Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 594218, 18 pages
doi:10.1155/2012/594218

*Research Article*

# Approximating Common Fixed Points of Nonspreading-Type Mappings and Nonexpansive Mappings in a Hilbert Space

**Kyung Soo Kim**

*Graduate School of Education, Mathematics Education, Kyungnam University, Changwon, Gyeongnam 631-701, Republic of Korea*

Correspondence should be addressed to Kyung Soo Kim, kksmj@kyungnam.ac.kr

Received 6 March 2012; Accepted 29 May 2012

Academic Editor: Paul Eloe



We obtain some fundamental properties for $k$-strictly pseudo-nonspreading mappings in a Hilbert space. We study approximation of common fixed points of $k$-strictly pseudo-nonspreading mappings and nonexpansive mappings in a Hilbert space by using a new iterative scheme. Furthermore, we suggest some open problems.

## 1. Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in C$.

Recently, Kohsaka and Takahashi [1] introduced a class of mappings called nonspreading mappings: Let $E$ be a real smooth, strictly convex, and reflexive Banach space, and let $J$ denote the duality mapping of $E$. Let $C$ be a nonempty closed convex subset of $E$. They called a maping $T : C \to C$ is said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \tag{1.2}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. They considered the class of nonspreading mappings to study the resolvents of a maximal monotone operator in the Banach space. Observe that, if $E$ is a real Hilbert space, then $J$ is the identity and

$$\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = \|x - y\|^2 \tag{1.3}$$

for all $x, y \in E$. Thus, if $C$ is a nonempty closed convex subset of a Hilbert space, then $T : C \to C$ is nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Tx - y\|^2 \tag{1.4}$$

for all $x, y \in C$. It is shown in [2] that (1.4) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle \tag{1.5}$$

for all $x, y \in C$.

Following the terminology of Browder-Petryshyn [3, (page 198)], a mapping $T : C \to H$ is *k-strictly pseudo-nonspreading* if there exists $k \in [0, 1)$ such that

$$\begin{aligned} \|Tx - Ty\|^2 \leq \|x - y\|^2 &+ 2\langle x - Tx, y - Ty\rangle \\ &+ k\|x - Tx - (y - Ty)\|^2, \end{aligned} \tag{1.6}$$

for all $x, y \in C$. Clearly, every nonspreading mapping is $k$-strictly pseudo-nonspreading.

The following is an example of nonspreading mapping which is not nonexpansive mapping.

*Example 1.1* (see [2]). Let $H$ be a Hilbert space. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $T : C \to C$ as follows:

$$Tx = \begin{cases} 0 & \text{if } x \in D, \\ P_E x & \text{if } x \in C \setminus D, \end{cases} \tag{1.7}$$

where $P_E$ is the metric projection of $H$ onto $E$. Then, $T$ is not nonexpansive but nonspreading mapping.

The following example shows that the class of $k$-strictly pseudo-nonspreading mapping is more general than the class of nonspreading mappings.

*Example 1.2* (see [4]). Let $\mathbb{R}$ denotes the real numbers with the usual norm. Let $T : \mathbb{R} \to \mathbb{R}$ be defined for each $x \in \mathbb{R}$ by

$$Tx = \begin{cases} x & \text{if } x \in (-\infty, 0), \\ -2x & \text{if } x \in [0, \infty). \end{cases} \tag{1.8}$$

Then, $T$ is $k$-strictly pseudo-nonspreading but not nonspreading mapping.

*Remark 1.3* (see [4]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $S : C \to C$ be a $k$-strictly pseudo-nonspreading mapping. If $F(S) \neq \emptyset$, then it is closed and convex.

Iemoto and Takahashi [2] introduced the following Moudafi iterative procedure [5]:

$$\begin{gathered} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{\beta_n S x_n + (1-\beta_n) T x_n\} \end{gathered} \tag{1.9}$$

for finding the approximation of common fixed points of nonspreading mapping $S$ and nonexpansive mapping $T$ in a Hilbert space.

In this paper, we obtain some fundamental properties for $k$-strictly pseudo-nonspreading mappings in a Hilbert space. We study approximation of common fixed points of $k$-strictly pseudo-nonspreading mappings and nonexpansive mappings in a Hilbert space by using a new iterative scheme. Furthermore, we suggest some open problems.

## 2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

*Definition 2.1.* Let $E$ be a real Banach space. A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be *demiclosed* at a point $p \in D(T)$ if whenever $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to a point $x \in D(T)$ and $\{Tx_n\}$ converges strongly to $p$, then $Tx = p$.

**Lemma 2.2** (see [2]). *Let $H$ be a real Hilbert space. Then, the following well known results hold:*

(1) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $x, y \in H$ *and for all* $t \in [0,1]$,

(2) $2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$ *for all* $x,y,z,w \in H$.

**Lemma 2.3** (see [6]). *Let $C$ be a nonempty closed convex subset of $H$. If $T : C \to C$ is nonexpansive, then $I - T : C \to H$ is $1/2$-inverse strongly monotone, that is,*

$$\frac{1}{2}\left\|(I-T)x - (I-T)y\right\|^2 \leq \langle x-y, (I-T)x - (I-T)y\rangle \tag{2.1}$$

*for all $x, y \in C$.*

The following lemma is one of the characterizations of a $k$-strictly pseudo-nonspreading mapping.

