

Research Article

Perturbation Bound of the Group Inverse and the Generalized Schur Complement in Banach Algebra

Xiaoji Liu,¹ Yonghui Qin,² and Hui Wei³

¹ Faculty of Science, Guangxi University for Nationalities and Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis, Nanning 530006, China

² College of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning 530006, China

³ Department of Computer Science, Fudan University, Shanghai 200433, China

Correspondence should be addressed to Xiaoji Liu, liuxiaoji.2003@yahoo.com.cn

Received 5 April 2012; Revised 26 June 2012; Accepted 11 July 2012

Academic Editor: Patricia J. Y. Wong

Copyright © 2012 Xiaoji Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the relative perturbation bound of the group inverse and also consider the perturbation bound of the generalized Schur complement in a Banach algebra.

1. Introduction

Let \mathcal{A} denote a Banach algebra with unit 1. The symbols \mathcal{A}^{-1} , \mathcal{A}^D , \mathcal{A}^d , \mathcal{A}_g , \mathcal{A}^{nil} , $\mathcal{A}^{\text{qnil}}$, and \mathcal{A}^\bullet stand for the sets of all invertible, Drazin invertible, generalized Drazin invertible, group invertible, nilpotent, quasinilpotent, and idempotent elements of a Banach algebra \mathcal{A} , respectively.

Some definitions will be given in the following.

Letting $a \in \mathcal{A}^D$, there is an unique element $x \in \mathcal{A}$ such that

$$a^{k+1}x = a^k, \quad xax = x, \quad ax = xa. \quad (1.1)$$

Then x is called the Drazin inverse of a , denoted by a^D . The smallest nonnegative integer k which satisfies (1.1) is called the index of a , denoted by $\text{Ind}(a) = k$. If $\text{Ind}(a) \leq 1$, then $a^D = a^\#$ (or a^g).

Let $a \in \mathcal{A}$, if the conditions (1.1) are replaced by

$$axa = a, \quad xax = x, \quad ax = xa. \quad (1.2)$$

Then x is called the group inverse of a , denoted by $x = a^\#$. If the conditions (1.1) are replaced by

$$xax = x, \quad ax = xa, \quad a(1 - ax) \text{ is quasinilpotent.} \quad (1.3)$$

Then x is called the generalized Drazin inverse of a , denoted by $x = a^d$.

Some notations of the Schur complement are given in the following.

For a 2×2 block complex matrix M is defined as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (1.4)$$

where $A \in C^{m \times m}$, $D \in C^{p \times p}$, $B \in C^{m \times p}$, and $C \in C^{p \times m}$. If A is nonsingular, then the classical Schur complement of A in M is given as follows (see [1]):

$$S = D - CA^{-1}B. \quad (1.5)$$

In [2], Benítez and Thome considered the expression

$$N = \begin{bmatrix} A^- + A^-BS^-CA^- & -A^-BS^- \\ -S^-CA^- & S^- \end{bmatrix}, \quad (1.6)$$

and N is called the generalized Schur form of the matrix M given in (1.4) being $S = D - CA^-B$ for some fixed generalized inverses $A^- \in A\{1\}$, $S^- \in S\{1\}$, where S is called generalized Schur complement of A in M . In [2, Theorem 2], Benítez and Thome investigated the expression of the group inverse of M in (1.4) by the generalized Schur complement, where (1.5) is replaced by

$$S = D - CA^\#B. \quad (1.7)$$

Similar results also were given by Sheng and Chen in [3, Theorem 3.2]. The Drazin inverse of a 2×2 block complex square matrix in (1.4) with a singular generalized Schur complement was considered in [4–6], where

$$S = A - CA^D D. \quad (1.8)$$

For the expression of a 2×2 block operator matrix was investigated by Deng and Wei in [7].

