

Research Article

Bifurcations of Nonconstant Solutions of the Ginzburg-Landau Equation

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We study local and global bifurcations of nonconstant solutions of the Ginzburg-Landau equation from the families of constant ones. As the topological tools we use the equivariant Conley index and the degree for equivariant gradient maps.

1. Introduction

Let us consider the following Ginzburg-Landau problem:

$$\begin{aligned} -(\nabla - iA(x))^2 u(x) &= \lambda(1 - |u(x)|^2)u(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) is an open, bounded domain with smooth boundary $\partial\Omega$, $\nu(x)$ is an outward normal to Ω at $x \in \partial\Omega$, $A \in C^0(\text{cl } \Omega, \mathbb{R}^N)$ and $u \in H^1(\Omega, \mathbb{C})$.

There is a vast literature on this problem. Bifurcations of solutions of the Ginzburg-Landau-type problems have been considered by many authors, see, for instance, [1–10] and references therein. Usually the authors study local bifurcations of nonzero solutions of problem (1.1) with Dirichlet boundary condition by using the Crandall-Rabinowitz bifurcation theorem, the Krasnosiel'ski bifurcation theorem for potential operators, the Lyapunov-Schmidt reduction, the center manifold theorem, the attractor bifurcation theorem,

or the implicit function theorem. On the other hand, the global bifurcations of solutions of the one-dimensional Ginzburg-Landau model have been studied in [3]. Using the Brouwer degree the authors have proved the existence of a closed connected set of asymmetric solutions which connect the global curve of symmetric solutions to an asymmetric normal state solution.

Our goal is to study the existence of nonconstant solutions of problem (1.1) with Neumann boundary condition. We apply the equivariant bifurcation theory technique. First of all, we study families of constant solutions of problem (1.1) and describe them assuming that the norm of magnetic field A is constant, that is, $\|A(x)\| = \text{const}$ for all $x \in \text{cl } \Omega$, where $\|\cdot\|$ is the usual Euclidean norm. We distinguish two cases $\|A\| = 0$ and $\|A\| = \text{const} \neq 0$. Next we find necessary and sufficient conditions for the existence of local and global bifurcation points of nonconstant solutions from these families. Problem (1.1) is S^1 -symmetric, that is, if $u \in H^1(\Omega, \mathbb{C})$ is a solution of this problem, then $e^{i\theta}u$ is. Therefore, we consider solutions of (1.1) as critical orbits of S^1 -invariant functionals. The basic idea is to apply the S^1 -equivariant Conley index, see [11, 12], and the degree for S^1 -equivariant gradient maps, see [13–18], to obtain a local and global bifurcation of critical S^1 -orbits of these functionals. The choice of these invariants seems to be the best adapted to our theory. Since the Leray-Schauder degree is not applicable (see the remarks under Corollary 4.2) we have chosen the invariants which are suitable for the study of critical orbits of invariant functionals.

After this introduction our paper is organized as follows.

In Section 2 we have summarized without proofs the relevant abstract material on the equivariant bifurcation theory. In the next sections we have applied these abstract results to the study of local and global bifurcation of nonconstant solutions of problem (1.1). Since the Ginzburg-Landau equation is S^1 -symmetric, we consider in this section only S^1 -symmetric variational bifurcation problems. The notion of a local and global bifurcation of critical S^1 -orbits of families of S^1 -invariant C^1 -functionals has been introduced in Definition 2.1. The necessary condition for the existence of bifurcation points of critical S^1 -orbits has been formulated in Lemma 2.3. The important point to note here is the form of the functional (2.2). Namely, we consider S^1 -invariant functionals whose gradients are of the form of compact perturbation of the identity. In Theorem 2.4 we have formulated sufficient conditions for the existence of global bifurcations of S^1 -orbits of critical points of S^1 -invariant functionals. Sufficient condition for the existence of local bifurcation of critical S^1 -orbits has been presented in Theorem 2.6. In Remarks 2.5 and 2.7, we have reformulated assumptions of Theorems 2.4 and 2.6, respectively, to make them easier to understand.

In Section 3 we study bifurcations of nonconstant solutions of the Ginzburg-Landau bifurcating from the set of constant solutions. In the first part of this section we show that the functional (F) corresponding to the Ginzburg-Landau equation satisfies all the assumptions of the functional considered in Section 2. We consider two cases of the Ginzburg-Landau equation.

In Section 3.1 we assume that the magnetic field A vanishes. In Lemma 3.2 we have described the set of constant solutions of the Ginzburg-Landau system (3.7), which consists of two families. Moreover, we have proved the necessary condition for the existence of bifurcation of nonconstant solutions from these families. The sets of local and global bifurcation points of nonconstant solutions of system (3.7) have been described in Theorem 3.3.

In Section 3.2 we assume that the norm of the magnetic field A is constant and different from 0. Without loss of generality we assume that this norm is equal to 1. The structure of this subsection is similar to that of Section 3.1. In Lemma 3.4 we have described the set

of constant solutions of system (3.2), which consists of three families. Moreover, we have proved sufficient conditions for the existence of local and global bifurcations of nonconstant solutions of system (3.2) from these families. The necessary conditions for the existence of local and global bifurcation of nonconstant solutions of system (3.2) from the families of constant solutions have been proved in Theorem 3.5.

In Section 4 we have shown that we cannot use the Leray-Schauder degree and the famous Rabinowitz alternative to study solutions of problem (1.1). Moreover, we have formulated an open question concerning bifurcations of nonconstant solutions of problem (1.1). This question is at present far from being solved. Finally, we have shown that for domains Ω with sufficiently small volume the first eigenvalue of the magnetic Laplace operator $-\Delta_A$ equals 1. This property has allowed us to simplify the formulation of Theorem 3.5, see Corollary 4.2.

In the appendix we have recalled for the convenience of the reader some material on equivariant algebraic topology thus making our presentation self-contained.

2. Bifurcations of Critical Orbits

In this section we summarize without proofs the relevant material on the equivariant bifurcation theory. In the next section we will apply these abstract results to the study of nonconstant solutions of the Ginzburg-Landau equation.

Throughout this paper S^1 stands for the group of complex numbers of module 1. We identify this group with the group of special orthogonal two-dimensional matrices $SO(2)$ as follows $e^{i\theta} \rightarrow \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. Consider a real Hilbert space $(\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}})$ which is an orthogonal S^1 -representation. The S^1 -action on the space $\mathbb{H} \times \mathbb{R}$ we define by $g(u, \lambda) = (gu, \lambda)$. For $u_0 \in \mathbb{H}$ define the orbit of u_0 by $S^1(u_0) = \{gu_0 : g \in S^1\}$ and the isotropy group of u_0 by $S^1_{u_0} = \{g \in S^1 : gu_0 = u_0\}$. Assume that $S^1_{u_0} = \begin{cases} S^1 & \text{if } u_0=0 \\ \{1\} & \text{if } u_0 \neq 0 \end{cases}$. Hence, $S^1(u_0)$ is a manifold such that $\dim S^1(u_0) = \begin{cases} 0 & \text{if } u_0=0 \\ 1 & \text{if } u_0 \neq 0 \end{cases}$. A functional $\Phi : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{R}$ is called S^1 -invariant provided that $\Phi(gu, \lambda) = \Phi(u, \lambda)$ for every $g \in S^1$ and $u \in \mathbb{H}$. The space of S^1 -invariant functionals of the class C^k will be denoted by $C^k_{S^1}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$. An operator $\Psi : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{H}$ is said to be S^1 -equivariant if $\Psi(gu, \lambda) = g\Psi(u, \lambda)$ for every $g \in S^1$ and $u \in \mathbb{H}$. It is a known fact that if $\Phi \in C^k_{S^1}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ then $\nabla_u \Phi \in C^{k-1}_{S^1}(\mathbb{H} \times \mathbb{R}, \mathbb{H})$, where $\nabla_u \Phi$ is the gradient of Φ with respect to the first coordinate. Note that if $\nabla_u \Phi(u_0, \lambda_0) = 0$, then the gradient $\nabla_u \Phi$ vanishes on the orbit $S^1(u_0) \times \{\lambda_0\}$.

Fix $\Phi \in C^1_{S^1}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$. It is of our interest to study solutions of the following equation:

$$\nabla_u \Phi(u, \lambda) = 0. \quad (2.1)$$

We are going to apply the bifurcation technique for S^1 -orbits of critical points of S^1 -invariant functionals. More precisely, we will apply the S^1 -equivariant Conley index [11, 12] and the degree for S^1 -equivariant gradient maps [16, 17] to prove a local and a global bifurcation of critical S^1 -orbits of problem (2.1).

Fix $k \in \mathbb{N}$ and $\{\lambda_{11}, \lambda_{21}, \dots, \lambda_{1k}, \lambda_{2k}\} \subset \mathbb{R} \cup \{\pm\infty\}$ such that $\lambda_{1i} < \lambda_{2i}$ for $i = 1, \dots, k$. For $i = 1, \dots, k$ define a connected family $\mathcal{F}_i = \{S^1(u_\lambda) \times \{\lambda\} : \lambda \in (\lambda_{1i}, \lambda_{2i})\}$, where $u_\lambda = \zeta_i(\lambda)$ and $\zeta_i \in C^0((\lambda_{1i}, \lambda_{2i}), \mathbb{H})$. Finally define $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$ and assume that $\mathcal{F} \subset (\nabla_u \Phi)^{-1}(0)$.

The set \mathcal{F} is called the set of trivial solutions of problem (2.1). Define $\mathcal{N} = \{(u, \lambda) \in \mathbb{H} \times \mathbb{R} : \nabla_u \Phi(u, \lambda) = 0 \text{ and } (u, \lambda) \notin \mathcal{F}\}$. Fix $(u_{\lambda_0}, \lambda_0) \in \mathcal{F}$ and denote by $\mathcal{C}(u_{\lambda_0}, \lambda_0) \subset \mathbb{H} \times \mathbb{R}$, a connected component of $\text{cl}(\mathcal{N})$ such that $(u_{\lambda_0}, \lambda_0) \in \mathcal{C}(u_{\lambda_0}, \lambda_0)$.