**Lemma 2.4.** *Let $C$ be a nonempty closed convex subset of $H$. Then, a mapping $S : C \to C$ is $k$-strictly pseudo-nonspreading if and only if*

$$2\left\|Sx - Sy\right\|^2 \leq \left\|Sx - y\right\|^2 + \left\|x - Sy\right\|^3 + k\left\|(I-S)x - (I-S)y\right\|^2 \tag{2.2}$$

*for all $x, y \in C$.*

*Proof.* We have that for all $x, y \in C$,

$$\begin{aligned}
&\|Sx - Sy\|^2 \le \|x-y\|^2 + 2\langle x - Sx, y - Sy\rangle + k\|x - Sx - (y - Sy)\|^2 \\
\Longleftrightarrow\ & 2\|Sx - Sy\|^2 \le \|Sx - Sy\|^2 + \|x-y\|^2 + 2\langle Sx - x, x - Sx - (y - Sy)\rangle \\
&\qquad + 2\|Sx - x\|^2 + k\|x - Sx - (y - Sy)\|^2 \\
&\qquad = \|Sx - Sy\|^2 + 2\langle x - Sx, Sx - Sy\rangle + \|x - Sx\|^2 \\
&\qquad\quad + \|x-y\|^2 + 2\langle Sx - x, x - y\rangle + \|Sx - x\|^2 + k\|x - Sx - (y - Sy)\|^2 \\
&\qquad = \|x - Sy\|^2 + \|Sx - y\|^2 + k\|(I - S)x - (I - S)y\|^2.
\end{aligned} \tag{2.3}$$

This completes the proof of Lemma 2.4. □

**Lemma 2.5.** *Let $C$ be a nonempty closed convex subset of $H$. Let $S$ be a $k$-strictly pseudo-nonspreading mapping of $C$ into itself and let $A = I - S$.*

*Then,*

$$(2-k)\|Ax - Ay\|^2 \le 2\langle x - y, Ax - Ay\rangle + \|Ax\|^2 + \|Ay\|^2 \tag{2.4}$$

*for all $x, y \in C$.*

*Proof.* Let $A = I - S$. We have

$$\begin{aligned}
\|Ax - Ay\|^2 &= \langle Ax - Ay, Ax - Ay\rangle \\
&= \langle x - y - (Sx - Sy), Ax - Ay\rangle \\
&= \langle x - y, Ax - Ay\rangle - \langle Sx - Sy, Ax - Ay\rangle
\end{aligned} \tag{2.5}$$

for all $x, y \in C$. From Lemma 2.2-(2) and Lemma 2.4, we get

$$\begin{aligned}
2\langle Sx - Sy, Ax - Ay\rangle &= 2\langle Sx - Sy, x - y - (Sx - Sy)\rangle \\
&= 2\langle Sx - Sy, x - y\rangle - 2\|Sx - Sy\|^2 \\
&\ge \|Sx - y\|^2 + \|Sy - x\|^2 - \|Sx - x\|^2 - \|Sy - y\|^2 \\
&\quad - \left(\|Sx - y\|^2 + \|x - Sy\|^2 + k\|Ax - Ay\|^2\right) \\
&= -\|x - Sx\|^2 - \|y - Sy\|^2 - k\|Ax - Ay\|^2 \\
&= -\|Ax\|^2 - \|Ay\|^2 - k\|Ax - Ay\|^2.
\end{aligned} \tag{2.6}$$

So, from (2.5) and (2.6), we have

$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle + \frac{1}{2}\left(\|Ax\|^2 + \|Ay\|^2\right) + \frac{k}{2}\|Ax - Ay\|^2. \tag{2.7}$$

Therefore, we get

$$(2 - k)\|Ax - Ay\|^2 \leq 2\langle x - y, Ax - Ay\rangle + \|Ax\|^2 + \|Ay\|^2 \tag{2.8}$$

for all $x, y \in C$. □

**Lemma 2.6** (see [4]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $S : C \to C$ be a $k$-strictly pseudo-nonspreading mapping. Then $I - S$ is demiclosed at 0.*

Tan and Xu [7] proved the following; see also [6, 8].

**Lemma 2.7.** *Let $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

**Lemma 2.8.** *Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. If $\sum_{n=1}^{\infty}\alpha_n\beta_n < \infty$, then $\liminf_{n\to\infty}\beta_n = 0$.*

## 3. Main Theorem

In this section, we prove our main theorem for finding common fixed points of $k$-strictly pseudo-nonspreading mapping and nonexpansive mapping in a Hilbert space.

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $S : C \to C$ be a $k$-strictly pseudo-nonspreading mapping and let $T : C \to C$ be a nonexpansive mapping such that $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0,1]$ such that $\beta_n \in (k, 1]$. Define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Sx_n) + \alpha_n(\gamma_n x_n + (1 - \gamma_n)Tx_n) \end{aligned} \tag{3.1}$$

*for all $n \in \mathbb{N}$. Then, the followings hold:*

(1) *if $\liminf_{n\to\infty}\alpha_n(\beta_n - \gamma_n) > 0$, $\sum_{n=1}^{\infty}\alpha_n(1 - \gamma_n) < \infty$, and $1 + k < (2 - \alpha_n)\beta_n + \alpha_n\gamma_n$, then $\{x_n\}$ converges weakly to $q \in F(S)$,*

(2) *if $\beta_n > \gamma_n$, $\sum_{n=1}^{\infty}(1 - \beta_n) < \infty$, $2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n) > 0$, and $\liminf_{n\to\infty}\alpha_n(\beta_n - \gamma_n)(2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n)) > 0$, then $\{x_n\}$ converges weakly to $q \in F(T)$,*

(3) *if $\liminf_{n\to\infty}\alpha_n > 0$, $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, $\liminf_{n\to\infty}(1 - \beta_n) > 0$, and $\liminf_{n\to\infty}\gamma_n(1 - \gamma_n) > 0$, then $\{x_n\}$ converges weakly to $q \in F(S) \cap F(T)$.*

