Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 497594, 6 pages
doi:10.1155/2012/497594

Research Article

Double Sequences and Selections

Dragan Djurčić,[1] Ljubiša D. R. Kočinac,[2] and Mališa R. Žižović[3]

[1] *Technical Faculty in Čačak, University of Kragujevac, Svetog Save 65, 32000 Čačak, Serbia*
[2] *Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia*
[3] *Department of Applied Mathematics, Singidunum University, 11070 Belgrade, Serbia*

Correspondence should be addressed to Ljubiša D. R. Kočinac, lkocinac@gmail.com

Received 6 May 2012; Accepted 18 June 2012

Academic Editor: Márcia Federson

Copyright © 2012 Dragan Djurčić et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We present some results about selection properties in the class of double sequences of real numbers.

1. Introduction

In 1900, Pringsheim introduced the concept of convergence of real double sequences: a double sequence $X = (x_{m,n})_{m,n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ (notation P-lim $X = a$ or P-lim $x_{m,n} = a$), if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|x_{m,n} - a| < \varepsilon$ for all $m, n > n_0$ (see [1], and also [2, 3]). The limit a is called the *Pringsheim limit* of X.

In this paper we denote by c_2^a the set of all double real sequences converging to a point $a \in \mathbb{R}$ in Pringsheim's sense.

A considerable number of papers which appeared in recent years study the set c_2^a and its subsets from various points of view (see [4–13]). Some results in this investigation are generalizations of known results concerning simple sequences to certain classes of double sequences, while other results reflect a specific nature of the Pringsheim convergence (e.g., the fact that a double sequence may converge without being bounded). In this paper we begin with a quite different investigation of double sequences related to selection principles (and games corresponded to them): for a given sequence $(X_n : n \in \mathbb{N})$ of double sequences that belong to one class $\mathcal{A}$ we select from each X_n a subset Y_n by a prescribed procedure so that Y_n's may be arranged to a new double sequence Y which belongs to another class $\mathcal{B}$ (not necessarily distinct from $\mathcal{A}$) of double sequences. (For selection principles theory see [14, 15], and for selection properties of some classes of simple sequences see [16–19]). Moreover, our investigation suggests also introduction of new selection principles: instead of a sequence of

double sequences from a class $\mathcal{A}$ we start with a double sequence of double sequences from $\mathcal{A}$ (see Definitions 2.1 and 2.7). The classes of double sequences considered in this article are subsets of the class c_2^a and will be defined below.

If P-lim$|X| = \infty$, (equivalently, for every $M > 0$ there are $n_1, n_2 \in \mathbb{N}$ such that $|x_{m,n}| > M$ whenever $m \geq n_1$, $n \geq n_2$), then X is said to be *definitely divergent*.

A double sequence $X = (x_{m,n})_{m,n\in\mathbb{N}}$ is *bounded* if there is $M > 0$ such that $|x_{m,n}| < M$ for all $m, n \in \mathbb{N}$.

Notice that a P-convergent double sequence need not be bounded.

A number $L \in \mathbb{R}$ is said to be a *Pringsheim limit point* of a double sequence $X = (x_{m,n})_{m,n\in\mathbb{N}}$ if there exist two increasing sequences $m_1 < m_2 \cdots < m_i, \cdots$ and $n_1 < n_2 \cdots < n_i, \cdots$ such that

$$\lim_{i\to\infty} x_{m_i,n_i} = L. \tag{1.1}$$

In [20], Hardy introduced the notion of *regular convergence* for double sequences: a double sequence $X = (x_{m,n})_{m,n\in\mathbb{N}}$ regularly converges to a point a if it P-converges to a and for each $m \in \mathbb{N}$ and each $n \in \mathbb{N}$ there exist the following two limits:

$$\begin{aligned} \lim_{n\to\infty} x_{m,n} &= R_m, \\ \lim_{m\to\infty} x_{m,n} &= C_n. \end{aligned} \tag{1.2}$$

The symbol $\overline{c}_2^a$ denotes the set of elements $(x_{m,n})_{m,n\in\mathbb{N}}$ in c_2^a which are bounded, regular and such that $\lim_{m\to\infty} x_{m,n} = \lim_{n\to\infty} x_{m,n} = a$.

2. Results

We begin with the following new selection principle for classes of double sequences.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be subclasses of c_2^a. Then $\mathsf{S}_1^{(\mathsf{d})}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each double sequence $(A_{m,n} : m, n \in \mathbb{N})$ of elements of $\mathcal{A}$ there are elements $a_{m,n} \in A_{m,n}$ such that the double sequence $(a_{m,n})_{m,n\in\mathbb{N}}$ belongs to $\mathcal{B}$.