Definition 2.1. A point $(u_{\lambda_0}, \lambda_0) \in \mathcal{F}$ is said to be a local bifurcation point of solutions of (2.1), if $(u_{\lambda_0}, \lambda_0) \in \text{cl}(\mathcal{N})$. The set of local bifurcation points will be denoted by \mathcal{BOF} . A point $(u_{\lambda_0}, \lambda_0) \in \mathcal{F}$ is said to be a global bifurcation point of solutions of (2.1), if either $\mathcal{C}(u_{\lambda_0}, \lambda_0) \cap (\mathcal{F} \setminus \{(u_{\lambda_0}, \lambda_0)\}) \neq \emptyset$ or $\mathcal{C}(u_{\lambda_0}, \lambda_0)$ is unbounded. The set of global bifurcation points will be denoted by \mathcal{GLOB} .

In other words a point $(u_{\lambda_0}, \lambda_0) \in \mathcal{F}$ is a bifurcation point of nontrivial solutions of (2.1) provided that it is an accumulation point of nontrivial solutions of this equation. A bifurcation point $(u_{\lambda_0}, \lambda_0) \in \mathcal{F}$ is a global bifurcation point of nontrivial solutions of (2.1) provided that a connected set $\mathcal{C}(u_{\lambda_0}, \lambda_0)$ of nontrivial solutions of (2.1) bifurcating from this point satisfies the Rabinowitz-type alternative, that is, either $\mathcal{C}(u_{\lambda_0}, \lambda_0)$ is unbounded or meets the set \mathcal{F} at least at two times.

Remark 2.2. Directly from the above definition it follows that $\mathcal{GLOB} \subset \mathcal{BOF}$. Moreover, if $(u_0, \lambda_0) \in \mathcal{BOF}$ ($(u_0, \lambda_0) \in \mathcal{GLOB}$), then $(gu_0, \lambda_0) \in \mathcal{BOF}$ ($(gu_0, \lambda_0) \in \mathcal{GLOB}$) for every $g \in S^1$.

From now on we assume that the functional $\Phi \in C_{S^1}^2(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ is of the following form:

$$\Phi(u, \lambda) = \frac{1}{2} \langle u, u \rangle_{\mathbb{H}} + \eta(u, \lambda), \quad (2.2)$$

where $\nabla_u \eta : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{H}$ is a compact operator.

The natural question is the following: what is the necessary condition for the existence of bifurcation points of solutions of (2.1)?

In the lemma below we answer the above-stated question.

Lemma 2.3. *If $(u_{\lambda_0}, \lambda_0) \in \mathcal{BOF}$, then $\dim \ker \nabla_u^2 \Phi(u_{\lambda_0}, \lambda_0) > \dim S^1(u_{\lambda_0})$.*

Fix $i_0 \in \{1, \dots, k\}$ and $(u_{\lambda_0}, \lambda_0) \in \mathcal{F}_{i_0}$ such that there is $\epsilon > 0$ satisfying the following conditions:

- (1) $[\lambda_0 - \epsilon, \lambda_0 + \epsilon] \subset (\lambda_{1i_0}, \lambda_{2i_0})$,
- (2) if $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ and $\dim \ker \nabla_u^2 \Phi(u_\lambda, \lambda) > \dim S^1(u_\lambda)$, then $\lambda = \lambda_0$.

Since $S^1(u_{\lambda_0 \pm \epsilon})$ is a nondegenerate critical S^1 -orbit of the functional $\Phi(\cdot, \lambda_0 \pm \epsilon)$, there is an open bounded S^1 -invariant subset $\Omega \subset \mathbb{H}$ satisfying $(\nabla_u \Phi(\cdot, \lambda_0 \pm \epsilon))^{-1} \cap \text{cl } \Omega = S^1(u_{\lambda_0 \pm \epsilon})$. Under these assumptions one can compute the index of an isolated critical orbit $S^1(u_{\lambda_0 \pm \epsilon})$ in terms of the degree for S^1 -equivariant gradient maps, see [13, 15–17], that is, $\nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_0 \pm \epsilon), \Omega) \in U(S^1)$, where $U(S^1)$ is the Euler ring of the group S^1 , see [19, 20]. For the convenience of the reader, one has reminded the definition of the Euler ring of the groups S^1 in appendix.

It is a known fact that change of any reasonable degree along the set of trivial solutions implies a global bifurcation of zeroes. In the theorem below we formulate sufficient condition for the existence of a global bifurcation of critical S^1 -orbits of problem (2.1). The proof of the following theorem is standard and therefore we omit it.

Theorem 2.4. *If $\nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_0 + \epsilon), \Omega) \neq \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_0 - \epsilon), \Omega)$, then $(u_{\lambda_0}, \lambda_0) \in \mathcal{GLOB}$.*

Finite-dimensional equivariant Conley index has been considered in [12]. Infinite-dimensional generalisation of equivariant Conley index one can find in [11]. In this paper one considers the S^1 -equivariant Conley index $\mathcal{C}\mathcal{O}_{S^1}(\cdot, \lambda)$ defined by a flow induces by $-\nabla_u \Phi(\cdot, \lambda)$.

Assume additionally that $S^1_{u_\lambda} = \{e\}$ for every $(u_\lambda, \lambda) \in \mathcal{F}_{i_0}$. Since $S^1(u_{\lambda_0 \pm \epsilon})$ is isolated in $\nabla_u \Phi(\cdot, \lambda_0 \pm \epsilon)^{-1}(0)$, it is an isolated invariant set in the sense of the S^1 -equivariant Conley index theory.

It is a known fact, see Lemma 5.7 of [16], that $\mathcal{C}\mathcal{O}_{S^1}(S^1(u_{\lambda_0 \pm \epsilon}), \lambda_0 \pm \epsilon)$ is an S^1 -CW-complex which consists of a base point and one S^1 -cell of dimension $m^- (\nabla_u^2 \Phi(u_{\lambda_0 \pm \epsilon}, \lambda_0 \pm \epsilon))$ and of isotropy group $\{e\}$, where $m^- (\cdot)$ is the Morse index.

Remark 2.5. Note that

$$\begin{aligned} \chi_{S^1} \mathcal{C}\mathcal{O}_{S^1}(S^1(u_{\lambda_0 \pm \epsilon}), \lambda_0 \pm \epsilon) &= \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_0 \pm \epsilon), \Omega) \\ &= (-1)^{m^- (\nabla_u^2 \Phi(u_{\lambda_0 \pm \epsilon}, \lambda_0 \pm \epsilon))} \chi_{S^1}(S^1 / \{e\}^+) \in U(S^1), \end{aligned} \quad (2.3)$$

where χ_{S^1} is the S^1 -equivariant Euler characteristic, see [19, 20]. The assumption of the above theorem is a little bit mysterious. Taking into account the above, it can be equivalently formulated in the following way: $m^- (\nabla_u^2 \Phi(u_{\lambda_0 + \epsilon}, \lambda_0 + \epsilon)) + m^- (\nabla_u^2 \Phi(u_{\lambda_0 - \epsilon}, \lambda_0 - \epsilon))$ is odd. We underline that since $\nabla_u \Phi$ is of the form of compact perturbation of the identity, these Morse indices are finite.

A finite-dimensional version of the following theorem has been proved in [12]. We can literally repeat this proof replacing the finite-dimensional S^1 -equivariant Conley index by its infinite-dimensional generalization.

Theorem 2.6. *If $\mathcal{C}\mathcal{O}_{S^1}(S^1(u_{\lambda_0 + \epsilon}), \lambda_0 + \epsilon) \neq \mathcal{C}\mathcal{O}_{S^1}(S^1(u_{\lambda_0 - \epsilon}), \lambda_0 - \epsilon)$, then $(u_{\lambda_0}, \lambda_0) \in \mathcal{BCF}$.*

Remark 2.7. Similarly as in the case of Theorem 2.4 one can reformulate the assumption of the above theorem. Equivalent but easier to understand formulation is the following:

$$m^- (\nabla_u^2 \Phi(u_{\lambda_0 + \epsilon}, \lambda_0 + \epsilon)) \neq m^- (\nabla_u^2 \Phi(u_{\lambda_0 - \epsilon}, \lambda_0 - \epsilon)). \quad (2.4)$$

3. Results

In this section we prove the main results of our paper. Namely, we study bifurcations of nonconstant solutions of the following Ginzburg-Landau equation:

$$\begin{aligned} -(\nabla - iA(x))^2 u(x) &= \lambda(1 - |u(x)|^2)u(x), \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (3.1)$$

where $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) is an open, bounded domain with smooth boundary $\partial\Omega$, $\nu(x)$ is an outward normal to Ω at $x \in \partial\Omega$, $A \in C^0(\text{cl } \Omega, \mathbb{R}^N)$ and $u \in H^1(\Omega, \mathbb{C})$.

After making in problem (3.1) a linear transformation $u = v + iw$ for $v, w \in H^1(\Omega, \mathbb{R})$ we obtain an equivalent system of real equations

$$\begin{aligned} -\Delta v(x) - 2\langle A(x), \nabla w(x) \rangle + \|A(x)\|^2 v(x) &= \lambda(1 - v(x)^2 - w(x)^2)v(x), \quad \text{in } \Omega, \\ -\Delta w(x) + 2\langle A(x), \nabla v(x) \rangle + \|A(x)\|^2 w(x) &= \lambda(1 - v(x)^2 - w(x)^2)w(x), \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} &= 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (3.2)$$

We are going to consider solutions of system (3.2) as critical points of S^1 -invariant functional of the class C^2 .