*Proof.* Putting $U_n = \beta_n I + (1 - \beta_n)S$ and $V_n = \gamma_n I + (1 - \gamma_n)T$. We first show that the sequence $\{x_n\}$ is bounded. Indeed, from Lemma 2.2-(1) and $S$ as a $k$-strictly pseudo-nonspreading mapping, we have

$$\begin{aligned}
\|U_nx - U_ny\|^2 &= \|\beta_n(x-y) + (1-\beta_n)(Sx - Sy)\|^2 \\
&= \beta_n\|x-y\|^2 + (1-\beta_n)\|Sx-Sy\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x - Sx - (y - Sy)\|^2 \\
&\le \beta_n\|x-y\|^2 + (1-\beta_n) \\
&\quad \times \left(\|x-y\|^2 + 2\langle x - Sx, y - Sy\rangle + k\|x - Sx - (y-Sy)\|^2\right) \\
&\quad - \beta_n(1-\beta_n)\|x - Sx - (y - Sy)\|^2 \\
&= \|x-y\|^2 + 2(1-\beta_n)\langle x - Sx, y - Sy\rangle \\
&\quad - (1-\beta_n)(\beta_n - k)\|x - Sx - (y-Sy)\|^2 \\
&\le \|x-y\|^2 + 2(1-\beta_n)\langle x - Sx, y - Sy\rangle \\
&= \|x-y\|^2 + \frac{2}{1-\beta_n}\langle x - U_nx, y - U_ny\rangle,
\end{aligned} \tag{3.2}$$

for all $x, y \in C$. Let $p \in F(S) \cap F(T)$, then

$$U_np = \beta_np + (1-\beta_n)Sp = p. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\|U_nx_n - p\| = \|U_nx_n - U_np\| \le \|x_n - p\|. \tag{3.4}$$

Since $T$ is a nonexpansive mapping and $F(T) \neq \emptyset$, we get

$$\begin{aligned}
\|V_nx_n - p\| &= \|\gamma_nx_n + (1-\gamma_n)Tx_n - p\| \\
&\le \gamma_n\|x_n - p\| + (1-\gamma_n)\|Tx_n - p\| \\
&\le \|x_n - p\|
\end{aligned} \tag{3.5}$$

for all $p \in F(S) \cap F(T)$. From (3.4) and (3.5), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)U_nx_n + \alpha_nV_nx_n - p\|^2 \\
&\le (1-\alpha_n)\|U_nx_n - p\|^2 + \alpha_n\|V_nx_n - p\|^2 \\
&\le \|x_n - p\|^2
\end{aligned} \tag{3.6}$$

for all $n \in \mathbb{N}$. Therefore, there exists $\lim_{n\to\infty}\|x_n - p\|$ and hence $\{x_n\}$ is bounded.

Let

$$\lim_{n\to\infty}\|x_n - p\| = c. \tag{3.7}$$

To prove (1), let

$$z_{n+1} = (1-\alpha_n)U_n x_n + \alpha_n\left(\gamma_n x_n + (1-\gamma_n)Sx_n\right) \tag{3.8}$$

and $A = I - S$. Then, we have

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\| &= \|(1-\alpha_n)U_n x_n + \alpha_n V_n x_n \\
&\quad -(1-\alpha_n)U_n x_n - \alpha_n\left(\gamma_n x_n + (1-\gamma_n)Sx_n\right)\| \\
&= \alpha_n\left\|\gamma_n x_n + (1-\gamma_n)Tx_n - \gamma_n x_n - (1-\gamma_n)Sx_n\right\| \\
&= \alpha_n(1-\gamma_n)\|Tx_n - Sx_n\|.
\end{aligned} \tag{3.9}$$

Since $\sum_{n=1}^{\infty}\alpha_n(1-\gamma_n) < \infty$, we have $\lim_{n\to\infty}\|x_n - z_n\| = 0$. From the continuity of $\|\cdot\|$, we get

$$\lim_{n\to\infty}\|z_n - p\| = \lim_{n\to\infty}\|x_n - p\| = c. \tag{3.10}$$

Since $U_n = \beta_n I + (1-\beta_n)S$, we get

$$\begin{aligned}
U_n x_n - Sx_n &= \beta_n x_n + (1-\beta_n)Sx_n - Sx_n \\
&= \beta_n(x_n - Sx_n) = \beta_n(I-S)x_n \\
&= \beta_n Ax_n, \\
U_n x_n - x_n &= \beta_n x_n + (1-\beta_n)Sx_n - x_n \\
&= -(1-\beta_n)(x_n - Sx_n) \\
&= -(1-\beta_n)Ax_n.
\end{aligned} \tag{3.11}$$

So, we have from Lemma 2.5, Lemma 2.6 ($;Ap = 0$), as well as (3.4), (3.11) that

$$\begin{aligned}
\left\|z_{n+1} - p\right\| &= \left\|(1-\alpha_n)U_n x_n + \alpha_n\left(\gamma_n x_n + (1-\gamma_n)Sx_n\right) - p\right\|^2 \\
&= \left\|U_n x_n - p - \alpha_n(U_n x_n - Sx_n) + \alpha_n\gamma_n(x_n - Sx_n)\right\|^2 \\
&= \left\|U_n x_n - p - \alpha_n\beta_n Ax_n + \alpha_n\gamma_n Ax_n\right\|^2
\end{aligned}$$