Theorem 2.2. *For $a \in \mathbb{R}$ the selection principle $\mathsf{S}_1^{(\mathsf{d})}(c_2^a, c_2^a)$ is true.*

Proof. Let $(X^{j,k} : j, k \in \mathbb{N})$ be a double sequence of elements in c_2^a. Suppose that $X^{j,k} = (x_{m,n}^{j,k})_{m,n\in\mathbb{N}}$ for all $j, k \in \mathbb{N}$. Let us construct a double sequence $Y = (y_{m,n})_{m,n\in\mathbb{N}}$ in the following way:

(1) $y_{1,1} = x_{m_1,m_1}^{1,1} \in X^{1,1}$, where $m_1 \in \mathbb{N}$ is such that $|x_{m,n}^{1,1} - a| \leq 1/2$ for each $m \geq m_1$ and each $n \geq m_1$.

(2) $y_{1,2} = x_{m_2,m_2}^{1,2} \in X^{1,2}$, where $m_2 \in \mathbb{N}$ is such that $|x_{m,n}^{1,2} - a| \leq 1/2^2$ for each $m \geq m_2$ and each $n \geq m_2$.

(3) $y_{2,1} = x^{2,1}_{m_3,m_3} \in X^{2,1}$ with $m_3 \in \mathbb{N}$ such that $|x^{2,1}_{m,n} - a| \leq 1/2^2$ for each $m \geq m_3$ and each $n \geq m_3$.

(4) $y_{2,2} = x^{2,2}_{m_4,m_4} \in X^{2,2}$ with $m_4 \in \mathbb{N}$ such that $|x^{2,2}_{m,n} - a| \leq 1/2^2$ for each $m \geq m_4$ and each $n \geq m_4$.

In general, for $s, t \in \mathbb{N}$, $q = \max\{s, t\} \geq 3$, we put $y_{s,t} = x^{s,t}_{m_p,m_p}$, where

$$p = \begin{cases} (q-1)^2 + t, & \text{if } q = s, \\ (q-1)^2 + 2t - s, & \text{if } q = t, \end{cases} \tag{2.1}$$

and $|x^{s,t}_{m,n} - a| \leq 1/2^q$ for each $m \geq m_p$ and each $n \geq m_p$.

We prove that the double sequence $Y = (y_{m,n})_{m,n\in\mathbb{N}} \in c_2^a$. Let $\varepsilon > 0$ be given. Pick $r \in \mathbb{N}$ such that $1/2^r < \varepsilon$. For each $m \geq r$ and each $n \geq r$, by construction of Y, we have $|y_{m,n} - a| \leq 1/2^r < \varepsilon$. This just means $Y \in c_2^a$. As $y_{m,n} \in X^{m,n}$ for all $m, n \in \mathbb{N}$, the theorem is proved. □

Remark 2.3. The double sequence Y from the proof of Theorem 2.2 has also the following properties: (i) Y is bounded; (ii) Y is regular and $\lim_{m\to\infty} y_{m,n} = \lim_{n\to\infty} y_{m,n} = a$ for each $m \in \mathbb{N}$ and each $n \in \mathbb{N}$, that is $Y \in \overline{c}_2^a$.

Definition 2.4 (see [15]). Let $\mathcal{A}$ and $\mathcal{B}$ be subclasses of c_2^a. Then $\alpha_2(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element B in $\mathcal{B}$ such that $B \cap A_n$ is infinite for all $n \in \mathbb{N}$.

Lemma 2.5. *For $a \in \mathbb{R}$, the selection principle $\alpha_2(c_2^a, c_2^a)$ is satisfied.*

Proof. Let $(S^k : k \in \mathbb{N})$ be a sequence of elements from c_2^a and let for each $k \in \mathbb{N}$, $S^k = (x^k_{m,n})_{m,n\in\mathbb{N}}$.

