}E_{n-k-j}\int_0^\infty e^{-x}x^{k+\alpha+j}dx\right\}\\
&= \frac{1}{2}\sum_{l=k}^{n-2}\binom{n+r-1}{l}\left(\sum_{\max\{0,l+r-n\}\le a\le r}\sum_{i_1+\cdots+i_a=n+a-l-r}B_{i_1}\cdots B_{i_a}+\sum_{i_1+\cdots+i_r=n-l}B_{i_1}\cdots B_{i_r}\right)\\
&\quad\times(-1)^k l!\sum_{j=0}^{l-k}\binom{\alpha+k+j}{j}\frac{E_{l-k-j}}{(l-k-j)!}\\
&\quad+\binom{n+r-1}{n}(-1)^k n!\sum_{j=0}^{n-k}\binom{\alpha+k+j}{j}\frac{E_{n-k-j}}{(n-k-j)!}.
\end{aligned}
\tag{2.40}
$$

Therefore, by (2.36), (2.39), and (2.40), we obtain the following theorem.

Theorem 2.10. *For $n\in\mathbb{Z}_+$, $r\in\mathbb{N}$, and $\alpha\in\mathbb{R}$ with $\alpha>-1$, one has*

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_r=n}B_{i_1}(x)B_{i_2}(x)\cdots B_{i_r}(x)\\
&\quad=\sum_{k=0}^{n-2}\frac{(-1)^k}{2}\left\{\sum_{l=k}^{n-2}l!\binom{n+r-1}{l}\left(\sum_{\max\{0,l+r-n\}\le a\le r}\binom{r}{a}\sum_{i_1+\cdots+i_a=n+a-l-r}B_{i_1}\cdots B_{i_a}\right.\right.\\
&\qquad\left.+\sum_{i_1+\cdots+i_r=n-l}B_{i_1}\cdots B_{i_r}\right)
\end{aligned}
$$

$$\times \sum_{j=0}^{l-k} \binom{\alpha+k+j}{j} \frac{E_{l-k-j}}{(l-k-j)!} + \binom{n+r-1}{n} (-1)^k n!$$
$$\times \sum_{j=0}^{n-k} \binom{\alpha+k+j}{j} \frac{E_{n-k-j}}{(n-k-j)!} \Bigg\} L_k^\alpha(x) + \binom{n+r-1}{n} (-1)^{n-1} n!$$
$$\times \left(n+\alpha-\frac{1}{2}\right) L_{n-1}^\alpha(x) + \binom{n+r-1}{n} (-1)^n n! L_n^\alpha(x). \tag{2.41}$$

For $m, s \in \mathbb{Z}_+$ with $m+s=n$, let one assumes that $p(x) = L_s^\alpha(x) L_m^\alpha(x) \in \mathbf{P}_n$. By Proposition 2.1, one sees that $p(x)$ can be written as

$$p(x) = L_s^\alpha(x) L_m^\alpha(x) = \sum_{k=0}^{n} C_k L_k^\alpha(x), \quad \alpha \in \mathbb{R} \text{ with } \alpha > -1. \tag{2.42}$$

From the orthogonality of $\{L_0^\alpha(x), L_1^\alpha(x), \ldots, L_n^\alpha(x)\}$, one has

$$C_k = \frac{1}{\Gamma(\alpha+k+1)} \int_0^\infty \left(\frac{d^k}{dx^k} e^{-x} x^{k+\alpha} \right) p(x) dx. \tag{2.43}$$

By (1.2), (1.3), and (1.8), one gets

$$\begin{aligned} L_s^\alpha(x) L_m^\alpha(x) &= \left(\sum_{r_1=0}^{s} \frac{(-1)^{r_1}}{r_1!} \binom{s+\alpha}{s-r_1} x^{r_1} \right) \left(\sum_{r_2=0}^{m} \frac{(-1)^{r_2}}{r_2!} \binom{m+\alpha}{m-r_2} x^{r_2} \right) \\ &= \sum_{r=0}^{n} \left(\sum_{r_1=0}^{r} (-1)^r \binom{r}{r_1} \binom{s+\alpha}{s-r_1} \binom{n-s+\alpha}{\alpha+r-r_1} \right) \frac{x^r}{r!}. \end{aligned} \tag{2.44}$$

Thus, from (2.44), one has

$$L_s^\alpha(x) L_m^\alpha(x) = \sum_{r=0}^{n} \left(\sum_{l=0}^{r} (-1)^r \binom{r}{l} \binom{s+\alpha}{\alpha+l} \binom{n-s+\alpha}{\alpha+r-l} \right) \frac{x^r}{r!}. \tag{2.45}$$

By (2.44) and (2.45), one gets

$$\begin{aligned} C_k &= \frac{1}{\Gamma(\alpha+k+1)} \sum_{r=0}^{n} (-1)^r \left\{ \sum_{l=0}^{r} \binom{r}{l} \binom{s+\alpha}{\alpha+l} \binom{n-s+\alpha}{\alpha+r-l} \right\} \frac{1}{r!} \\ &\quad \times \int_0^\infty \left(\frac{d^k}{dx^k} e^{-x} x^{k+\alpha} \right) x^r dx \\ &= \frac{1}{\Gamma(\alpha+k+1)} \sum_{r=k}^{n} (-1)^r \left\{ \sum_{l=0}^{r} \binom{r}{l} \binom{s+\alpha}{\alpha+l} \binom{n-s+\alpha}{\alpha+r-l} \frac{1}{r!} \right\} \end{aligned}$$

$$
\begin{aligned}
&\times (-1)^k r(r-1)\cdots(r-k+1)\int_0^\infty e^{-x}x^{r+\alpha}dx \\
&= \frac{1}{\Gamma(\alpha+k+1)}\sum_{r=k}^{n}(-1)^r\left\{\sum_{l=0}^{r}\binom{r}{l}\binom{s+\alpha}{\alpha+l}\binom{n-s+\alpha}{\alpha+r-l}\frac{1}{r!}\right\} \\
&\quad \times \frac{(-1)^k r!}{(r-k)!}\Gamma(\alpha+r+1) \\
&= \sum_{r=k}^{n}(-1)^{r+k}\left(\sum_{l=0}^{r}\binom{r}{l}\binom{s+\alpha}{\alpha+l}\binom{n-s+\alpha}{\alpha+r-l}\frac{(\alpha+r)(\alpha+r-1)\cdots(\alpha+k+1)}{(r-k)!}\right) \\
&= \sum_{r=k}^{n}(-1)^{r+k}\left(\sum_{l=0}^{r}\binom{r}{l}\binom{s+\alpha}{\alpha+l}\binom{n-s+\alpha}{\alpha+r-l}\binom{\alpha+r}{r-k}\right).
\end{aligned}
\tag{2.46}
$$

Therefore, by (2.42) and (2.46), we obtain the following theorem.

Theorem 2.11. *For $s, m \in \mathbb{Z}_+$ with $s + m = n$, $\alpha \in \mathbb{R}$ with $\alpha > -1$, one has*

$$
L_s^\alpha(x)L_m^\alpha(x) = \sum_{k=0}^{n}\sum_{r=k}^{n}(-1)^{r+k}\left(\sum_{l=0}^{r}\binom{r}{l}\binom{s+\alpha}{\alpha+l}\binom{n-s+\alpha}{\alpha+r-l}\binom{\alpha+r}{r-k}\right)L_k^\alpha(x). \tag{2.47}
$$

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