$$\begin{aligned}
&= \left\|U_n x_n - p - \alpha_n(\beta_n - \gamma_n)Ax_n\right\|^2 \\
&\leq \left\|U_n x_n - p\right\|^2 - 2\alpha_n(\beta_n - \gamma_n)\langle U_n x_n - p, Ax_n\rangle + \alpha_n^2(\beta_n - \gamma_n)^2\|Ax_n\|^2 \\
&\leq \left\|x_n - p\right\|^2 - 2\alpha_n(\beta_n - \gamma_n)\langle U_n x_n - p, Ax_n - Ap\rangle + \alpha_n^2(\beta_n - \gamma_n)^2\|Ax_n\|^2 \\
&= \left\|x_n - p\right\|^2 - 2\alpha_n(\beta_n - \gamma_n)\langle U_n x_n - x_n, Ax_n - Ap\rangle \\
&\quad - 2\alpha_n(\beta_n - \gamma_n)\langle x_n - p, Ax_n - Ap\rangle + \alpha_n^2(\beta_n - \gamma_n)^2\|Ax_n\|^2 \\
&\leq \left\|x_n - p\right\|^2 + 2\alpha_n(1-\beta_n)(\beta_n - \gamma_n)\langle Ax_n, Ax_n - Ap\rangle \\
&\quad + \alpha_n(\beta_n - \gamma_n)\left(\|Ax_n\|^2 + \left\|Ap\right\|^2 - (2-k)\left\|Ax_n - Ap\right\|^2\right) \\
&\quad + \alpha_n^2(\beta_n - \gamma_n)^2\ \|Ax_n\|^2 \\
&= \left\|x_n - p\right\|^2 + 2\alpha_n(1-\beta_n)(\beta_n - \gamma_n)\|Ax_n\|^2 \\
&\quad - (1-k)\alpha_n(\beta_n - \gamma_n)\|Ax_n\|^2 + \alpha_n^2(\beta_n - \gamma_n)^2\|Ax_n\|^2 \\
&= \left\|x_n - p\right\|^2 - \alpha_n(\beta_n - \gamma_n)(1 - k - 2(1-\beta_n) - \alpha_n(\beta_n - \gamma_n))\|Ax_n\|^2.
\end{aligned} \tag{3.12}$$

Hence

$$\alpha_n(\beta_n - \gamma_n)(1 - k - 2(1-\beta_n) - \alpha_n(\beta_n - \gamma_n))\|Ax_n\|^2 \leq \left\|x_n - p\right\|^2 - \left\|z_{n+1} - p\right\|^2. \tag{3.13}$$

Since $\liminf_{n\to\infty}\alpha_n(\beta_n - \gamma_n) > 0$, we get

$$\lim_{n\to\infty}\|x_n - Sx_n\| = \lim_{n\to\infty}\|Ax_n\| = 0. \tag{3.14}$$

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $q$. From Lemma 2.6, we obtain $q \in F(S)$. To show our conclusion, it is sufficient to show that for another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $v \in F(S)$, $q = v$. Before proving this, we show that for any $z \in F(S)$, $\lim_{n\to\infty}\|x_n - z\|$ exists. Since

$$U_n z = \beta_n z + (1-\beta_n)Sz = z \tag{3.15}$$

and, from (3.2), we get

$$\|U_n x_n - z\|^2 = \|U_n x_n - U_n z\|^2 \leq \|x_n - z\|^2 \tag{3.16}$$

for all $z \in F(S)$. Hence, we have

$$\begin{aligned}
\|z_{n+1} - z\| &= \left\|(1-\alpha_n)U_n x_n + \alpha_n\left(\gamma_n x_n + (1-\gamma_n)Sx_n\right) - z\right\| \\
&= \left\|(1-\alpha_n)(U_n x_n - z) + \alpha_n\left(\gamma_n(x_n - z) + (1-\gamma_n)(Sx_n - z)\right)\right\| \\
&\le (1-\alpha_n)\|U_n x_n - z\| + \alpha_n\gamma_n\|x_n - z\| + \alpha_n(1-\gamma_n)\|Sx_n - z\| \\
&\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|x_n - z\| + \alpha_n(1-\gamma_n)\|Sx_n - z\| \\
&= \|x_n - z\| + \alpha_n(1-\gamma_n)\|Sx_n - z\| \\
&\le \|z_n - z\| + \|x_n - z_n\| + \alpha_n(1-\gamma_n)\|Sx_n - z\|
\end{aligned} \tag{3.17}$$

for all $z \in F(S)$. From Lemma 2.7 and (3.9), $\lim_{n\to\infty}\|z_n - z\|$ exists. So, there exists $\lim_{n\to\infty}\|x_n - z\|$ for all $z \in F(S)$ because $\lim_{n\to\infty}(x_n - z_n) = 0$. Suppose that $q \ne v$. We have from Opial's theorem [9] that

$$\begin{aligned}
\lim_{n\to\infty}\left\|x_n - q\right\| = \lim_{j\to\infty}\left\|x_{n_i} - q\right\| &< \lim_{j\to\infty}\|x_{n_i} - v\| = \lim_{n\to\infty}\|x_n - v\| \\
&= \lim_{j\to\infty}\left\|x_{n_j} - v\right\| \\
&< \lim_{j\to\infty}\left\|x_{n_j} - q\right\| = \lim_{n\to\infty}\left\|x_n - q\right\|.
\end{aligned} \tag{3.18}$$

This is a contradiction. So, $\{x_n\}$ converges weakly to $q \in F(S)$.

To prove (2), let

$$z_{n+1} = (1-\alpha_n)\left(\beta_n x_n + (1-\beta_n)Tx_n\right) + \alpha_n V_n x_n \tag{3.19}$$

and $B = I - T$. It follows that

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\| &= \|(1-\alpha_n)U_n x_n + \alpha_n V_n x_n \\
&\quad -(1-\alpha_n)\left(\beta_n x_n + (1-\beta_n)Tx_n\right) - \alpha_n V_n x_n\| \\
&= (1-\alpha_n)\left\|\beta_n x_n + (1-\beta_n)Sx_n - \beta_n x_n - (1-\beta_n)Tx_n\right\| \\
&\le (1-\beta_n)\|Sx_n - Tx_n\|.
\end{aligned} \tag{3.20}$$