Some notations for the block matrix form of a given element $a \in \mathcal{A}$ are introduced in [8]. Let $a \in \mathcal{A}$ and $s \in \mathcal{A}^\bullet$ (see [8, Chapter VII]) which denotes the set of all idempotent elements in \mathcal{A} . Then we write

$$a = sas + sa(1-s) + (1-s)as + (1-s)a(1-s) \quad (1.9)$$

and use the notations

$$a_{11} = sas, \quad a_{12} = sa(1-s), \quad a_{21} = (1-s)as, \quad a_{22} = (1-s)a(1-s). \quad (1.10)$$

For a representation of arbitrary element $a \in \mathcal{A}$ is given as the following matrix form:

$$a = \begin{bmatrix} sas & sa(1-s) \\ (1-s)as & (1-s)a(1-s) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_s. \quad (1.11)$$

In this paper, we will consider some results on the relative perturbation bounds of group inverse and also give the perturbation bounds of the generalized Schur complement of an element $a \in \mathcal{A}$ under some certain conditions in a Banach algebra.

2. Perturbation Bound of $\|(a+b)^\# - a^d\|$ in Banach Algebra

In recent years, perturbation theory for the Drazin inverse of a given matrix $A \in \mathbb{C}^{n \times n}$ and its applications have been considered in [9–20]. In [12], Yimin and Guorong gave a perturbation result for the Drazin inverse under condition (\mathcal{W}) (see [12] for details). In [8, Ch 5], Djordjević and Rakočević extended the perturbation bound of Yimin and Guorong [12] to Banach algebra. In [13], Wei had discussed the upper perturbation bound of $\|B^\# - A^\#\|/\|A^\#\|$ with $B = A + E$ and had answered the question of Campbell and Meyer [21] when $\text{Ind}(A) = 1$. In [14], Wei and Wu presented the perturbation upper bounds of $\|B^D - A^D\|/\|A^D\|$ under the weaker condition $\text{core-rank } B = \text{core-rank } A$ and completely answered the question of Campbell and Meyer in [21]. In [16], Wei derived a relative perturbation upper bound of $\|B^\# - A^D\|/\|A^D\|$ by Jordan canonical of A . In [5], Li gave sharper upper bounds for $\|B^\# - A^D\|$ under weaker conditions: $\text{rank}(B) = \text{rank}(A^k)$ and $\|A^D\|\|E\| < 1/(1 + \|A^D\|\|A\|)$. In [17], Wei et al. derived constructive perturbation bound of the Drazin inverse of a square matrix by using a technique proposed by Stewart and based on perturbation theory for invariant subspaces. In [18], Xu et al. gave some upper bounds for $\|B^D - A^D\|/\|A^D\|$ only under the condition that B is a stable perturbation of A . In [22], González and Koliha investigated the perturbation of the Drazin inverse of a closed linear operator and derived explicit bounds for the perturbations under certain restrictions on the perturbing operators. In [23], González and Vélez-Cerrada analyzed the perturbation of the Drazin inverse and also gave explicit upper bounds of $\|B^\# - A^D\|$ and $\|BB^\# - AA^D\|$ and obtained a result on the continuity of the group inverse for operators on Banach space.

In this section, we will investigate the relative perturbation bound of the group inverse in Banach algebra.

At first, we will give some concepts and lemmas as follows.

For $a \in \mathcal{A}^d$, let $p = aa^d$ and $p \in \mathcal{A}^\bullet$ (see [24]):

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \quad a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p, \quad a^\pi = 1 - p = \begin{bmatrix} 0 & 0 \\ 0 & 1 - p \end{bmatrix}_p, \quad (2.1)$$

where $a_1 \in p\mathcal{A}p$ is invertible and $a_2 \in (1 - p)\mathcal{A}(1 - p)$ is quasinilpotent.

For any $a \in \mathcal{A}^d$, we write $\sigma(a)$, $\rho(a)$, and $r(a)$ for the spectrum, the resolvent set, and the spectral radius of a , respectively. For $\lambda \in \rho(a)$ and let $R(\lambda, a) = (\lambda - a)^{-1}$. If 0 is an isolated point of $\sigma(a)$, then the spectral idempotent corresponding to the set $\{0\}$ is defined by

$$a^\pi = \frac{1}{2\pi i} \int_\gamma R(\lambda, a) d\lambda, \quad (2.2)$$

where γ is a small circle surrounding 0 and separating 0 from $\sigma(a) \setminus \{0\}$.

Some lemmas will be useful for the following proof in this paper.