(1) Form first an increasing sequence $j_1 < j_2 < \cdots < j_i < \cdots$ in $\mathbb{N}$ so that:

(a) $j_1 = \min\{n_0 \in \mathbb{N} : |x^1_{m,n} - a| \leq 1/2 \text{ for all } m \geq n_0 \text{ and for all } n \geq n_0\}$;

(b) Let $i \geq 2$. Find $p_i = \min\{n_0 \in \mathbb{N} : |x^i_{m,n} - a| \leq 1/2^i \text{ for all } m, n \geq n_0\}$, and then define

$$j_i = \begin{cases} p_i, & \text{if } p_i > j_{i-1}; \\ j_{i-1} + 1, & \text{if } p_i \leq j_{i-1}. \end{cases} \tag{2.2}$$

(2) Define now a double sequence $Y = (y_{s,t})_{s,t\in\mathbb{N}}$ in this way:

(a) $y_{s,t} = x^1_{s,t}$ for each $1 \leq s < j_2$, $t \in \mathbb{N}$, and each $1 \leq t < j_2$, $s \in \mathbb{N}$;

(b) for $i \geq 2$, $y_{s,t} = x^i_{s,t}$, for $j_i \leq s < j_{i+1}$, $t \geq j_i$, and $j_i \leq t < j_{i+1}$, $s \geq j_i$.

By construction, $Y \in c_2^a$ and Y has infinitely many common elements with each X^k, $k \in \mathbb{N}$; that is, the selection principle $\alpha_2(c_2^a, c_2^a)$ is satisfied. □

Remark 2.6. Using the technique from [17] we can prove that the double sequence Y in the proof of the previous lemma can be chosen in such a way that Y has infinitely many common elements with each X^k, $k \in \mathbb{N}$, but on the same (corresponding) positions.

Let for each $k \in \mathbb{N}$, x^k denote the sequence $(x^k_{m,m})_{m\in\mathbb{N}}$. Then each x^k converges to a, so that we have the sequence $(x^k : k \in \mathbb{N})$ of sequences converging to a. Let $2 = p_1 < p_2 < p_3 < \cdots$ be a sequence of prime natural numbers. Take sequence $x^1 = (x^1_{m,m})_{m\in\mathbb{N}}$. For each $i \in \mathbb{N}$, replace the elements of x^1 on the positions p^h_i, $h \in \mathbb{N}$, by the corresponding elements of the sequence x^{i+1}. One obtains the sequence $(z_m)_{m\in\mathbb{N}}$ converging to a which has infinitely many common elements with each x^k on the same positions as in x^k. Define now the double sequence $Y = (y_{s,t})_{s,t\in\mathbb{N}}$ so that $y_{s,s} = z_s$, $s \in \mathbb{N}$, and $y_{s,t} = a$ whenever $s \neq t$. By construction, $Y \in \overline{c}^a_2$ and has infinitely many common positions with each X^k.

The following definition gives a double sequence version of the selection property $\alpha_2(\mathcal{A}, \mathcal{B})$.

Definition 2.7. Let $\mathcal{A}$ and $\mathcal{B}$ be subclasses of c^a_2. Then $\alpha^{(d)}_2(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each double sequence $(A_{m,n} : m, n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element B in $\mathcal{B}$ such that $B \cap A_{m,n}$ is infinite for all $(m, n) \in \mathbb{N} \times \mathbb{N}$.

Theorem 2.8. *Let $a \in \mathbb{R}$ be given. The selection principle $\alpha^{(d)}_2(c^a_2, c^a_2)$ is true.*

Proof. Let $(X^{j,k} : j, k \in \mathbb{N})$ be a double sequence of elements in c^a_2 and let $X^{j,k} = (x^{j,k}_{m,n})_{m,n\in\mathbb{N}}$. In a standard way (see [2]) form from this double sequence a sequence $(X^i : i \in \mathbb{N})$ of double sequences $X^i = (x^i_{m,n})_{m,n\in\mathbb{N}}$. Apply now Lemma 2.5 to this sequence and find a double sequence $Y \in c^a_2$ such that $Y \cap X^i$ is infinite for each $i \in \mathbb{N}$. But then $Y \cap X^{j,k}$ is infinite for all $j, k \in \mathbb{N}$. □

Remark 2.9. Notice that the double sequence Y from the proofs of Lemma 2.5 and Theorem 2.8 satisfies: (a) Y is bounded; (b) Y is regular, and $\lim_{n\to\infty} y_{m,n} = \lim_{m\to\infty} y_{m,n} = a$ for each $m \in \mathbb{N}$ and each $n \in \mathbb{N}$.

Theorem 2.10. *Let $a \in \mathbb{R}$ and let $(X^k : k \in \mathbb{N})$ be a sequence of double sequences in c^a_2, $X^k = (x^k_{m,n})_{m,n\in\mathbb{N}}$. Then there is a double sequence $Y = (y_{s,t})_{s,t\in\mathbb{N}}$ in c^a_2 such that for each $k \in \mathbb{N}$ the set $\{(s, t) \in \mathbb{N} \times \mathbb{N} : y_{s,t} = x^k_{m,n}$ for some $(m, n) \in \mathbb{N} \times \mathbb{N}\}$ is infinite.*

Proof. The double sequence Y is defined in the following way.