So, from the boundedness of $\{x_n\}$, $\{z_n\}$ is also bounded. Since $T$ is a nonexpansive, by Lemma 2.3, $B$ is $1/2$-inverse strongly monotone and $Bp = 0$, we have

$$\begin{aligned}
\|z_{n+1} - p\|^2 &= \|(1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n) + \alpha_n V_n x_n - p\|^2 \\
&= \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p) - \alpha_n(\beta_n - \gamma_n)Bx_n\|^2 \\
&= \|\beta_n(x_n - p) + (1-\beta_n)(Tx_n - p)\|^2 \\
&\quad - 2\alpha_n(\beta_n - \gamma_n)\langle \beta_n(x_n - p) + (1-\beta_n)(Tx_n - p), Bx_n\rangle \\
&\quad + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&\le (\beta_n\|x_n - p\| + (1-\beta_n)\|Tx_n - p\|)^2 \\
&\quad - 2\alpha_n\beta_n(\beta_n - \gamma_n)\langle x_n - p, Bx_n - Bp\rangle \\
&\quad - 2\alpha_n(1-\beta_n)(\beta_n - \gamma_n)\langle Tx_n - p, Bx_n - Bp\rangle \\
&\quad + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&\le \|x_n - p\|^2 - 2\alpha_n\beta_n(\beta_n - \gamma_n)\langle x_n - p, Bx_n - Bp\rangle \\
&\quad - 2\alpha_n(1-\beta_n)(\beta_n - \gamma_n)\langle Tx_n - x_n, Bx_n - Bp\rangle \\
&\quad - 2\alpha_n(1-\beta_n)(\beta_n - \gamma_n)\langle x_n - p, Bx_n - Bp\rangle + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&= \|x_n - p\|^2 + 2\alpha_n(1-\beta_n)(\beta_n - \gamma_n)\langle Bx_n, Bx_n\rangle \\
&\quad - 2\alpha_n(\beta_n - \gamma_n)\langle x_n - p, Bx_n - Bp\rangle + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&\le \|x_n - p\|^2 + 2\alpha_n(1-\beta_n)(\beta_n - \gamma_n)\|Bx_n\|^2 \\
&\quad - \alpha_n(\beta_n - \gamma_n)\|Bx_n - Bp\|^2 + \alpha_n^2(\beta_n - \gamma_n)^2\|Bx_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_n(\beta_n - \gamma_n)(1 - 2(1-\beta_n) - \alpha_n(\beta_n - \gamma_n))\|Bx_n\|^2
\end{aligned} \tag{3.21}$$

and hence

$$\alpha_n(\beta_n - \gamma_n)(1 - 2(1-\beta_n) - \alpha_n(\beta_n - \gamma_n))\|Bx_n\|^2 \le \|x_n - p\|^2 - \|z_{n+1} - p\|^2 \tag{3.22}$$

for $p \in F(S) \cap F(T)$. Summing from $n = 1$ to $N$, from (3.20), we have

$$\begin{aligned}
&\sum_{n=1}^{N}\alpha_n(\beta_n - \gamma_n)(2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n))\|Bx_n\|^2 \\
&\quad \le \|x_1 - p\|^2 + \sum_{n=1}^{N-1}\left(\|x_{n+1} - p\|^2 - \|z_{n+1} - p\|^2\right) - \|z_{N+1} - p\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|x_1 - p\|^2 + \sum_{n=1}^{N-1} (\|x_{n+1} - p\| + \|z_{n+1} - p\|) \|x_{n+1} - z_{n+1}\| \\
&\leq \|x_1 - p\|^2 + \sum_{n=1}^{N-1} (1 - \beta_n) (\|x_{n+1} - p\| + \|z_{n+1} - p\|) \|Sx_n - Tx_n\| \\
&\leq \|x_1 - p\|^2 + M \sum_{n=1}^{N-1} (1 - \beta_n),
\end{aligned} \tag{3.23}$$

where $M = \sup_{n\in\mathbb{N}}\{(\|x_{n+1} - p\| + \|z_{n+1} - p\|)\|Sx_n - Tx_n\|\}$. Letting $N \to \infty$, from $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \alpha_n (\beta_n - \gamma_n) (2\beta_n - 1 - \alpha_n (\beta_n - \gamma_n)) \|Bx_n\|^2 \\
&\quad \leq \|x_1 - p\|^2 + M \sum_{n=1}^{\infty} (1 - \beta_n) \\
&\quad < \infty.
\end{aligned} \tag{3.24}$$

Since $\sum_{n=1}^{\infty} \alpha_n (\beta_n - \gamma_n)(2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n)) = \infty$, from Lemma 2.8, we get

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = \liminf_{n\to\infty} \|Bx_n\| = 0. \tag{3.25}$$

Since $T$ is a nonexpansive mapping, from (3.1), we get

$$\begin{aligned}
\|Tx_{n+1} - x_{n+1}\| &= \|Tx_{n+1} - (1 - \alpha_n)U_n x_n - \alpha_n V_n x_n\| \\
&\leq \|Tx_{n+1} - Tx_n\| + (1 - \alpha_n)\|Tx_n - U_n x_n\| + \alpha_n \|Tx_n - V_n x_n\| \\
&\leq \|x_{n+1} - x_n\| + (1 - \alpha_n)\|Tx_n - U_n x_n\| + \alpha_n \|Tx_n - V_n x_n\| \\
&\leq (1 - \alpha_n)\|U_n x_n - x_n\| + \alpha_n \|V_n x_n - x_n\| \\
&\quad + (1 - \alpha_n)\|Tx_n - \beta_n x_n - (1 - \beta_n)Sx_n\| \\
&\quad + \alpha_n \|Tx_n - \gamma_n x_n - (1 - \gamma_n)Tx_n\| \\
&= (1 - \alpha_n)\|\beta_n x_n + (1 - \beta_n)Sx_n - x_n\| + \alpha_n \|\gamma_n x_n + (1 - \gamma_n)Tx_n - x_n\| \\
&\quad + (1 - \alpha_n)\|\beta_n(Tx_n - x_n) + (1 - \beta_n)(Tx_n - Sx_n)\| + \alpha_n \|\gamma_n(Tx_n - x_n)\| \\
&\leq (1 - \alpha_n)(1 - \beta_n)\|Sx_n - x_n\| + \alpha_n (1 - \gamma_n)\|Tx_n - x_n\| \\
&\quad + (1 - \alpha_n)\|Tx_n - x_n\| + (1 - \alpha_n)(1 - \beta_n)\|Tx_n - Sx_n\| + \alpha_n \gamma_n \|Tx_n - x_n\| \\
&\leq \|Tx_n - x_n\| + (1 - \beta_n)(\|Sx_n - x_n\| + \|Tx_n - Sx_n\|).
\end{aligned} \tag{3.26}$$

Since $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, from Lemma 2.7, there exists the limit of $\{\|Tx_n - x_n\|\}$. Therefore, from (3.25), we get

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{3.27}$$

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $q$. Since a nonexpansive mapping $T$ is demiclosed, we have $q \in F(T)$. As in the proof of (1), $\{x_n\}$ converges weakly to $q \in F(T)$.