Lemma 2.1 (see [24, Theorem 2.3]). *Let $x, y \in \mathcal{A}$, and let $p \in \mathcal{A}^\bullet$. Assume that*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p, \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-p}. \quad (2.3)$$

(i) *If $a \in (p\mathcal{A}p)^d$ and $b \in ((1 - p)\mathcal{A}(1 - p))^d$, then $x, y \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & u \\ 0 & b^d \end{bmatrix}_p, \quad y^d = \begin{bmatrix} b^d & 0 \\ u & a^d \end{bmatrix}_{1-p}, \quad (2.4)$$

where $u = \sum_{n=0}^{\infty} (a^d)^{n+2} cb^n b^\pi + \sum_{n=0}^{\infty} a^\pi a^n c (b^d)^{n+2} - a^d cb^d$.

(ii) *If $x \in \mathcal{A}^d$ and $a \in (p\mathcal{A}p)^d$, then $b \in [(1 - p)\mathcal{A}(1 - p)]^d$ and x^d is given by (2.4).*

Lemma 2.2 (see [24, Corollary 3.4]). *If $a, b \in \mathcal{A}$ are generalized Drazin invertible, b is quasinilpotent, and $ab = 0$, then $a + b$ is generalized Drazin invertible and*

$$(a + b)^d = \sum_{n=0}^{\infty} b^n (a^d)^{n+1}. \quad (2.5)$$

The following lemma is a generalization of [25, Theorem 1].

Lemma 2.3. *Let $a, b \in \mathcal{A}^d$ such that $ab = ba$. Then $a + b \in \mathcal{A}^d$ if and only if $1 + a^d b \in \mathcal{A}^d$. In this case*

$$(a + b)^d = a^d (1 + a^d b)^d b b^d + \sum_{n=0}^{\infty} b^n (-b)^n (a^d)^{n+1} + \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n a^\pi. \quad (2.6)$$

Now we will state a lemma for the representation of the group inverse of an element $a \in \mathcal{A}^d$ with 2×2 block form in Banach algebra (see [26, Theorem 7.7.7] and [23, Theorem 2.2.] which were established for a finite dimensional case and partitioned operators matrix, resp.).

Lemma 2.4. *Let $z \in \mathcal{A}$, and it has the block matrix form as $z = \begin{bmatrix} z_1 & z_{12} \\ z_{21} & z_2 \end{bmatrix}_p$, where $p \in \mathcal{A}^\bullet$ is an idempotent element, z_1 is invertible in $p\mathcal{A}p$, and $z_2 = z_{21}z_1^{-1}z_{12}$. Let $\delta = p + z_1^{-1}z_{12}z_{21}z_1^{-1}$. Then z is group invertible if and only if δ is an invertible element in $p\mathcal{A}p$. In this case*

$$\begin{aligned} z^\# &= \begin{bmatrix} (\delta z_1 \delta)^{-1} & (\delta z_1 \delta)^{-1} z_1^{-1} z_{12} \\ z_{21} z_1^{-1} (\delta z_1 \delta)^{-1} & z_{21} z_1^{-1} (\delta z_1 \delta)^{-1} z_1^{-1} z_{12} \end{bmatrix}_p, \\ z^\pi &= \begin{bmatrix} p - \delta^{-1} & -\delta^{-1} z_1^{-1} z_{12} \\ -z_{21} z_1^{-1} \delta^{-1} & 1 - p - z_{21} z_1^{-1} \delta^{-1} z_1^{-1} z_{12} \end{bmatrix}_p. \end{aligned} \quad (2.7)$$

Let $b \in \mathcal{A}$ be a perturbation element of a . According to (2.1), we obtain

$$b = \begin{bmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{bmatrix}_p, \quad a + b = \begin{bmatrix} a_1 + b_1 & b_{12} \\ b_{21} & a_2 + b_2 \end{bmatrix}_p, \quad (2.8)$$

where $p = aa^d$.