Let $k \in \mathbb{N}$. There is $i_k \in \mathbb{N}$ such that $|x^k_{m,n} - a| < 2^{-k}$ for all $m, n \geq i_k$. Let

$$s^* = \begin{cases} i_k, & \text{for } s = k, \\ i_k + p, & \text{for } s = k + p,\ p \in \mathbb{N}, \end{cases}$$
$$t^* = \begin{cases} i_k, & \text{for } t = k, \\ i_k + p, & \text{for } t = k + p,\ p \in \mathbb{N}. \end{cases} \tag{2.3}$$

For $t \geq k$ let $y_{k,t} = x^k_{i_k,t^*}$, and for $s \geq k$ let $y_{s,k} = x^k_{s^*,i_k}$. The double sequence $Y = (y_{s,t})_{s,t\in\mathbb{N}}$ constructed in this way is as required, because Y has the following properties:

(1) $Y \in c_2^a$;

(2) the set $B^k = \{y_{k,t} : t \geq k\} \cup \{y_{s,k} : s \geq k\}$ is a subset of $A^k = \{x_{m,n}^k : m, n \in \mathbb{N}\}$;

(3) for each $k \in \mathbb{N}$, B^k is countable;

(4) $\bigcup_{k\in\mathbb{N}} B^k = \{y_{s,t} : s, t \in \mathbb{N}\}$.

□

Another similar result is given in the next theorem.

Theorem 2.11. *Let $a \in \mathbb{R}$ and let $(X^k : k \in \mathbb{N})$ be a sequence of double sequences in c_2^a, $X^k = (x_{m,n}^k)_{m,n\in\mathbb{N}}$. Then there is a double sequence $Y = (y_{s,t})_{s,t\in\mathbb{N}}$ in c_2^a which has one common row with X^k for each $k \in \mathbb{N}$.*

Proof. For each $k \in \mathbb{N}$ there is $n_0(k) \in \mathbb{N}$ such that $|x_{m,n}^k - a| < 2^{-k}$ for all $m, n \geq n_0(k)$, $n_0(k_1) > n_0(k_2)$ whenever $k_1 > k_2$, and $n_0(k) \geq \min\{i(k) \in \mathbb{N} : |x_{m,n}^{k+1} - a| < 2^{-k} \text{ for all } m, n \geq i(k)\}$. Then the desired double sequence Y is defined in such a way that its $n_0(k)$th row is the $n_0(k)$th row of X^k, that is $y_{n_0(k),n} = x_{n_0(k),n}^k$ $(n \in \mathbb{N})$, and $y_{s,t} = a$ otherwise. Let us prove that $Y \in c_2^a$. Indeed, if $\varepsilon > 0$ is given, then choose $p \in \mathbb{N}$ such that $2^{-p} < \varepsilon$. Then for each $k \in \mathbb{N}$ we have $|x_{m,n}^k - a| < \varepsilon$ for all $m, n \geq p$. By construction of Y we have actually that $|y_{m,n} - a| < \varepsilon$ for all $m, n \geq p$, that is $Y \in c_2^a$. □

Consider now an order on the set $\mathbb{N} \times \mathbb{N}$. Let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. Set $(m_1, n_1) \leq_\varphi (m_2, n_2) \Leftrightarrow \varphi(m_1, n_1) \leq \varphi(m_2, n_2)$, where $\leq$ is the natural order in $\mathbb{N}$.

Definition 2.12. Let $\mathcal{A}$ and $\mathcal{B}$ be subclasses of c_2^a. Then $\mathsf{S}_1^\varphi(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element $B = (b_{\varphi^{-1}(n)})_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that $b_{\varphi^{-1}(n)} \in A_n$ for all $n \in \mathbb{N}$.

Theorem 2.13. *Let $a \in \mathbb{R}$ and let $\leq_\varphi$ be as previously mentioned. Then the selection hypothesis $\mathsf{S}_1^\varphi(c_2^a, \overline{c}_2^a)$ is satisfied.*

Proof. Let $(X^k : k \in \mathbb{N})$, $X^k = (x_{m,n}^k)_{m,n\in\mathbb{N}}$, be a sequence in c_2^a. Construct a double sequence $Y = (y_{s,t})_{s,t\in\mathbb{N}}$ as follows.