Define scalar products $\langle \cdot, \cdot \rangle_{H_A^1(\Omega, \mathbb{C})}, \langle \cdot, \cdot \rangle_{H^1(\Omega, \mathbb{C})} : H^1(\Omega, \mathbb{C}) \oplus H^1(\Omega, \mathbb{C}) \rightarrow \mathbb{C}$ as follows:

$$\begin{aligned} \langle u_1, u_2 \rangle_{H_A^1(\Omega, \mathbb{C})} &= \int_{\Omega} \left(\nabla u_1(x) - iA(x)u_1(x), \overline{\nabla u_2(x) - iA(x)u_2(x)} \right) + u_1(x)\overline{u_2(x)} dx, \\ \langle u_1, u_2 \rangle_{H^1(\Omega, \mathbb{C})} &= \int_{\Omega} \left(\nabla u_1(x), \overline{\nabla u_2(x)} \right) + u_1(x)\overline{u_2(x)} dx. \end{aligned} \quad (3.3)$$

We underline that norms $\|\cdot\|_{H_A^1(\Omega, \mathbb{C})}, \|\cdot\|_{H^1(\Omega, \mathbb{C})}$ are equivalent, see [10, 21]. Now define scalar products $\langle \cdot, \cdot \rangle_{\mathbb{H}_A}, \langle \cdot, \cdot \rangle_{\mathbb{H}} : \mathbb{H} \oplus \mathbb{H} \rightarrow \mathbb{R}$ by

$$\begin{aligned} \langle (v_1, w_1), (v_2, w_2) \rangle_{\mathbb{H}_A} &= \Re \langle v_1 + iw_1, v_2 + iw_2 \rangle_{H_A^1(\Omega, \mathbb{C})}, \\ \langle (v_1, w_1), (v_2, w_2) \rangle_{\mathbb{H}} &= \Re \langle v_1 + iw_1, v_2 + iw_2 \rangle_{H^1(\Omega, \mathbb{C})}, \end{aligned} \quad (3.4)$$

where $\mathbb{H} = H^1(\Omega, \mathbb{R}) \oplus H^1(\Omega, \mathbb{R})$. For simplicity of notation put $\mathbf{0} = (0, 0) \in \mathbb{H}$. For $v, w \in H^1(\Omega, \mathbb{R})$ put $u = v + iw \in H^1(\Omega, \mathbb{C})$. Since $\|u\|_{H^1(\Omega, \mathbb{C})} = \|(v, w)\|_{\mathbb{H}}$ and $\|u\|_{H_A^1(\Omega, \mathbb{C})} = \|(v, w)\|_{\mathbb{H}_A}$, norms $\|\cdot\|_{\mathbb{H}_A}, \|\cdot\|_{\mathbb{H}}$ are equivalent. From now on we consider \mathbb{H} as a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_{\mathbb{H}_A}$. It is easy to check that \mathbb{H} is an orthogonal S^1 -representation with S^1 -action given by $gu = (gu^t)^t$, where u^t is the transposition of u . Define a map $F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ by $F((v, w), \lambda) = \lambda((1/2)(v^2 + w^2) - (1/4)(v^2 + w^2)^2)$ and a functional $\Phi \in C^2(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ as follows:

$$\begin{aligned} \Phi((v, w), \lambda) &= \frac{1}{2} \|(v, w)\|_{\mathbb{H}_A}^2 - \int_{\Omega} F((v(x), w(x)), \lambda) + \frac{1}{2} (v(x)^2 + w(x)^2) dx \\ &= \frac{1}{2} \int_{\Omega} |\nabla z(x) - iA(x)z(x)|^2 dx - \int_{\Omega} \lambda \left(\frac{1}{2} |z|^2 - \frac{1}{4} |z|^4 \right) dx \\ &= \frac{1}{2} \|z\|_{H_A^1(\Omega, \mathbb{C})}^2 - \int_{\Omega} \frac{\lambda + 1}{2} |z|^2 - \frac{\lambda}{4} |z|^4 dx, \end{aligned} \quad (F)$$

where $z = v + iw$.

Remark 3.1. It is easy to verify that $\Phi \in C_{S^1}^2(\mathbb{H} \times \mathbb{R}, \mathbb{R})$. Indeed, this is a standard fact that the functional Φ is of the class C^2 . What is left is to show that the functional Φ is S^1 -invariant.

For $(v, w) \in \mathbb{H}$ put $z = v + iw$ and note that $\Phi(e^{i\theta}(v, w), \lambda) = (1/2)\|e^{i\theta}z\|_{H_A^1(\Omega, \mathbb{C})}^2 - \int_{\Omega}((\lambda + 1)/2)|e^{i\theta}z|^2 - (\lambda/4)|e^{i\theta}z|^4 dx = \Phi((v, w), \lambda)$. Moreover, it is clear that for $u = (v, w)$ we have $\Phi(u, \lambda) = (1/2)\|u\|_{\mathbb{H}_A}^2 - ((\lambda + 1)/2)\langle Lu, u \rangle_{\mathbb{H}_A} - \eta(u, \lambda)$, where

- (1) $L : \mathbb{H} \rightarrow \mathbb{H}$ is a linear, compact, bounded, self-adjoint, positively definite, and S^1 -equivariant operator,
- (2) $\eta : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{R}$ is a S^1 -invariant functional of the class C^2 such that

- (a) $\nabla_u \eta : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{H}$ is a compact, S^1 -equivariant operator,
- (b) $\nabla_u \eta(0, \lambda) = 0$,
- (c) $\nabla_u^2 \eta(u, \lambda) = o(\|u\|_{\mathbb{H}})$ at $u = 0$, uniformly on bounded λ -intervals.

It is easy to verify that the gradient $\nabla_u \Phi : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{H}$ is an S^1 -equivariant operator of the class C^1 of the form $\nabla_u \Phi(u, \lambda) = u - (\lambda + 1)Lu - \nabla_u \eta(u, \lambda)$. Fix $u_0 = (v_0, w_0) \in \mathbb{H}$, $\lambda' \in \mathbb{R}$ and note that the study of $\ker \nabla_u^2 \Phi(u_0, \lambda')$ is equivalent to the study of solutions of the following system:

$$L_A \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \lambda' \begin{bmatrix} 1 - 3v_0^2 - w_0^2 & -2v_0w_0 \\ -2v_0w_0 & 1 - v_0^2 - 3w_0^2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \quad (3.5)$$

where $L_A \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -\Delta\phi_1(x) - 2\langle A(x), \nabla\phi_2(x) \rangle + \|A(x)\|^2\phi_1 \\ -\Delta\phi_2(x) + 2\langle A(x), \nabla\phi_1(x) \rangle + \|A(x)\|^2\phi_2 \end{bmatrix}$.

It is a known fact that S^1 -orbits of solutions of system (3.2) are in one to one correspondence with the critical S^1 -orbits of the functional Φ , that is, with the S^1 -orbits of solutions of the following equation:

$$\nabla_u \Phi(u, \lambda) = 0. \quad (3.6)$$

From now on we study bifurcations of solutions of the above equation. For simplicity of notations we write Δ_A instead of $(\nabla - iA(x))^2$. Let $\sigma(-\Delta_A) = \{0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots\}$ denote the set of eigenvalues of the following eigenvalue problem $\{-\Delta_A \phi = \lambda \phi \text{ in } \Omega; \partial\phi/\partial\nu = 0 \text{ on } \partial\Omega\}$. It is known that $\lambda_1 = 0$ if and only if $A = 0$, see [8]. It is clear that $\sigma(-\Delta_A) = \sigma(L_A)$.

3.1. Case $\|A\| = 0$

In this section we study bifurcations of solutions of the simplified Ginzburg-Landau equation, that is, we assume that the magnetic field A vanishes. Such an equation has been considered in [22]. To underline that $A = 0$ we write Δ_θ instead Δ_A . Note that $\Delta_\theta \phi = \Delta\phi_1 + i\Delta\phi_2$ for $\phi = \phi_1 + i\phi_2 \in H^1(\Omega, \mathbb{C})$, where Δ is the usual Laplace operator on $H^1(\Omega, \mathbb{R})$. Note that $\sigma(-\Delta_\theta) = \sigma(-\Delta)$ (with Neumann boundary data). If $\lambda_k \in \sigma(-\Delta)$, then $\mathbb{V}_{-\Delta}(\lambda_k)$ denotes the eigenspace of $-\Delta$ corresponding to λ_k . Finally put $\sigma^{\text{odd}}(-\Delta) = \{\lambda_k \in \sigma(-\Delta) : \dim \mathbb{V}_{-\Delta}(\lambda_k) \text{ is odd}\}$.

Since $A = \theta$, system (3.2) has the following form

$$\begin{aligned} -\Delta v &= \lambda(1 - v^2 - w^2)v, \quad \text{in } \Omega, \\ -\Delta w &= \lambda(1 - v^2 - w^2)w, \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} &= \frac{\partial w}{\partial \nu} = 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (3.7)$$

The simplest solutions of system (3.7) are constant solutions. We call them trivial and denote \mathcal{F} . The set of trivial solutions consists of two families $\mathcal{F}_1 = \{0\} \times \mathbb{R}$ and $\mathcal{F}_2 = S^1((1, 0)) \times \mathbb{R}$. In the lemma below we have described the set of trivial solutions of system (3.7) and have proved the necessary conditions for the existence of bifurcation points of solutions of this problem.

Lemma 3.2. *Under the above assumptions,*

$$(\nabla_u \Phi)^{-1}(\mathbf{0}) \cap \{(v, w), \lambda\} \in \mathbb{H} \times \mathbb{R} : v = \text{const and } w = \text{const}\} = \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2, \quad (3.8)$$

where $\mathcal{F}_1 = \{0\} \times \mathbb{R}$ and $\mathcal{F}_2 = S^1((1, 0)) \times \mathbb{R}$. Moreover,

- (1) if $(\mathbf{0}, \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F}$, then $\lambda' \in \sigma(-\Delta)$ and $\ker \nabla_u^2 \Phi(\mathbf{0}, \lambda') = \mathbb{V}_{-\Delta}(\lambda') \oplus \mathbb{V}_{-\Delta}(\lambda')$,
- (2) if $((1, 0), \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F}$, then $-2\lambda' \in \sigma(-\Delta)$.

Proof. It is easy to check that the set of constant solutions of problem (3.7) is equal \mathcal{F} .

- (1) By Lemma 2.3, if $(\mathbf{0}, \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F}$, then $\dim \ker \nabla_u^2 \Phi(\mathbf{0}, \lambda') > 0$. Putting in (3.5) $((v_0, w_0), \lambda_0) = (\mathbf{0}, \lambda')$ we obtain $\begin{cases} -\Delta \phi_1 = \lambda' \phi_1 \\ -\Delta \phi_2 = \lambda' \phi_2 \end{cases}$, which completes the proof.
- (2) By Lemma 2.3, if $((1, 0), \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F}$, then $\dim \ker \nabla_u^2 \Phi((1, 0), \lambda') > 1$. Putting in (3.5) $((v_0, w_0), \lambda_0) = ((1, 0), \lambda')$ we obtain $\begin{cases} -\Delta \phi_1 = -2\lambda' \phi_1 \\ -\Delta \phi_2 = 0 \end{cases}$, which completes the proof. \square

In the theorem below we study local and global bifurcations of nonconstant solutions of system (3.7) from the set of constant solutions. The idea of proof is natural. We compute the S^1 -equivariant Conley index and the degree for S^1 -equivariant gradient maps along the families of constant solutions and determine levels at which these invariants change.