Theorem 2.5. *Let $a \in \mathcal{A}^d$ and $b \in \mathcal{A}$ be a perturbation element of a , a and $a + b$ which are defined as (2.1) and (2.8), respectively. If $\|a^d b a a^d\| < 1$, then $a_1 + b_1$ is invertible in subalgebra $p\mathcal{A}p$. Furthermore let $a_2 + b_2 = b_{21}(a_1 + b_1)^{-1}b_{12}$ and $\delta = p + [p(a + b)p]^d b(1 - p)b[p(a + b)p]^d \in p\mathcal{A}p$. Then $a + b$ is group invertible if and only if $\delta \in p\mathcal{A}p$ is invertible and δ is invertible if and only if $[p(a + b)p]^2 + pb(1 - p)bp \in p\mathcal{A}p$ is invertible. In this case,*

$$\begin{aligned} \frac{\|(a + b)^\# - a^d\|}{\|a^d\|} &\leq \frac{T_1^2}{1 - \|a_1^{-1}b_1\|} (\|b_{12}\| + \|b_{21}\|) + T_1^2 \frac{\|a_1^{-1}\|^3}{(1 - \|a_1^{-1}b_1\|)^4} \|b_{12}\| \|b_{21}\| \\ &+ \left(T_1 T_2 + T_2 + \frac{\|a_1^{-1}b_1\|}{1 - \|a_1^{-1}b_1\|} T_1 \right) T_1, \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} a_1 &= pap, & b_1 &= pbp, & b_{12} &= pb(1 - p), & b_{21} &= (1 - p)bp, \\ T_1 &= \|a_1 + b_1\|^2 \left\| \left((a_1 + b_1)^2 + b_{12}b_{21} \right)^{-1} \right\|, & T_2 &= \frac{\|a_1^{-1}\| \|b_{12}b_{21}\|}{1 - \|a_1^{-1}b_1\|}. \end{aligned} \quad (2.10)$$

Proof. Let $p = aa^d$. Then a , a^d , b and $a + b$ have the matrix form as (2.1) and (2.8), respectively, where a_1 is invertible in $p\mathcal{A}p$ and a_2 is quasiniptent in $(1 - p)\mathcal{A}(1 - p)$.

It follows from the hypothesis $\|a^d b a a^d\| < 1$ that $\|a_1^{-1} b_1\| < 1$. Thus, it implies that $p + a_1^{-1} b_1 \in p\mathcal{A}p$ is invertible. It is easy to see that $(a_1 + b_1)^{-1} = (p + a_1^{-1} b_1)^{-1} a_1^{-1}$. Let $\delta = p + [p(a+b)p]^d b(1-p)b[p(a+b)p]^d \in p\mathcal{A}p$; that is, we have $\delta = p + (a_1 + b_1)^{-1} b_{12} b_{21} (a_1 + b_1)^{-1}$. Therefore, we have

$$\begin{aligned} \delta &= (a_1 + b_1)^{-1} [(a_1 + b_1)p(a_1 + b_1) + b_{12} b_{21}] (a_1 + b_1)^{-1} \\ &= (a_1 + b_1)^{-1} [(a_1 + b_1)^2 + b_{12} b_{21}] (a_1 + b_1)^{-1} \\ &= [p(a+b)p]^d \left[[p(a+b)p]^2 + pb(1-p)bp \right] [p(a+b)p]^d. \end{aligned} \quad (2.11)$$

From the previous equations, we get that δ is invertible if and only if $[p(a+b)p]^2 + pb(1-p)bp$ is invertible. Since $a_2 + b_2 = b_{21}(a_1 + b_1)^{-1} b_{12}$ and by Lemma 2.4, we obtain that $a + b$ is group invertible if and only if $\delta \in p\mathcal{A}p$ is invertible.

In the following, we consider the upper bound of $\|(a+b)^\# - a^d\|/\|a^d\|$.

Applying Lemma 2.4, we obtain

$$(a+b)^\# = \begin{bmatrix} \eta & \eta(a_1 + b_1)^{-1} b_{12} \\ b_{21}(a_1 + b_1)^{-1} \eta & b_{21}(a_1 + b_1)^{-1} \eta(a_1 + b_1)^{-1} b_{12} \end{bmatrix}_p, \quad (2.12)$$

where $\eta = (\delta(a_1 + b_1)\delta)^{-1}$.