(3) From (3.6) and (3.7), we have that, for any $p \in F(S) \cap F(T)$,

$$0 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \longrightarrow c^2 - c^2 = 0 \tag{3.28}$$

as $n \to \infty$. We first show that $\{x_n\}$ converges weakly to some point in $F(S)$. Actually, from (3.4) and (3.5), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|U_n x_n - p\|^2 + \alpha_n \|V_n x_n - p\|^2 \\ &\le (1-\alpha_n)\|U_n x_n - p\|^2 + \alpha_n\|x_n - p\|^2 \\ &\le \|x_n - p\|^2\end{aligned} \tag{3.29}$$

for $p \in F(S) \cap F(T)$. Hence, we get

$$\begin{aligned}0 &\le \|x_n - p\|^2 - (1-\alpha_n)\|U_n x_n - p\|^2 - \alpha_n\|x_n - p\|^2 \\ &= (1-\alpha_n)\left(\|x_n - p\|^2 - \|\beta_n x_n + (1-\beta_n)Sx_n - p\|^2\right) \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.\end{aligned} \tag{3.30}$$

Since $\liminf_{n\to\infty}(1-\alpha_n) > 0$, it follows from (3.28) and (3.30) that

$$\lim_{n\to\infty}\left(\|x_n - p\|^2 - \|\beta_n x_n + (1-\beta_n)Sx_n - p\|^2\right) = 0. \tag{3.31}$$

From Lemma 2.2-(1), we have

$$\begin{aligned}&\|\beta_n x_n + (1-\beta_n)Sx_n - p\|^2 \\ &\quad = \|\beta_n(x_n - p) + (1-\beta_n)(Sx_n - p)\|^2 \\ &\quad = \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Sx_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Sx_n\|^2.\end{aligned} \tag{3.32}$$

Since $S$ is a $k$-strictly pseudo-nonspreading mapping, from (1.6), we get

$$\begin{aligned}
&\beta_n(1-\beta_n)\|x_n - Sx_n\|^2 \\
&\quad = \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Sx_n - p\|^2 - \|\beta_n x_n + (1-\beta_n)Sx_n - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\left(\|x_n - p\|^2 + k\|x_n - Sx_n\|^2\right) \\
&\qquad - \|\beta_n x_n + (1-\beta_n)Sx_n - p\|^2 \\
&\quad = \|x_n - p\|^2 + k(1-\beta_n)\|x_n - Sx_n\|^2 - \|\beta_n x_n + (1-\beta_n)Sx_n - p\|^2
\end{aligned} \tag{3.33}$$

and hence

$$(1-\beta_n)(\beta_n - k)\|x_n - p\|^2 \le \|x_n - p\|^2 - \|\beta_n x_n + (1-\beta_n)Sx_n - p\|^2 \tag{3.34}$$

for $p \in F(S) \cap F(T)$. Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, from (3.31), we get

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.35}$$

As in the proof of (1), from Lemma 2.6, we obtain that if $\{x_{n_i}\}$ converges weakly to $v$, then $v \in F(S)$. We also show that such $v$ is in $F(T)$. In fact, from (3.4) and (3.5), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|U_n x_n - p\|^2 + \alpha_n\|V_n x_n - p\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|\gamma_n x_n + (1-\gamma_n)Tx_n - p\|^2 \\
&\le \|x_n - p\|^2
\end{aligned} \tag{3.36}$$

for $p \in F(S) \cap F(T)$. Hence, we have

$$\begin{aligned}
0 &\le \|x_n - p\|^2 - (1-\alpha_n)\|x_n - p\|^2 - \alpha_n\|\gamma_n x_n + (1-\gamma_n)Tx_n - p\|^2 \\
&= \alpha_n\left(\|x_n - p\|^2 - \|\gamma_n x_n + (1-\gamma_n)Tx_n - p\|^2\right) \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
\end{aligned} \tag{3.37}$$

Since $\liminf_{n\to\infty}\alpha_n > 0$, it follows from (3.28) that

$$\lim_{n\to\infty}\left(\|x_n - p\|^2 - \|\gamma_n x_n + (1-\gamma_n)Tx_n - p\|^2\right) = 0. \tag{3.38}$$

Since $T$ is a nonexpansive mapping, we have

$$\begin{aligned}
&\left\|\gamma_n x_n + (1-\gamma_n)Tx_n - p\right\|^2 \\
&\quad = \gamma_n\left\|x_n - p\right\|^2 + (1-\gamma_n)\left\|Tx_n - p\right\|^2 - \gamma_n(1-\gamma_n)\|x_n - Tx_n\|^2 \\
&\quad \le \left\|x_n - p\right\|^2 - \gamma_n(1-\gamma_n)\|x_n - Tx_n\|^2,
\end{aligned} \tag{3.39}$$

and hence

$$\gamma_n(1-\gamma_n)\|x_n - Tx_n\|^2 \le \left\|x_n - p\right\|^2 - \left\|\gamma_n x_n + (1-\gamma_n)Tx_n - p\right\|^2. \tag{3.40}$$

Since $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$, from (3.38), we get

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.41}$$

Since $\{x_{n_i}\}$ converges weakly to $q$, we have $q \in F(T)$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $v$. Then, we have $q = v$. In fact, if $q \neq v$, then we obtain

$$\begin{aligned}
\lim_{n\to\infty}\left\|x_n - q\right\| &= \lim_{i\to\infty}\left\|x_{n_i} - q\right\| \\
&< \lim_{i\to\infty}\|x_{n_i} - v\| = \lim_{n\to\infty}\|x_n - v\| = \lim_{j\to\infty}\left\|x_{n_j} - v\right\| \\
&< \lim_{j\to\infty}\left\|x_{n_j} - q\right\| = \lim_{n\to\infty}\left\|x_n - q\right\|.
\end{aligned} \tag{3.42}$$

This is a contradiction. So, we have $q = v$. Therefore, we conclude that $\{x_n\}$ converges weakly to $q \in F(S) \cap F(T)$. □

For nonspreading mapping $S$, we have $k = 0$ reduces the followings.