Theorem 3.3. *Under the above assumptions,*

(a)

$$\begin{aligned} \mathcal{B}\mathcal{O}\mathcal{F} &= (\mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_1) \cup (\mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_2) \\ &= \bigcup_{\lambda_k \in \sigma(-\Delta)} \{(\mathbf{0}, \lambda_k) \in \mathbb{H} \times \mathbb{R}\} \cup \bigcup_{\lambda_k \in \sigma(-\Delta)} \{S^1((1, 0)) \times \{-\lambda_k/2\} \subset \mathbb{H} \times \mathbb{R}\}, \end{aligned} \quad (3.9)$$

(b)

$$\begin{aligned} \mathcal{GL}\mathcal{OB} &= (\mathcal{GL}\mathcal{OB} \cap \mathcal{F}_1) \cup (\mathcal{GL}\mathcal{OB} \cap \mathcal{F}_2) \\ &= \bigcup_{\lambda_k \in \sigma(-\Delta)} \{(\mathbf{0}, \lambda_k) \in \mathbb{H} \times \mathbb{R}\} \cup \bigcup_{\lambda_k \in \sigma^{\text{odd}}(-\Delta)} \left\{ S^1((1, 0)) \times \{-\lambda_k/2\} \subset \mathbb{H} \times \mathbb{R} \right\}. \end{aligned} \quad (3.10)$$

Moreover, if $\lambda_{k_0} > 0$, then the continuum $\mathcal{C}(\mathbf{0}, \lambda_{k_0})$ is either unbounded or $\mathcal{C}(\mathbf{0}, \lambda_{k_0}) \cap (\mathbb{H} \times \{0\}) \neq \emptyset$.

Proof. Fix an arbitrary $\lambda_{k_0} \in \sigma(-\Delta)$. By Lemma 3.2 we have $\ker \nabla_u^2 \Phi(\mathbf{0}, \lambda_{k_0}) = \mathbb{V}_{-\Delta}(\lambda_{k_0}) \oplus \mathbb{V}_{-\Delta}(\lambda_{k_0})$. It is simple matter to check that $\ker \nabla_u^2 \Phi(\mathbf{0}, \lambda_{k_0})$ is a nontrivial even-dimensional orthogonal S^1 -representation. Combining Theorems 4.5 and 4.7 of [20] we obtain $(\mathbf{0}, \lambda_{k_0}) \in \mathcal{GL}\mathcal{OB}$. If $\lambda_{k_0} > 0$ and $\mathcal{C}(\mathbf{0}, \lambda_{k_0}) \cap \mathbb{H} \times \{0\} = \emptyset$, then applying Theorem 4.7 of [20] we obtain that the continuum $\mathcal{C}(\mathbf{0}, \lambda_{k_0}) \subset \mathbb{H} \times (0, +\infty)$ is unbounded.

Choose $\epsilon > 0$ such that

$$\left[-\frac{\lambda_{k_0} - \epsilon}{2}, -\frac{\lambda_{k_0} + \epsilon}{2} \right] \cap \left\{ -\frac{\lambda_k}{2} : \lambda_k \in \sigma(-\Delta) \right\} = \left\{ -\frac{\lambda_{k_0}}{2} \right\}. \quad (3.11)$$

To shorten notation set $\lambda_{k_0}^\pm = -((\lambda_{k_0} \pm \epsilon)/2)$, $\hat{\lambda}_{k_0} = -(\lambda_{k_0}/2)$, $u_0 = (1, 0)$, $T_{\pm\epsilon} = \nabla_u^2 \Phi(u_0, \lambda_{k_0}^\pm)$ and define $\mathbb{V}_{k_0} = \mathbb{V}_{-\Delta}(\lambda_1) \oplus \cdots \oplus \mathbb{V}_{-\Delta}(\lambda_{k_0})$, $\mathbb{V}_{k_0-1} = \mathbb{V}_{-\Delta}(\lambda_1) \oplus \cdots \oplus \mathbb{V}_{-\Delta}(\lambda_{k_0-1})$. It is understood that $\mathbb{V}_0 = \{0\}$.

We claim that

- (c1) $\dim \ker T_{\pm\epsilon} = 1$,
- (c2) $T_{+\epsilon}$ is negatively defined on $\mathbb{V}_{k_0} \oplus \{0\} \subset \mathbb{H}$,
- (c3) $T_{-\epsilon}$ is negatively defined on $\mathbb{V}_{k_0-1} \oplus \{0\} \subset \mathbb{H}$.

Indeed, like in the proof of Lemma 3.2 the study of $\ker T_{\pm\epsilon}$ is equivalent to the study of solutions of the following system:

$$\begin{aligned} -\Delta \phi_1 &= (\lambda_{k_0} \pm \epsilon) \phi_1, \\ -\Delta \phi_2 &= 0. \end{aligned} \quad (3.12)$$

From condition (3.11) it follows that the linear space of solutions of (3.12) is spanned by $(\phi_1, \phi_2) = (0, 1)$. Hence, $\dim \ker T_{\pm\epsilon} = 1$. Examining (3.12) we obtain that

- (1) the operator $T_{+\epsilon}$ is negatively defined on the space $\mathbb{V}_{k_0} \oplus \{0\} \subset \mathbb{H}$,
- (2) the operator $T_{-\epsilon}$ is negatively defined on the space $\mathbb{V}_{k_0-1} \oplus \{0\} \subset \mathbb{H}$,

which completes the proof of (c1), (c2), and (c3).

Since $\dim \ker T_{\pm e} = 1$, there is an open, bounded, and S^1 -invariant subset $\Omega \subset \mathbb{H}$ such that $\nabla_u \Phi(\cdot, \lambda_{k_0}^\pm)^{-1}(0) \cap \Omega = S^1((1, 0))$. Moreover, since the isotropy group of every $(v_0, w_0) \in \mathbb{H} \setminus \{0\}$ is trivial, we obtain

- (1) $\nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^+), \Omega) = (-1)^{\dim \mathbb{V}_{k_0}} \cdot \chi_{S^1}(S^1/\{e\}^+) \in U(S^1),$
- (2) $\nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^-), \Omega) = (-1)^{\dim \mathbb{V}_{k_0-1}} \cdot \chi_{S^1}(S^1/\{e\}^+) \in U(S^1).$

Taking into account that $\dim \mathbb{V}_{k_0} = \mu_{-\Delta}(\lambda_{k_0}) + \dim \mathbb{V}_{k_0-1}$ and the above we obtain

$$\nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^+), \Omega) = (-1)^{\mu_{-\Delta}(\lambda_{k_0})} \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^-), \Omega). \quad (3.13)$$

From the above it follows that

$$\lambda_{k_0} \in \sigma^{\text{odd}}(-\Delta) \text{ iff } \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^+), \Omega) \neq \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^-), \Omega). \quad (3.14)$$

Hence, by Theorem 2.4 we obtain $(u_0, \hat{\lambda}_{k_0}) \in \mathcal{GL}\mathcal{OB}$ provided that $\lambda_{k_0} \in \sigma^{\text{odd}}(-\Delta)$, which completes the proof of (b). Fix $\lambda_{k_0} \in \sigma(-\Delta)$ and note that

- (i) the Conley index $\mathcal{C}\mathcal{J}_{S^1}(S^1(u_0), \lambda_{k_0}^+)$ is an S^1 -CW-complex with S^1 -CW-decomposition $\{(0, S^1)\}, \{(\dim \mathbb{V}_{k_0}, \{e\})\},$
- (ii) the Conley index $\mathcal{C}\mathcal{J}_{S^1}(S^1(u_0), \lambda_{k_0}^-)$ is an S^1 -CW-complex with S^1 -CW-decomposition $\{(0, S^1)\}, \{(\dim \mathbb{V}_{k_0-1}, \{e\})\}.$

Note that $\chi_{S^1}(\mathcal{C}\mathcal{J}_{S^1}(S^1 u_0, \lambda_{k_0}^\pm)) = \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^\pm), \Omega)$, where χ_{S^1} is the S^1 -equivariant Euler characteristic, see [19, 20].

Since the Conley indices $\mathcal{C}\mathcal{J}_{S^1}(S^1(u_0), \lambda_{k_0}^+), \mathcal{C}\mathcal{J}_{S^1}(S^1 u_0, \lambda_{k_0}^-)$ are not S^1 -homotopically equivalent, applying Theorem 2.6 we obtain $(u_0, \hat{\lambda}_{k_0}) \in \mathcal{B}\mathcal{J}\mathcal{F}$, which completes the proof of (a). \square

3.2. Case $\|A\| = \text{const} \neq 0$

In this section we study bifurcations of solutions of the Ginzburg-Landau equation (3.1) assuming that $\|A(x)\| = 1$ for every $x \in \text{cl } \Omega$.

In the lemma below we have described the set of constant solutions of system (3.2) and have proved the necessary conditions for the existence of bifurcation points of solutions of this problem. To simplify notation we put $u_\lambda = (\sqrt{((\lambda - 1)/\lambda)}, 0)$.

Lemma 3.4. *Under the above assumptions,*

$$(\nabla_u \Phi)^{-1}(0) \cap \{((v, w), \lambda) \in \mathbb{H} \times \mathbb{R} : v = \text{const and } w = \text{const}\} = \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3, \quad (3.15)$$

where $\mathcal{F}_1 = \{0\} \times \mathbb{R}, \mathcal{F}_2 = \{S^1(u_\lambda) \times \{\lambda\} : \lambda > 1\}, \mathcal{F}_3 = \{S^1(u_\lambda) \times \{\lambda\} : \lambda < 0\}.$

Moreover,

- (1) if $(0, \lambda') \in \mathcal{B}\mathcal{J}\mathcal{F} \cap \mathcal{F}_1$, then $\lambda' \in \sigma(-\Delta_A),$

(2) if $\lambda' \in (1, +\infty)$ and $(u_{\lambda'}, \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_2$, then $\lambda' \in (1, 3/2)$ and $3 - 2\lambda' \in \sigma(-\Delta_A) \cap (0, 1)$,

(3) if $\lambda' \in (-\infty, 0)$ and $(u_{\lambda'}, \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_3$, then $3 - 2\lambda' \in \sigma(-\Delta_A)$.