Note that

$$\begin{aligned} \eta - a_1^{-1} &= (\delta(a_1 + b_1)\delta)^{-1} - a_1^{-1} = \delta^{-1}(a_1 + b_1)^{-1} \delta^{-1} - a_1^{-1} \\ &= \delta^{-1} a_1^{-1} \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} - a_1^{-1} \\ &= \delta^{-1} (a_1^{-1} - \delta a_1^{-1} \delta) \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= \delta^{-1} (a_1^{-1} - (p + \theta) a_1^{-1} (p + \theta)) \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= \delta^{-1} \left[a_1^{-1} - (p a_1^{-1} p + \theta a_1^{-1} p + p a_1^{-1} \theta + \theta a_1^{-1} \theta) \right] \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= -\delta^{-1} (\theta a_1^{-1} p + p a_1^{-1} \theta + \theta a_1^{-1} \theta) \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= -\delta^{-1} (\theta a_1^{-1} p + \delta a_1^{-1} \theta) \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \\ &= -\delta^{-1} \theta a_1^{-1} \delta^{-1} - a_1^{-1} \theta \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1}, \end{aligned} \quad (2.13)$$

$$\begin{aligned}\|\delta^{-1}\| &= \left\| \left[p + (a_1 + b_1)^{-1} b_{12} b_{21} (a_1 + b_1)^{-1} \right]^{-1} \right\| \\ &\leq \|a_1 + b_1\|^2 \left\| \left[(a_1 + b_1)^2 + b_{12} b_{21} \right]^{-1} \right\| = T_1,\end{aligned}\quad (2.14)$$

where $\theta = (a_1 + b_1)^{-1} b_{12} b_{21} (a_1 + b_1)^{-1}$.

It shows from $\|a^d b a a^d\| < 1$ (i.e., $\|a_1^{-1} b_1\| < 1$) that

$$\begin{aligned}\|\theta\| &= \left\| (a_1 + b_1)^{-1} b_{12} b_{21} (a_1 + b_1)^{-1} \right\| \\ &\leq \frac{\|a_1^{-1}\| \|b_{12} b_{21}\|}{1 - \|a_1^{-1} b_1\|} = T_2.\end{aligned}\quad (2.15)$$

From (2.13), (2.14), (2.15), and by $\|a_1^{-1} b_1\| < 1$, we obtain that

$$\begin{aligned}\|\eta - a_1^{-1}\| &= \left\| -\delta^{-1} \theta a_1^{-1} \delta^{-1} - a_1^{-1} \theta \delta^{-1} + \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \right\| \\ &\leq \left\| \delta^{-1} \theta a_1^{-1} \delta^{-1} \right\| + \left\| a_1^{-1} \theta \delta^{-1} \right\| + \left\| \delta^{-1} \sum_{n=1}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \delta^{-1} \right\| \\ &\leq \left(T_1 T_2 + T_2 + \frac{\|a_1^{-1} b_1\|}{1 - \|a_1^{-1} b_1\|} T_1 \right) T_1 \|a_1^{-1}\|.\end{aligned}\quad (2.16)$$

It follows from (2.12) that

$$(a + b)^{\#} - a^d = \begin{bmatrix} \eta - a_1^{-1} & \eta(a_1 + b_1)^{-1} b_{12} \\ b_{21}(a_1 + b_1)^{-1} \eta & b_{21}(a_1 + b_1)^{-1} \eta(a_1 + b_1)^{-1} b_{12} \end{bmatrix}_p. \quad (2.17)$$

Therefore, according to (2.14), (2.15), and (2.16), we obtain

$$\begin{aligned}\|(a + b)^{\#} - a^d\| &= \left\| \begin{bmatrix} \eta - a_1^{-1} & \eta(a_1 + b_1)^{-1} b_{12} \\ b_{21}(a_1 + b_1)^{-1} \eta & b_{21}(a_1 + b_1)^{-1} \eta(a_1 + b_1)^{-1} b_{12} \end{bmatrix}_p \right\| \\ &\leq \left\| \eta - a_1^{-1} \right\| + \left\| \eta(a_1 + b_1)^{-1} b_{12} \right\| + \left\| b_{21}(a_1 + b_1)^{-1} \eta \right\| \\ &\quad + \left\| b_{21}(a_1 + b_1)^{-1} \eta(a_1 + b_1)^{-1} b_{12} \right\| \\ &\leq \frac{\|a_1^{-1}\| T_1^2}{1 - \|a_1^{-1} b_1\|} (\|b_{12}\| + \|b_{21}\|) + T_1^2 \left(\frac{\|a_1^{-1}\|}{1 - \|a_1^{-1} b_1\|} \right)^4 \|b_{12}\| \|b_{21}\| \\ &\quad + \left(T_1 T_2 + T_2 + \frac{\|a_1^{-1} b_1\|}{1 - \|a_1^{-1} b_1\|} T_1 \right) T_1 \|a_1^{-1}\|.\end{aligned}\quad (2.18)$$