**Corollary 3.2** (cf. [2]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $S : C \to C$ be a nonspreading mapping and let $T : C \to C$ be a nonexpansive mapping such that $F(S) \cap F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
&x_1 \in C, \\
&x_{n+1} = (1-\alpha_n)\left(\beta_n x_n + (1-\beta_n)Sx_n\right) + \alpha_n\left(\gamma_n x_n + (1-\gamma_n)Tx_n\right)
\end{aligned} \tag{3.43}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$. Then, the followings hold:*

(1) *if $\liminf_{n\to\infty}\alpha_n(\beta_n - \gamma_n) > 0$, $\sum_{n=1}^{\infty}\alpha_n(1-\gamma_n) < \infty$ and $1 < (2-\alpha_n)\beta_n + \alpha_n\gamma_n$, then $\{x_n\}$ converges weakly to $q \in F(S)$,*

(2) *if $\beta_n > \gamma_n$, $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, $2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n) > 0$ and $\liminf_{n\to\infty}\alpha_n(\beta_n - \gamma_n)(2\beta_n - 1 - \alpha_n(\beta_n - \gamma_n)) > 0$, then $\{x_n\}$ converges weakly to $q \in F(T)$,*

(3) *if* $\liminf_{n\to\infty}\alpha_n > 0$, $\liminf_{n\to\infty}(1-\alpha_n) > 0$, $\liminf_{n\to\infty}(1-\beta_n) > 0$ *and* $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$, *then* $\{x_n\}$ *converges weakly to* $q \in F(S)\cap F(T)$.

**Corollary 3.3** (cf. [2, 10]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $S : C \to C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n \end{aligned} \tag{3.44}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$. If $\liminf_{n\to\infty}\alpha_n > 0$, then $\{x_n\}$ converges weakly to $q \in F(S)$.*

*Proof.* Putting $\beta_n = 0, \gamma_n = 1$ for $n \in \mathbb{N}$ in Theorem 3.1, we get the conclusion. □

**Corollary 3.4.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \end{aligned} \tag{3.45}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$. If $\sum_{n=1}^{\infty}\alpha_n = \infty$, then $\{x_n\}$ converges weakly to $q \in F(T)$.*

*Proof.* Putting $\beta_n = 1, \gamma_n = 0$ for $n \in \mathbb{N}$ in Theorem 3.1, we get the conclusion. □

## 4. Some Open Problems

Let $S$ be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real-valued functional on $S$ with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$\big(l(s)f\big)(t) = f(st), \qquad \big(r(s)f\big)(t) = f(ts) \tag{4.1}$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. Let $X$ be a subspace of $l^\infty(S)$ containing 1. An element $\mu$ in the dual space $X^*$ of $X$ is said to be a *mean* on $X$ if $\|\mu\| = \mu(1) = 1$. For $s \in S$, we can define a point evaluation $\delta_s$ by $\delta_s(f) = f(s)$ for each $f \in X$. It is well known that $\mu$ is mean on $X$ if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s) \tag{4.2}$$

for each $f \in X$.

Let $X$ be a translation invariant subspace of $l^\infty(S)$ (i.e., $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then, a mean $\mu$ on $X$ is said to be *left invariant* (resp. *right invariant*) if

$$\mu(l(s)f) = \mu(f) \quad (\text{resp.} \quad \mu(r(s)f) = \mu(f)) \tag{4.3}$$

for each $s \in S$ and $f \in X$. A mean $\mu$ on X is said to be *invariant* if $\mu$ is both left and right invariant [11–14]. $X$ is said to be *left* (resp., *right*) *amenable* if $X$ has a left (resp. right) invariant mean. $X$ is amenable if $X$ is left and right amenable. Moreover, $l^\infty(S)$ is amenable when $S$ is a commutative semigroup or a solvable group. However, the free group or semigroup of two generators is not left or right amenable (see [15–17]). A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically left* (resp., *right*) *invariant* if

$$\lim_\alpha(\mu_\alpha(l(s)f) - \mu_\alpha(f)) = 0 \quad \left(\text{resp. } \lim_\alpha(\mu_\alpha(r(s)f) - \mu_\alpha(f)) = 0\right) \tag{4.4}$$

for each $f \in X$ and $s \in S$, and it is said to be left (resp., right) strongly asymptotically invariant (or strong regular) if

$$\lim_\alpha\|l^*(s)\mu_\alpha - \mu_\alpha\| = 0 \quad \left(\text{resp. } \lim_\alpha\|r^*(s)\mu_\alpha - \mu_\alpha\| = 0\right) \tag{4.5}$$

for each $s \in S$, where $l^*(s)$ and $r^*(s)$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day in [15] where they were called *weak* invariant* and *norm invariant*, respectively.

It is easy to see that if a semigroup $S$ is left (resp. right) amenable, then the semigroup $S' = S \cup \{e\}$ where $es' = s'e = s'$ for all $s' \in S$ is also left (resp. right) amenable and conversely.