Proof. First of all we are looking for constant solutions of system (3.2). Put $(v, w) \equiv (c_1, c_2) \in \mathbb{R}^2$ in (3.2). Taking into account that $\|A(x)\| = 1$ we obtain system (3.2) in the following form $\{c_1 = \lambda(1 - c_1^2 - c_2^2)c_1; c_2 = \lambda(1 - c_1^2 - c_2^2)c_2$. Solving this system we obtain the following set of solutions:

$$\{0\} \times \mathbb{R} \cup \bigcup_{\lambda \in (-\infty, 0) \cup (1, +\infty)} S^1(u_\lambda) \times \{\lambda\} \subset (\nabla_u \Phi)^{-1}(0) \subset \mathbb{H} \times \mathbb{R}, \quad (3.16)$$

which completes the proof.

(1) Applying Lemma 2.3 we obtain that if $(0, \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_1$, then $\dim \ker \nabla_u^2 \Phi(0, \lambda') > 0$. Putting in (3.5) $((v_0, w_0), \lambda_0) = (0, \lambda')$ we obtain the following system $L_A \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \lambda' \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$. Since $\sigma(L_A) = \sigma(-\Delta_A)$, the proof is completed.

(2) Fix $\lambda' \in (1, +\infty)$. Putting in (3.5) $((v_0, w_0), \lambda_0) = (u_{\lambda'}, \lambda')$ we obtain the following system $L_A \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} (3-2\lambda')\phi_1 \\ \phi_2 \end{bmatrix}$. Note that the vector $(\phi_1, \phi_2) = (0, 1)$ solves this system. Hence, $\dim \ker \nabla_u^2 \Phi(u_{\lambda'}, \lambda') \geq 1$. By Lemma 2.3, if $(u_{\lambda'}, \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_2$, then $\dim \ker \nabla_u^2 \Phi(u_{\lambda'}, \lambda') > 1$. Since $\sigma(-\Delta_A) \cap (-\infty, 0] = \emptyset$, $\dim \ker \nabla_u^2 \Phi(u_{\lambda'}, \lambda') > 1$ if and only if $3 - 2\lambda' \in \sigma(-\Delta_A) \cap (0, 1)$ and $\lambda' \in (1, 3/2)$ which completes the proof.

(3) Fix $\lambda' \in (-\infty, 0)$. Putting in (3.5) $((v_0, w_0), \lambda_0) = (u_{\lambda'}, \lambda')$ we obtain the following system $L_A \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} (3-2\lambda')\phi_1 \\ \phi_2 \end{bmatrix}$. Note that the vector $(\phi_1, \phi_2) = (0, 1)$ solves this system, that is, $\dim \ker \nabla_u^2 \Phi(u_{\lambda'}, \lambda') \geq 1$. By Lemma 2.3, if $(u_{\lambda'}, \lambda') \in \mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_3$, then $\dim \ker \nabla_u^2 \Phi(u_{\lambda'}, \lambda') > 1$. Since $\lambda' < 0$, $\dim \ker \nabla_u^2 \Phi(u_{\lambda'}, \lambda') > 1$ if and only if $3 - 2\lambda' \in \sigma(-\Delta_A)$, which completes the proof. \square

Put $\sigma^{\text{odd}}(-\Delta_A) = \{\lambda_k \in \sigma(-\Delta_A) : \dim_{\mathbb{C}} \mathbb{V}_{-\Delta_A}(\lambda_k) \text{ is odd}\}$. In the theorem below we study local and global bifurcations of nonconstant solutions of Ginzburg-Landau equation (3.2). The proof is similar in spirit to that of Theorem 3.3. For simplicity of notation put $u_k = (\sqrt{(2 - \lambda_k)/(3 - \lambda_k)}, 0)$ and $\hat{\lambda}_k = ((3 - \lambda_k)/2)$.

Theorem 3.5. *Under the above assumptions,*

(a)

$$\begin{aligned} \mathcal{B}\mathcal{O}\mathcal{F} &= (\mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_1) \cup (\mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_2) \cup (\mathcal{B}\mathcal{O}\mathcal{F} \cap \mathcal{F}_3) \\ &= \bigcup_{\lambda_k \in \sigma(-\Delta_A)} \{(0, \lambda_k) \in \mathbb{H} \times \mathbb{R}\} \cup \bigcup_{\lambda_k \in \sigma(-\Delta_A) \cap (0, 1)} S^1(u_k) \times \{\hat{\lambda}_k\} \cup \bigcup_{\lambda_k \in \sigma(-\Delta_A) \cap (3, +\infty)} S^1(u_k) \times \{\hat{\lambda}_k\}, \end{aligned} \quad (3.17)$$

(b)

$$\begin{aligned}
\mathcal{GL}\mathcal{OB} &= (\mathcal{GL}\mathcal{OB} \cap \mathcal{F}_1) \cup (\mathcal{GL}\mathcal{OB} \cap \mathcal{F}_2) \cup (\mathcal{GL}\mathcal{OB} \cap \mathcal{F}_3) \\
&= \bigcup_{\lambda_k \in \sigma(-\Delta_A)} \{(\mathbf{0}, \lambda_k) \in \mathbb{H} \times \mathbb{R}\} \cup \bigcup_{\lambda_k \in \sigma^{odd}(-\Delta_A) \cap (0,1)} S^1(u_k) \\
&\quad \times \{\hat{\lambda}_k\} \cup \bigcup_{\lambda_k \in \sigma^{odd}(-\Delta_A) \cap (3,+\infty)} S^1(u_k) \times \{\hat{\lambda}_k\}.
\end{aligned} \tag{3.18}$$

Moreover, for every $\lambda_k \in \sigma(-\Delta_A)$

(1) $\mathcal{C}(\mathbf{0}, \lambda_k) \cap \mathbb{H} \times \{0\} = \emptyset$,

(2) either the continuum $\mathcal{C}(\mathbf{0}, \lambda_k) \subset \mathbb{H} \times (0, +\infty)$ is unbounded or $\mathcal{C}(\mathbf{0}, \lambda_k) \cap \mathcal{GL}\mathcal{OB} \cap \mathcal{F}_2 \neq \emptyset$.

Proof. Bifurcations from the family \mathcal{F}_1 . Note that $\ker \nabla_u^2 \Phi(\mathbf{0}, \lambda_k)$ is a nontrivial even-dimensional orthogonal S^1 -representation for every $\lambda_k \in \sigma(-\Delta_A)$. Combining Theorems 4.5 and 4.7 of [20] we obtain $(\mathbf{0}, \lambda_k) \in \mathcal{GL}\mathcal{OB} \cap \mathcal{F}_1$ and for every $\lambda_k \in \sigma(-\Delta_A)$ the continuum $\mathcal{C}(\mathbf{0}, \lambda_k) \subset \mathbb{H} \times \mathbb{R}$ is unbounded or $\mathcal{C}(\mathbf{0}, \lambda_k) \cap \mathcal{F}_2 \neq \emptyset$. Taking into account that $\nabla_u^2 \Phi(\mathbf{0}, 0)$ is an isomorphism we obtain $\mathcal{C}(\mathbf{0}, \lambda_k) \subset \mathbb{H} \times (0, +\infty)$.

Bifurcations from the family \mathcal{F}_3 . Fix an arbitrary $\lambda_{k_0} \in \sigma(-\Delta_A) \cap (3, +\infty)$ and choose $\epsilon > 0$ such that

$$\left[\frac{3 - \lambda_{k_0} - \epsilon}{2}, \frac{3 - \lambda_{k_0} + \epsilon}{2} \right] \cap \left\{ \frac{3 - \lambda_k}{2} : \lambda_k \in \sigma(-\Delta_A) \right\} = \left\{ \frac{3 - \lambda_{k_0}}{2} \right\}. \tag{3.19}$$

For brevity set $\lambda_{k_0}^\pm = (3 - \lambda_{k_0} \pm \epsilon)/2$, $\hat{\lambda}_{k_0} = (3 - \lambda_{k_0})/2$, $u_{\lambda_{k_0}}^\pm = (\sqrt{(2 - \lambda_{k_0} \pm \epsilon)/(3 - \lambda_{k_0} \pm \epsilon)}, 0)$, $\hat{u}_{\lambda_{k_0}} = (\sqrt{(2 - \lambda_{k_0})/(3 - \lambda_{k_0})}, 0)$, $T_{\pm\epsilon} = \nabla_u^2 \Phi(u_{\lambda_{k_0}}^\pm, \lambda_{k_0}^\pm)$, $\mathbb{V}_k = \mathbb{V}_{-\Delta_A}(\lambda_1) \oplus \cdots \oplus \mathbb{V}_{-\Delta_A}(\lambda_k)$ and $\mathbb{W} = \oplus_{\lambda_k < 1} \mathbb{V}_{-\Delta_A}(\lambda_k)$. It is understood that $\mathbb{V}_0 = \{0\}$ and $\mathbb{W} = \{0\}$ provided that $\sigma(-\Delta_A) \cap (-\infty, 0) = \emptyset$. We claim that

(c1) $\dim \ker T_{\pm\epsilon} = 1$,

(c2) $T_{+\epsilon}$ is negatively defined on the space of dimension $\dim_{\mathbb{C}} \mathbb{V}_{k_0-1} + \dim_{\mathbb{C}} \mathbb{W}$,

(c3) $T_{-\epsilon}$ is negatively defined on the space of dimension $\dim_{\mathbb{C}} \mathbb{V}_{k_0} + \dim_{\mathbb{C}} \mathbb{W}$.

Indeed, like in the proof of Lemma 3.4 the study of $\ker T_{\pm\epsilon}$ is equivalent to the study of solutions of the following system:

$$L_A \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} (\lambda_{k_0} \mp \epsilon) \phi_1 \\ \phi_2 \end{bmatrix}. \tag{3.20}$$

From condition (3.19) it follows that the linear space of solutions of (3.20) is spanned by $(\phi_1, \phi_2) = (0, 1)$. Hence, $\dim \ker T_{\pm} = 1$. Fix $\lambda_k \in \sigma(-\Delta_A)$ and $\phi_1 + i\phi_2 \in \mathbb{V}_{-\Delta_A}(\lambda_k)$. Then (3.20) is equivalent to the following system $\{(\lambda_k - \lambda_{k_0} \pm \epsilon)\phi_1 = 0; (\lambda_k - 1)\phi_2 = 0$. That is why

(1) the operator $T_+(\phi_1, \phi_2)$ is negatively definite if and only if $\phi_1 \in \mathbb{V}_{k_0-1}$ and $\phi_2 \in \mathbb{W}$,

(2) the operator $T_-(\phi_1, \phi_2)$ is negatively definite if and only if $\phi_1 \in \mathbb{V}_{k_0}$ and $\phi_2 \in \mathbb{W}$,

which completes the proof of (c1), (c2), and (c3).