Since $\|a_1^{-1}\| = \|a^d\|$ and by (2.18), it is easy to see that the conclusion holds.

Thus, we complete the proof. \square

Let $A, E \in B(X)$ be both bounded linear operators with $B = A + E$ on Banach space, where X denotes Banach space. If $\|A^D E\| + \|B^\pi - A^\pi\| < 1$ is satisfied, (it implies that $\|A^D E\| < 1$ and $\|A^D E A A^D\| < 1$), then we have the remark.

Remark 2.6 (see [23, Theorem 4.2]). Let $A, B \in B(X)$ be Drazin invertible and group invertible, respectively. If $\|A^D E\| + \|B^\pi - A^\pi\| < 1$, then

$$\frac{\|B^\# - A^D\|}{\|A^D\|} \leq \frac{\|A^D E\| + 2\|B^\pi - A^\pi\|}{1 - \|A^D E\| - \|B^\pi - A^\pi\|}. \quad (2.19)$$

Let $a = a_1 \oplus a_2$ and $a^d = [a_1]_p^{-1} = a_1^\#$; if we put $\delta a = b + a_2$, then $1 + \delta a a$ is invertible in \mathcal{A} when $\|a^d b a a^d\| < 1$. From the Proposition 2.2 (5) of [20], we have $a_2 + b_2 = b_{21}(a_1 + b_1)^{-1}b_{12}$ when $(a_1 + \delta a)\mathcal{A} \cap (1 - a a^d)\mathcal{A} = \{0\}$ for $[a_1]_p [a_1]_p^{-1} = p = a a^d$. Therefore, for $\|a_1^\#\| \|b + a_2\| < (1 + \|1 - a a^d\|)^{-1}$, we arrive at [20, Theorem 4.2]. In fact, the following remark implies that Theorem 2.5 improves the upper bound of $\|(a + b)^\# - a^\#\|$ of [20, Theorem 4.2].

Remark 2.7 (see [20, Theorem 4.2]). Let $a \in G(\mathcal{A})$ and let $\bar{a} = a + \delta a \in \mathcal{A}$ with $\mathcal{K}_\# \epsilon_a < (1 + \|a^\pi\|)$. Assume that $\bar{a}\mathcal{A} \cap (1 - a a^\#)\mathcal{A} = \{0\}$. Then $\bar{a} \in G(\mathcal{A})$ and

$$\|\bar{a}^\# - a^\#\| \leq \frac{(1 + 2\|a^\pi\|)\|a^\#\|\mathcal{K}_\#(a)\epsilon_a}{[1 - (1 + \|a^\pi\|)\|a^\#\|\mathcal{K}_\#(a)\epsilon_a]^2}, \quad (2.20)$$

where $\mathcal{K}_\# = \|a\|\|a^\#\|$ and $\epsilon_a = \|\delta a\|\|a\|^{-1}$.

Theorem 2.8. Let $a, b \in \mathcal{A}$ be generalized Drazin invertible and satisfy the conditions

$$\|a^d b a a^d\| < 1, \quad a^\pi b a = 0. \quad (2.21)$$

Then $(a + b)^\#$ exists if and only if $a^\pi(a + b)$ is group invertible. In this case,

$$\begin{aligned} \frac{\|(a + b)^\# - a^d\|}{\|a^d\|} &\leq \frac{\|a^d\|}{(1 - \|a^d b\|)^2} \left[\|b\| + \|b a^\pi\| \sum_{n=0}^{\infty} \|a^{n+1}\| \|(b^d)^{n+1}\| \right] \\ &\quad + \left[\frac{\|a^d\|}{(1 - \|a^d b\|)^2} \|b a^\pi b\| + \frac{\|b a^\pi\|}{1 - \|a^d b\|} \right] \sum_{n=0}^{\infty} \|a^n\| \|(b^d)^{n+1}\| \\ &\quad + \|a^\pi\| \|a^d\|^{-1} \sum_{n=0}^{\infty} \|a^n\| \|(b^d)^{n+1}\| + \frac{\|a^d b\|}{1 - \|a^d b\|}. \end{aligned} \quad (2.22)$$