From now on $S$ denotes a semigroup with an identity $e$. $S$ is called *left reversible* if any two right ideals of $S$ have nonvoid intersection, that is, $aS \cap bS \neq \emptyset$ for $a, b \in S$. In this case, $(S, \preceq)$ is a directed system when the binary relation "$\preceq$" on $S$ is defined by $a \preceq b$ if and only if $\{a\} \cup aS \supseteq \{b\} \cup bS$ for $a, b \in S$. It is easy to see that $t \preceq ts$ for all $t, s \in S$. Further, if $t \preceq s$, then $pt \preceq ps$ for all $p \in S$. The class of left reversible semigroup includes all groups and commutative semigroups. If a semigroup $S$ is left amenable, then $S$ is left reversible. But the converse is not true ([18–23]).

Let $S$ be a semigroup and let $C$ be a closed and convex subset of $E$. Let $F(T)$ denote the fixed point set of $T$. Then, $\Im = \{T(s) : s \in S\}$ is called a *representation of S as nonexpansive mappings on* C if $T(s)$ is nonexpansive with $T(e) = I$ and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\Im)$ the set of common fixed points of $\{T(s) : s \in S\}$, that is,

$$F(\Im) = \bigcap_{s\in S} F(T(s)) = \bigcap_{s\in S}\{x \in C : T(s)x = x\}. \tag{4.6}$$

We know that if $\mu$ is a mean on $X$ and if for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^*\rangle$ is contained in $X$ and $C$ is weakly compact, then there exists a unique point $x_0$ of $E$ such that $\mu\langle T(\cdot)x, x^*\rangle = \langle x_0, x^*\rangle$ for each $x^* \in E^*$. We denote such a point $x_0$ by $T_\mu x$. Note that $T_\mu x$ is contained in the closure of the convex hull of $\{T(s)x : s \in S\}$ for each $x \in C$. Note that $T_\mu z = z$ for each $z \in F(\Im)$; see [12–14, 17, 20, 22].

*Open Problems.*

(1) Does Theorem 3.1 hold for a Banach space? (cf. [24, 25])

(2) Does Theorem 3.1 hold a commutative or amenable or reversible semigroup of nonexpansive mappings in place of $T$ using asymptotically invariant nets?

## Acknowledgments

The author would like to thank Prof. Anthony To-Ming Lau for his helpful suggestions. Also, special thanks are due to the referee for his/her deep insight which improved the paper. This work was supported by Kyungnm University Researsh Fund, 2012.


## References


[1] F. Kohsaka and W. Takahashi, "Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces," *Archiv der Mathematik*, vol. 91, no. 2, pp. 166–177, 2008.

[2] S. Iemoto and W. Takahashi, "Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 71, no. 12, pp. e2082–e2089, 2009.

[3] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[4] M. O. Osilike and F. O. Isiogugu, "Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 74, no. 5, pp. 1814–1822, 2011.

[5] A. Moudafi, "Krasnoselski-Mann iteration for hierarchical fixed-point problems," *Inverse Problems*, vol. 23, no. 4, pp. 1635–1640, 2007.

[6] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama, Yokohama, Japan, 2009.

[7] K.-K. Tan and H. K. Xu, "Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.

[8] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama, Yokohama, Japan, 2000.

[9] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[10] Y. Kurokawa and W. Takahashi, "Weak and strong convergence theorems for nonspreading mappings in Hilbert spaces," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 73, no. 6, pp. 1562–1568, 2010.

[11] J. I. Kang, "Fixed points of non-expansive mappings associated with invariant means in a Banach space," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 68, no. 11, pp. 3316–3324, 2008.

[12] A. T.-M. Lau, "Invariant means on almost periodic functions and fixed point properties," *The Rocky Mountain Journal of Mathematics*, vol. 3, pp. 69–76, 1973.

[13] A. T.-M. Lau, "Invariant means and fixed point properties of semigroup of nonexpansive mappings," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1525–1542, 2008.

[14] A. T.-M. Lau and W. Takahashi, "Invariant means and fixed point properties for non-expansive representations of topological semigroups," *Topological Methods in Nonlinear Analysis*, vol. 5, no. 1, pp. 39–57, 1995.

[15] M. M. Day, "Amenable semigroups," *Illinois Journal of Mathematics*, vol. 1, pp. 509–544, 1957.

[16] J. I. Kang, "Fixed point set of semigroups of non-expansive mappings and amenability," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1445–1456, 2008.

[17] A. T.-M. Lau, N. Shioji, and W. Takahashi, "Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces," *Journal of Functional Analysis*, vol. 161, no. 1, pp. 62–75, 1999.

[18] R. D. Holmes and A. T.-M. Lau, "Non-expansive actions of topological semigroups and fixed points," *Journal of the London Mathematical Society. Second Series*, vol. 5, no. 2, pp. 330–336, 1972.

[19] K. S. Kim, "Nonlinear ergodic theorems of nonexpansive type mappings," *Journal of Mathematical Analysis and Applications*, vol. 358, no. 2, pp. 261–272, 2009.

[20] A. T.-M. Lau, "Semigroup of nonexpansive mappings on a Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 105, no. 2, pp. 514–522, 1985.

[21] A. T.-M. Lau and P. F. Mah, "Fixed point property for Banach algebras associated to locally compact groups," *Journal of Functional Analysis*, vol. 258, no. 2, pp. 357–372, 2010.

[22] A. T.-M. Lau and W. Takahashi, "Weak convergence and nonlinear ergodic theorems for reversible semigroups of nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 126, no. 2, pp. 277–294, 1987.

[23] A. T.-M. Lau and Y. Zhang, "Fixed point properties of semigroups of non-expansive mappings," *Journal of Functional Analysis*, vol. 254, no. 10, pp. 2534–2554, 2008.

[24] K. S. Kim, "Convergence of a hybrid algorithm for a reversible semigroup of nonlinear operators in Banach spaces," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 73, no. 10, pp. 3413–3419, 2010.

[25] K. S. Kim, "Approximation of common fixed points for an implicit iteration with errors of finite family of nonself nonexpansive mappings in Banach spaces," *Dynamics of Continuous, Discrete and Impulsive Systems B*, vol. 15, no. 6, pp. 807–815, 2008.