Since $\dim \ker T_{\pm\epsilon} = 1$, there is an open, bounded, and S^1 -invariant subset $\Omega \subset \mathbb{H}$ such that $\nabla_u \Phi(\cdot, \lambda_{k_0}^\pm)^{-1}(0) \cap \Omega = S^1(u_{\lambda_{k_0}}^\pm) \times \{\lambda_{k_0}^\pm\}$. Moreover, since the isotropy group of every $(v_0, w_0) \in \mathbb{H} \setminus \{0\}$ is trivial, we obtain

$$\begin{aligned} (1) \quad & \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^+), \Omega) = (-1)^{\dim_{\mathbb{C}}(\mathbb{V}_{k_0-1} \oplus \mathbb{W})} \cdot \chi_{S^1}(S^1/\{e\}^+) \in U(S^1), \\ (2) \quad & \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^-), \Omega) = (-1)^{\dim_{\mathbb{C}}(\mathbb{V}_{k_0} \oplus \mathbb{W})} \cdot \chi_{S^1}(S^1/\{e\}^+) \in U(S^1). \end{aligned}$$

Taking into account that $\dim_{\mathbb{C}} \mathbb{V}_{k_0} = \mu_{-\Delta_A}^{\mathbb{C}}(\lambda_{k_0}) + \dim_{\mathbb{C}} \mathbb{V}_{k_0-1}$ and the above we obtain

$$\nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^-), \Omega) = (-1)^{\mu_{-\Delta_A}^{\mathbb{C}}(\lambda_{k_0})} \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^+), \Omega). \quad (3.21)$$

From the above it follows that

$$\mu_{-\Delta_A}^{\mathbb{C}}(\lambda_{k_0}) \text{ is odd if and only if } \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^-), \Omega) \neq \nabla_{S^1}\text{-deg}(\nabla_u \Phi(\cdot, \lambda_{k_0}^+), \Omega). \quad (3.22)$$

Hence, by Theorem 2.4 we obtain $(\hat{u}_{\lambda_{k_0}}, \hat{\lambda}_{k_0}) \in \mathcal{GL}\mathcal{OB} \cap \mathcal{F}_3$, provided that $\lambda_{k_0} \in \sigma^{\text{odd}}(-\Delta_A)$. Fix $\lambda_{k_0} \in \sigma(-\Delta_A)$ and note that

- (i) the Conley index $\mathcal{CO}_{S^1}(S^1(u_{\lambda_{k_0}}^+), \lambda_{k_0} + \epsilon)$ is an S^1 -CW-complex with S^1 -CW-decomposition $\{(0, S^1)\}, \{(\dim_{\mathbb{C}}(\mathbb{V}_{k_0-1} \oplus \mathbb{W}), \{e\})\}$,
- (ii) the Conley index $\mathcal{CO}_{S^1}(S^1(u_{\lambda_{k_0}}^-), \lambda_{k_0} - \epsilon)$ is an S^1 -CW-complex with the following S^1 -CW-decomposition $\{(0, S^1)\}, \{(\dim_{\mathbb{C}}(\mathbb{V}_{k_0} \oplus \mathbb{W}), \{e\})\}$.

Since the Conley indices $\mathcal{CO}_{S^1}(S^1(u_{\lambda_{k_0}}^+), \lambda_{k_0} + \epsilon)$, $\mathcal{CO}_{S^1}(S^1(u_{\lambda_{k_0}}^-), \lambda_{k_0} - \epsilon)$ are not S^1 -homotopically equivalent, applying Theorem 2.6 we obtain $(\hat{u}_{\lambda_{k_0}}, \hat{\lambda}_{k_0}) \in \mathcal{BOF} \cap \mathcal{F}_3$.

Bifurcations from the family \mathcal{F}_2 . This case can be handled in much the same way as bifurcations from the family \mathcal{F}_3 . \square

4. Remarks and Open Questions

We can substantially simplify Theorem 3.5 assuming that the first eigenvalue of the magnetic Laplace operator $-\Delta_A$ is equal to 1. In the lemma below we show that for a domain Ω of sufficiently small volume this assumption is satisfied.

Lemma 4.1. *Assume that $\|A(x)\| = 1$ for all $x \in \Omega$. Then there is $\epsilon > 0$ such that if $\text{vol } \Omega < \epsilon$, then the first eigenvalue of the magnetic Laplace operator $-\Delta_A$ is equal to 1, that is, $\sigma(-\Delta_A) = \{\lambda_1 = 1 < \lambda_2 < \dots\}$.*

Proof. Let us consider an eigenvalue problem $-\Delta_A \phi = \lambda \phi$. Putting $\phi = \phi_1 + i\phi_2$ we obtain equivalent system of the form

$$\begin{aligned} -\Delta \phi_1(x) - 2\langle A(x), \nabla \phi_2(x) \rangle + \|A(x)\|^2 \phi_1 &= \lambda \phi_1, \\ -\Delta \phi_2(x) + 2\langle A(x), \nabla \phi_1(x) \rangle + \|A(x)\|^2 \phi_2 &= \lambda \phi_2. \end{aligned} \quad (4.1)$$

Multiplying equations by ϕ_1, ϕ_2 , respectively, and integrating by parts we obtain the following equality:

$$\begin{aligned} & \int_{\Omega} \|\nabla \phi_1(x)\|^2 + \|\nabla \phi_2(x)\|^2 - 2\langle A(x), \phi_1(x) \nabla \phi_2(x) - \phi_2(x) \nabla \phi_1(x) \rangle dx \\ &= (\lambda - 1) \int_{\Omega} \phi_1^2(x) + \phi_2^2(x) dx. \end{aligned} \quad (4.2)$$

We remind that $1 \in \sigma(-\Delta_A)$. To complete the proof it is enough to show that for every $\phi_1 + i\phi_2 \in \mathbb{V}_{-\Delta_A}(1)^\perp \subset \mathbb{H}$ the following inequality holds true:

$$\int_{\Omega} \|\nabla \phi_1(x)\|^2 + \|\nabla \phi_2(x)\|^2 - 2\langle A(x), \phi_1(x) \nabla \phi_2(x) - \phi_2(x) \nabla \phi_1(x) \rangle dx > 0. \quad (4.3)$$

Let $\mu_2(\Omega)$ be the first positive eigenvalue of the Laplace operator $-\Delta$ under the Neumann boundary condition. Taking into account that $\|A(x)\| = 1$ we obtain the following estimation:

$$\begin{aligned} & \int_{\Omega} \|\nabla \phi_1(x)\|^2 + \|\nabla \phi_2(x)\|^2 - 2\langle A(x), \phi_1(x) \nabla \phi_2(x) - \phi_2(x) \nabla \phi_1(x) \rangle dx \\ & \geq \int_{\Omega} \|\nabla \phi_1(x)\|^2 + \|\nabla \phi_2(x)\|^2 - 2\|\phi_1(x) \nabla \phi_2(x) - \phi_2(x) \nabla \phi_1(x)\| dx \\ & \geq \int_{\Omega} \|\nabla \phi_1(x)\|^2 + \|\nabla \phi_2(x)\|^2 - 2(|\phi_1(x)| \|\nabla \phi_2(x)\| + |\phi_2(x)| \|\nabla \phi_1(x)\|) dx \\ & \geq |\nabla \phi_1|_2^2 + |\nabla \phi_2|_2^2 - 2(|\phi_1|_2 |\nabla \phi_2|_2 + |\phi_2|_2 |\nabla \phi_1|_2) \\ & \geq |\nabla \phi_1|_2^2 + |\nabla \phi_2|_2^2 - \frac{4}{\sqrt{\mu_2(\Omega)}} |\nabla \phi_1|_2 |\nabla \phi_2|_2 \\ & = (|\nabla \phi_1|_2 - |\nabla \phi_2|_2)^2 + \left(2 - \frac{4}{\sqrt{\mu_2(\Omega)}}\right) |\nabla \phi_1|_2 |\nabla \phi_2|_2, \end{aligned} \quad (4.4)$$

where $|\cdot|_2$ is the $L_2(\Omega)$ norm. By the Cheeger's inequality we have $\mu_2(\Omega) \rightarrow \infty$ as $\text{vol } \Omega \rightarrow 0$, see [23], that is, for any $M > 0$ exists $\epsilon > 0$ such that if $\text{vol } \Omega < \epsilon$, then $\mu_2(\Omega) > M$. To complete the proof it is enough to choose $\epsilon > 0$ for $M = 4$. \square

Combining Lemmas 3.4 and 4.1 with Theorem 3.5 we obtain the following corollary. In this corollary we use the notation of Theorem 3.5. Since $\sigma(-\Delta_A) \cap (0, 1) = \emptyset$, the sets $\mathcal{BCF}, \mathcal{GLCB}$ are less complicated. What is important is that we obtain unbounded continua $\mathcal{C}(0, \lambda_k)$, that is, we have excluded one possibility of behavior of bifurcating continua in the famous Rabinowitz alternative.

Corollary 4.2. *Choose a domain Ω such that $\text{vol } \Omega < \epsilon$, where ϵ is given by Lemma 4.1. Then $\lambda_1 = 1$ and $\sigma(-\Delta_A) \cap (0, 1) = \emptyset$. By Lemma 3.4 one obtains $\mathcal{BCF} \cap \mathcal{F}_2 = \emptyset$. Applying Theorem 3.5 one obtains*

(a)

$$\mathcal{B}\mathcal{O}\mathcal{F} = \bigcup_{\lambda_k \in \sigma(-\Delta_A)} \{(\mathbf{0}, \lambda_k) \in \mathbb{H} \times \mathbb{R}\} \cup \bigcup_{\lambda_k \in \sigma(-\Delta_A) \cap (3, +\infty)} S^1(u_k) \times \{\hat{\lambda}_k\}, \quad (4.5)$$

(b)

$$\mathcal{G}\mathcal{L}\mathcal{O}\mathcal{B} = \bigcup_{\lambda_k \in \sigma(-\Delta_A)} \{(\mathbf{0}, \lambda_k) \in \mathbb{H} \times \mathbb{R}\} \cup \bigcup_{\lambda_k \in \sigma^{\text{odd}}(-\Delta_A) \cap (3, +\infty)} S^1(u_k) \times \{\hat{\lambda}_k\}. \quad (4.6)$$

Moreover, for every $\lambda_k \in \sigma(-\Delta_A)$ the continuum $\mathcal{C}(\mathbf{0}, \lambda_k) \subset \mathbb{H} \times (0, +\infty)$ is unbounded.