Fix $k \in \mathbb{N}$. Let $(s(k), t(k)) = \varphi^{-1}(k)$, and let $p(k) = \max\{s(k), t(k)\}$. There is $n_0(k) \in \mathbb{N}$ such that $|x_{m,n}^k - a| < 2^{-p(k)}$ for all $m, n \geq n_0(k)$. Set $y_{s(k),t(k)} = x_{n_0(k),n_0(k)}^k$ and $Y = (y_{s(k),t(k)})_{k\in\mathbb{N}}$. Then, by the construction, $Y \in \overline{c}_2^a$ and Y have exactly one common element with X^k for each $k \in \mathbb{N}$, that is Y is the desired selector. □

3. Concluding Remarks

We considered here selection properties of some classes of convergent double sequences. It would be interesting also to study similar properties for classes of divergent double sequences, as well as selections related to the Pringsheim limit points instead of the P-limits.

Acknowledgments

The authors are supported by MES RS.

References

[1] A. Pringsheim, "Zur Theorie der zweifach unendlichen Zahlenfolgen," *Mathematische Annalen*, vol. 53, no. 3, pp. 289–321, 1900.

[2] E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. 2, Cambridge University Press, Cambridge, UK, 2nd edition, 1926.

[3] G. M. Robison, "Divergent double sequences and series," *Transactions of the American Mathematical Society*, vol. 28, no. 1, pp. 50–73, 1926.

[4] H. Çakalli and E. Savaş, "Statistical convergence of double sequences in topological groups," *Journal of Computational Analysis and Applications*, vol. 12, no. 2, pp. 421–426, 2010.

[5] K. Kayaduman and C. Çakan, "The Cesàro core of double sequences," *Abstract and Applied Analysis*, vol. 2011, Article ID 950364, 9 pages, 2011.

[6] Mursaleen and O. H. H. Edely, "Statistical convergence of double sequences," *Journal of Mathematical Analysis and Applications*, vol. 288, no. 1, pp. 223–231, 2003.

[7] M. Mursaleen and S. A. Mohiuddine, "Statistical convergence of double sequences in intuitionistic fuzzy normed spaces," *Chaos, Solitons & Fractals*, vol. 41, no. 5, pp. 2414–2421, 2009.

[8] M. Mursaleen and S. A. Mohiuddine, "On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space," *Journal of Computational and Applied Mathematics*, vol. 233, no. 2, pp. 142–149, 2009.

[9] M. Mursaleen, S. A. Mohiuddine, and O. H. H. Edely, "On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 603–611, 2010.

[10] R. F. Patterson and E. Savaş, "A category theorem for double sequences," *Applied Mathematics Letters of Rapid Publication*, vol. 24, no. 10, pp. 1680–1684, 2011.

[11] E. Savaş, "(A_σ)-double sequence spaces defined by Orlicz function and double statistical convergence," *Computers & Mathematics with Applications*, vol. 55, no. 6, pp. 1293–1301, 2008.

[12] E. Savaş and R. F. Patterson, "Double sequence spaces defined by Orlicz functions," *Iranian Journal of Science and Technology*, vol. 31, no. 2, pp. 183–188, 2007.

[13] E. Savaş and R. F. Patterson, "Double sequence spaces defined by a modulus," *Mathematica Slovaca*, vol. 61, no. 2, pp. 245–256, 2011.

[14] L. D. R. Kočinac, "Selected results on selection principles," in *Proceedings of the 3rd Seminar on Geometry & Topology*, pp. 71–104, Tabriz, Iran, July 2004.

[15] L. D. R. Kočinac, "α_i-selection principles and games," in *Set Theory and Its Applications*, vol. 533 of *Contemporary Mathematics*, pp. 107–124, American Mathematical Society, Providence, RI, USA, 2011.

[16] D. Djurčić, Lj. D. R. Kočinac, and M. R. Žižović, "Some properties of rapidly varying sequences," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 1297–1306, 2007.

[17] D. Djurčić, Lj. D. R. Kočinac, and M. R. Žižović, "Relations between sequences and selection properties," *Abstract and Applied Analysis*, vol. 2007, Article ID 43081, 8 pages, 2007.

[18] D. Djurčić, Lj. D. R. Kočinac, and M. R. Žižović, "A few remarks on divergent sequences: rates of divergence. II," *Journal of Mathematical Analysis and Applications*, vol. 367, no. 2, pp. 705–709, 2010.

[19] D. Djurčić, L. D. R. Kočinac, and M. R. Žižović, "Summability of sequences and selection properties," *Abstract and Applied Analysis*, vol. 2011, Article ID 213816, 7 pages, 2011.

[20] G. H. Hardy, "On the convergence of certain multiple series," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 19, pp. 86–95, 1917.