Note that in the proofs of Theorems 3.3 and 3.5 one cannot replace the degree for S^1 -equivariant gradient maps by the Leray-Schauder degree. Indeed, since $\Omega^{S^1} = \emptyset$, $\deg_{\text{LS}}(\nabla_u \Phi(\cdot, \lambda_{k_0}^\pm), \Omega) = 0 \in \mathbb{Z}$, see [24, 25]. The famous Rabinowitz alternative, see [26], is not applicable in our paper because $\ker \nabla_u^2 \Phi(\mathbf{0}, \hat{\lambda}_{k_0})$ is even. Moreover, if the multiplicity $\mu_{-\Delta}(\lambda_{k_0})(\mu_{-\Delta_A}^{\mathbb{C}}(\lambda_{k_0}))$ is even, then one cannot apply the degree for S^1 -equivariant gradient maps to prove the existence of a global bifurcation, see assumption (3.14) (see assumption (3.22)). On the other hand, to prove the existence of a local bifurcation one applies the S^1 -equivariant Conley index.

It is a known fact that change of the Conley index implies only a local bifurcation of critical points, see [27–30] for examples and discussion. Therefore, it is natural to rise the following question. Fix $\lambda_{k_0} \in \sigma(-\Delta) \setminus \sigma^{\text{odd}}(-\Delta)$ and $\lambda'_{k_0} \in (\sigma(-\Delta) \setminus \sigma^{\text{odd}}(-\Delta)) \cap ((1, 3/2) \cup (3, +\infty))$.

Is it true that $((1, 0), -(\lambda_{k_0}/2)), ((\sqrt{(2 - \lambda'_{k_0})/(3 - \lambda'_{k_0})}, 0), (3 - \lambda'_{k_0})/2) \in \mathcal{G}\mathcal{L}\mathcal{O}\mathcal{B}$?

This question is at present far from being solved.

From now on one replaces the group S^1 with an arbitrary compact connected Lie group G and assume that $\Phi \in C_G^2(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ is of the form $\Phi(u, \lambda) = (1/2)\|u\|_{\mathbb{H}}^2 - (\lambda/2)\langle Lu, u \rangle_{\mathbb{H}} - \eta(u, \lambda)$, where

(1) $L : \mathbb{H} \rightarrow \mathbb{H}$ is a linear, compact, bounded, self-adjoint, positively definite, and G -equivariant operator,

(2) $\eta : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{R}$ is a G -invariant functional of the class C^2 such that

(a) $\nabla_u \eta : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{H}$ is a compact, G -equivariant operator,

(b) $\nabla_u \eta(\mathbf{0}, \lambda) = \mathbf{0}$,

(c) $\nabla_u^2 \eta(u, \lambda) = o(\|u\|_{\mathbb{H}})$ at $u = \mathbf{0}$, uniformly on bounded λ -intervals.

The gradient $\nabla_u \Phi : \mathbb{H} \times \mathbb{R} \rightarrow \mathbb{H}$ is a G -equivariant operator of the class C^1 of the form $\nabla_u \Phi(u, \lambda) = u - \lambda Lu - \nabla_u \eta(u, \lambda)$. It is clear that $\nabla_u \Phi(\mathbf{0}, \lambda) = \mathbf{0}$ for any $\lambda \in \mathbb{R}$. Define $\mathcal{L}(\lambda) = Id - \lambda L$. Following [31] one can introduce a notion of nonlinear eigenvalue of $\mathcal{L}(\lambda)$.

Definition 4.3. $\lambda_0 > 0$ is a nonlinear eigenvalue of $\mathcal{L}(\lambda)$, if $(0, \lambda_0) \in \mathbb{H} \times \mathbb{R}$ is a bifurcation point of solutions of equation $\nabla_u \Phi(u, \lambda) = 0$ from the curve $\{0\} \times \mathbb{R} \subset \mathbb{H} \times \mathbb{R}$ for any $\eta \in C_G^2(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ satisfying the above conditions.

Forgetting about variational structure and G -symmetries one can study bifurcations of solutions of the equation $\nabla_u \Phi(u, \lambda) = 0$ applying the Leray-Schauder degree. It is clear that

λ_0 is a nonlinear eigenvalue of $\mathcal{L}(\lambda)$ if and only if λ_0^{-1} is an eigenvalue of L of odd multiplicity, see, for instance, Theorem 1.2.1 of [31].

Computing the bifurcation index in term of the degree for G -equivariant gradient maps we obtain the following theorem, see Theorem 4.5 of [20].

Theorem 4.4. $\lambda_0 > 0$ is a nonlinear eigenvalue of $\mathcal{L}(\lambda)$ if and only if λ_0^{-1} is an eigenvalue of odd multiplicity or the eigenspace $\mathbb{V}_L(\lambda_0^{-1})$ of L corresponding to the eigenvalue λ_0^{-1} is a nontrivial G -representation.

In this paper $G = S^1$. All the eigenvalues of L are of even multiplicity and corresponding eigenspaces of L are nontrivial S^1 -representations. Summing up, inverse of any eigenvalue of L is a nonlinear eigenvalue of $\mathcal{L}(\lambda)$. Therefore, one obtains global bifurcations of critical S^1 -orbits from the family \mathcal{F}_1 in Theorems 3.3 and 3.5. Note that if Ω is of sufficiently small volume, then continua bifurcating from family \mathcal{F}_1 are unbounded, see Corollary 4.2.

Finally we underline that we have proved local and global bifurcations of critical S^1 -orbits from connected sets of trivial solutions which are not of the form $\{0\} \times (\alpha, \beta) \subset \mathbb{H} \times \mathbb{R}$, see local and global bifurcations from families $\mathcal{F}_2, \mathcal{F}_3$ in Theorems 3.3 and 3.5.

Appendix

In this section for the convenience of the reader we repeat the relevant material from [19], without proofs, thus making our exposition self-contained.

Let $S^{k-1} = \{x \in \mathbb{R}^k : |x| = 1\}$, $D^k = \{x \in \mathbb{R}^k : |x| \leq 1\}$ and $B^k = D^k \setminus S^{k-1}$. For $H \in \overline{\text{sub}}(S^1)$ define a H -action $H \times S^1 \rightarrow S^1$ by $(h, g) \rightarrow gh^{-1}$. The set of orbits will be denoted S^1/H .

Definition A.1. Let (\mathbb{X}, \mathbb{A}) be a pair of compact S^1 -spaces and let $H_1, \dots, H_q \in \overline{\text{sub}}(S^1)$. We say that \mathbb{X} is obtained from \mathbb{A} by attaching the family of equivariant k -cells of orbit type $\{(k, (H_j)) : j = 1, \dots, q\}$ if there exists a S^1 -equivariant map

$$\varphi : \left(\coprod_{j=1}^q D^k \times S^1/H_j, \coprod_{j=1}^q S^{k-1} \times S^1/H_j \right) \rightarrow (\mathbb{X}, \mathbb{A}), \quad (\text{A.1})$$

which maps $\coprod_{j=1}^q B^k \times S^1/H_j$ S^1 -homeomorphically on $\mathbb{X} \setminus \mathbb{A}$.

Definition A.2. Let (\mathbb{X}, \star) be a pointed compact S^1 -space. If there is a finite sequence of S^1 -spaces $\mathbb{X}_{-1} \subset \mathbb{X}_0 \subset \mathbb{X}_1 \subset \dots \subset \mathbb{X}_{p-1} \subset \mathbb{X}_p = \mathbb{X}$ such that

- (1) $\mathbb{X}_{-1} = \{\star\}$,
- (2) $\mathbb{X}_0 \approx \{\star\} \sqcup \coprod_{j=1}^{q(0)} G/H_{j,0}$, where $H_{1,0}, \dots, H_{q(0),0} \in \overline{\text{sub}}(S^1)$,
- (3) \mathbb{X}_k is obtained from \mathbb{X}_{k-1} by attaching a family of equivariant k -cells of orbit type $\{(k, (H_{j,k})) : j = 1, \dots, q(k)\}$, for $k = 1, \dots, p$,

then S^1 -space \mathbb{X} is said to be a finite S^1 -CW-complex. The set of subspaces $\{\mathbb{X}_0, \dots, \mathbb{X}_p\}$ is said to be the cell decomposition of \mathbb{X} and the set $\bigcup_{k=0}^p \{(k, (H_{j,k})) : j = 1, \dots, q(k)\}$ is said to be the orbit type of the cell decomposition of \mathbb{X} .

We denote by $\mathcal{F}_*(S^1)$ the full subcategory of $\mathcal{T}_*(S^1)$ whose objects are finite S^1 -CW-complexes and by $\mathcal{F}_*[S^1]$ subset of $\mathcal{T}_*[S^1]$ consisting of S^1 -homotopy types of finite S^1 -CW-complexes. Let \mathbf{F} be the free abelian group generated by the pointed S^1 -homotopy types of finite S^1 -CW-complexes and let \mathbf{N} be the subgroup of \mathbf{F} generated by all elements $[\mathbb{A}] - [\mathbb{X}] + [\mathbb{X}/\mathbb{A}]$ for pointed S^1 -CW-subcomplexes \mathbb{A} of a pointed S^1 -CW-complex \mathbb{X} .

Definition A.3. Put $U(S^1) = \mathbf{F}/\mathbf{N}$ and let $\chi_{S^1}(\mathbb{X}) \in U(S^1)$ be the class of $[\mathbb{X}]$ in $U(S^1)$. The element $\chi_{S^1}(\mathbb{X})$ is said to be an S^1 -equivariant Euler characteristic of a pointed S^1 -CW-complex \mathbb{X} .

For $\mathbb{X}, \mathbb{Y} \in \mathcal{F}_*(S^1)$ let $[\mathbb{X} \vee \mathbb{Y}] \in \mathcal{F}_*(S^1)$ denote an S^1 -homotopy type of the wedge $\mathbb{X} \vee \mathbb{Y} \in \mathcal{F}_*(S^1)$. Since $[\mathbb{X}] - [\mathbb{X} \vee \mathbb{Y}] + [(\mathbb{X} \vee \mathbb{Y})/\mathbb{X}] = [\mathbb{X}] - [\mathbb{X} \vee \mathbb{Y}] + [\mathbb{Y}] \in \mathbf{N}$,

$$\chi_{S^1}(\mathbb{X}) + \chi_{S^1}(\mathbb{Y}) = \chi_{S^1}(\mathbb{X} \vee \mathbb{Y}). \quad (\text{A.2})$$

For $\mathbb{X}, \mathbb{Y} \in \mathcal{F}_*(S^1)$ let $\mathbb{X} \wedge \mathbb{Y} = \mathbb{X} \times \mathbb{Y} / \mathbb{X} \vee \mathbb{Y}$. The assignment $(\mathbb{X}, \mathbb{Y}) \rightarrow \mathbb{X} \wedge \mathbb{Y}$ induces a product $U(S^1) \times U(S^1) \rightarrow U(S^1)$ given by

$$\chi_{S^1}(\mathbb{X}) \star \chi_{S^1}(\mathbb{Y}) = \chi_{S^1}(\mathbb{X} \wedge \mathbb{Y}). \quad (\text{A.3})$$

Lemma A.4. $(U(S^1), +, \star)$ with an additive and multiplicative structures given by (A.2) and (A.3), respectively, is a commutative ring with unit $\mathbb{1} = \chi_{S^1}(S^1/S^{1+})$.

One calls $(U(S^1), +, \star)$ the Euler ring of S^1 . If \mathbb{X} is a S^1 -CW-complex without base point, then by \mathbb{X}^+ one denotes pointed S^1 -CW-complex $\mathbb{X}^+ = \mathbb{X} \cup \{\star\}$ and consequently one puts $\chi_{S^1}(\mathbb{X}) = \chi_{S^1}(\mathbb{X}^+)$.

Lemma A.5. $(U(S^1), +)$ is the free abelian group with basis $\chi_{S^1}(S^1/H^+)$, $(H) \in \overline{\text{sub}}[S^1]$. Moreover, if $\mathbb{X} \in \mathcal{F}_*(S^1)$ and $\bigcup_{k=0}^p \bigcup_{j=1}^{q(k)} \{k, (H_{j,k})\}$ is the orbit type of the cell decomposition of \mathbb{X} , then $\chi_{S^1}(\mathbb{X}) = \sum_{(H) \in \overline{\text{sub}}[S^1]} n_{(H)}^{S^1}(\mathbb{X}) \cdot \chi_{S^1}(S^1/H^+)$, where $n_{(H)}^{S^1}(\mathbb{X}) = \sum_{k=0}^q (-1)^k v((H), k)$ and $v((H), k)$ is the number of k -dimensional cells of orbit type (H) .

Below we present additive and multiplicative structures in the Euler ring $U(S^1)$. It is easy to see that for $k, k' \in \mathbb{N}$, the following holds true

- (1) $\chi_{S^1}(S^1/S^{1+}) \star \chi_{S^1}(S^1/\mathbb{Z}_k^+) = \chi_{S^1}(S^1/\mathbb{Z}_k^+)$,
- (2) $\chi_{S^1}(S^1/\mathbb{Z}_k^+) \star \chi_{S^1}(S^1/\mathbb{Z}_{k'}^+) = \Theta = 0 \cdot \chi_{S^1}(S^1/S^{1+}) \in U(S^1)$.

That is why if $\alpha, \beta \in U(S^1)$, then

- (1) $\alpha = \alpha_0 \chi_{S^1}(S^1/S^{1+}) + \sum_{k=1}^{\infty} \alpha_k \chi_{S^1}(S^1/\mathbb{Z}_k^+)$,
- (2) $\beta = \beta_0 \chi_{S^1}(S^1/S^{1+}) + \sum_{k=1}^{\infty} \beta_k \chi_{S^1}(S^1/\mathbb{Z}_k^+)$,
- (3) $\alpha \star \beta = (\alpha_0 \beta_0) \chi_{S^1}(S^1/S^{1+}) + \sum_{k=1}^{\infty} (\alpha_0 \beta_k + \alpha_k \beta_0) \chi_{S^1}(S^1/\mathbb{Z}_k^+)$,
- (4) $\alpha + \beta = (\alpha_0 + \beta_0) \chi_{S^1}(S^1/S^{1+}) + \sum_{k=1}^{\infty} (\alpha_k + \beta_k) \chi_{S^1}(S^1/\mathbb{Z}_k^+)$.

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References

- [1] Y. Almog, "On the bifurcation and stability of periodic solutions of the Ginzburg-Landau equations in the plane," *SIAM Journal on Applied Mathematics*, vol. 61, no. 1, pp. 149–171, 2000.
- [2] P. Bauman, D. Phillips, and Q. Tang, "Stable nucleation for the Ginzburg-Landau system with an applied magnetic field," *Archive for Rational Mechanics and Analysis*, vol. 142, no. 1, pp. 1–43, 1998.
- [3] E. N. Dancer and S. P. Hastings, "On the global bifurcation diagram for the one-dimensional Ginzburg-Landau model of superconductivity," *European Journal of Applied Mathematics*, vol. 11, no. 3, pp. 271–291, 2000.
- [4] M. Dutour and B. Helffer, "On bifurcations from normal solutions to superconducting states," *The Rendiconti del Seminario Matematico della Università e Politecnico di Torino*, vol. 58, no. 3, pp. 259–279, 2000.
- [5] S. Fournais and B. Helffer, *Spectral methods in surface superconductivity*, PNDLEA 77, Birkhäuser Boston, Boston, Mass, USA, 2010.
- [6] B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and M. Owen, *Nodal Sets, Multiplicity and Superconductivity in Non Simply Connected Domains*, 2001.
- [7] N. Hirano and S. M. Rybicki, "A remark on global bifurcations of solutions of Ginzburg-Landau equation," *Nonlinear Analysis*, vol. 12, no. 6, pp. 2943–2946, 2011.
- [8] T. Ma and S. Wang, *Bifurcation Theory and Applications*, vol. 53 of *World Scientific Series on Nonlinear Science A*, 2005.
- [9] T. Ma and S. Wang, "Bifurcation and stability of superconductivity," *Journal of Mathematical Physics*, vol. 56, no. 9, pp. 95112–95143, 2005.
- [10] A. Pushnitski and G. Rozenblum, "Eigenvalue clusters of the Landau Hamiltonian in the exterior of a compact domain," *Documenta Mathematica*, vol. 12, pp. 569–586, 2007.
- [11] M. Izydorek, "Equivariant Conley index in Hilbert spaces and applications to strongly indefinite problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 51, no. 1, pp. 33–66, 2002.
- [12] J. Smoller and A. Wasserman, "Bifurcation and symmetry-breaking," *Inventiones Mathematicae*, vol. 100, no. 1, pp. 63–95, 1990.
- [13] Z. Balanov, W. Krawcewicz, and H. Steinlein, *Applied Equivariant Degree*, vol. 1 of *AIMS Series on Differential Equations & Dynamical Systems*, Springfield, Ill, USA, 2006.
- [14] Z. Balanov, W. Krawcewicz, S. Rybicki, and H. Steinlein, "A short treatise on the equivariant degree theory and its applications," *Journal of Fixed Point Theory and Applications*, vol. 8, no. 1, pp. 1–74, 2010.
- [15] J. Fura and S. Rybicki, "Bifurcation from infinity of periodic solutions of second order Hamiltonian systems," *Annales de l'Institut Henri Poincaré C*, vol. 24, no. 3, pp. 471–490, 2007.
- [16] K. Gęba, "Degree for gradient equivariant maps and equivariant Conley index," in *Topological Nonlinear Analysis II: Degree, Singularity, and Variations*, M. Matzeu and A. Vignoli, Eds., PNDLE 27, pp. 247–272, Birkhäuser, 1997.
- [17] S. Rybicki, " S^1 -degree for orthogonal maps and its applications to bifurcation theory," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 23, no. 1, pp. 83–102, 1994.
- [18] S. Rybicki, "Degree for equivariant gradient maps," *Milan Journal of Mathematics*, vol. 73, pp. 103–144, 2005.
- [19] T. Tom Dieck, *Transformation Groups*, Walter de Gruyter, Berlin, Germany, 1987.
- [20] G. López Garza and S. Rybicki, "Equivariant bifurcation index," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 7, pp. 2779–2791, 2010.
- [21] G. Arioli and A. Szulkin, "A semilinear Schrödinger equation in the presence of a magnetic field," *Archive for Rational Mechanics and Analysis*, vol. 170, no. 4, pp. 277–295, 2003.
- [22] S. Jimbo, Y. Morita, and J. Zhai, "Ginzburg-Landau equation and stable steady state solutions in a non-trivial domain," *Communications in Partial Differential Equations*, vol. 20, no. 11-12, pp. 2093–2112, 1995.
- [23] J. Cheeger, "A lower bound for the smallest eigenvalue of the Laplacian," in *Problems in Analysis*, pp. 195–199, Princeton University Press, Princeton, NJ, USA, 1970, Papers dedicated to Salomon Bochner, 1969.
- [24] E. N. Dancer, "Symmetries, degree, homotopy indices and asymptotically homogeneous problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 6, no. 7, pp. 667–686, 1982.
- [25] A. Kushkuley and Z. Balanov, *Geometric methods in degree theory for equivariant maps*, vol. 1632 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1996.
- [26] P. H. Rabinowitz, "Some global results for nonlinear eigenvalue problems," *Journal of Functional Analysis*, vol. 7, pp. 487–513, 1971.

- [27] A. Ambrosetti, "Branching points for a class of variational operators," *Journal d'Analyse Mathématique*, vol. 76, pp. 321–335, 1998.
- [28] R. Baohme, "Die laosung der Verzweigungsgleichungen für nichtlineare eigenwert-probleme," *Mathematische Zeitschrift*, vol. 127, pp. 105–126, 1972.
- [29] A. Marino, "La biforcazione nel caso variazionale," in *Proceedings of the Conferenze del Seminario di Matematica dell'Università di Bari*, no. 132, 1977.
- [30] F. Takens, "Some remarks on the Böhme-Berger bifurcation theorem," *Mathematische Zeitschrift*, vol. 129, pp. 359–364, 1972.
- [31] J. Lopez-Gomez, *Spectral Theory and Nonlinear Analysis*, Chapman and Hall, Boca Raton, UK, 2001.