Proof. Since a^d exists, a^d is defined as (2.1). Let b have the block matrix form as

$$b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p. \quad (2.23)$$

Applying the condition $a^\pi ba = 0$, we have $b_2 a_2 = 0$ and

$$a^\pi b a a^d a = \begin{bmatrix} 0 & 0 \\ b_4 a_1 & 0 \end{bmatrix}_p = 0. \quad (2.24)$$

It follows from (2.24) that $a_1 \in p\mathcal{A}p$ is invertible, $b_4 = 0$, and

$$b = \begin{bmatrix} b_1 & b_3 \\ 0 & b_2 \end{bmatrix}_p. \quad (2.25)$$

Combining (2.1) and (2.25), we obtain

$$a + b = \begin{bmatrix} a_1 + b_1 & b_3 \\ 0 & a_2 + b_2 \end{bmatrix}_p. \quad (2.26)$$

The condition $\|a^d b a a^d\| < 1$ implies $\|a_1^{-1} b_1\| < 1$ in the subalgebra $p\mathcal{A}p$. Therefore, we conclude that $a_1 + b_1 \in p\mathcal{A}p$ is invertible and $\text{Ind}(a_1 + b_1) = 0$. According to (2.26) and by Lemma 2.1, one observes that $(a + b)^d$ exists if and only if $(a_2 + b_2)^d$ also. Thus, $(a + b)^\#$ exists if and only if $a^\pi(a + b)$ is group invertible.

If $a^\pi(a + b)$ is group invertible and by Lemma 2.1, we obtain

$$(a + b)^\# = \begin{bmatrix} (a_1 + b_1)^{-1} & x \\ 0 & (a_2 + b_2)^\# \end{bmatrix}_p, \quad (2.27)$$

where $x = (a_1 + b_1)^{-2} b_3 (a_2 + b_2)^\pi - (a_1 + b_1)^{-1} b_3 (a_2 + b_2)^\#$.

Since $b_2 a_2 = 0$ and a_2 is quasinilpotent, by Lemma 2.2, we obtain

$$(a_2 + b_2)^\# = \sum_{n=0}^{\infty} a_2^n (b_2^d)^{n+1} = a^\pi \sum_{n=0}^{\infty} a^n (b^d)^{n+1}. \quad (2.28)$$

From $\|a^d b a a^d\| < 1$, one easily has

$$\begin{aligned} (a_1 + b_1)^{-1} \oplus 0 &= \sum_{n=0}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} \oplus 0 = \sum_{n=0}^{\infty} (a^d b)^n a^d \\ &= a_1^{-1} (1 + b_1 a_1^{-1})^{-1} \oplus 0 = a^d \sum_{n=0}^{\infty} (b a^d)^n. \end{aligned} \quad (2.29)$$

It follows from (2.27) and (2.29) that

$$\begin{aligned}
 x &= (a_1 + b_1)^{-2} b_3 (a_2 + b_2)^\pi - (a_1 + b_1)^{-1} b_3 (a_2 + b_2)^\sharp \\
 &= \left(\sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 \left[b - b a^\pi \sum_{n=0}^{\infty} a^{n+1} (b^d)^{n+1} \right] \\
 &\quad + \left[\left(\sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 b a^\pi b - \sum_{n=0}^{\infty} (a^d b)^n a^d b a^\pi \right] \sum_{n=0}^{\infty} a^n (b^d)^{n+1}.
 \end{aligned} \tag{2.30}$$

Combining (2.27), (2.28), and (2.29), we obtain

$$\begin{aligned}
 (a + b)^\sharp &= \begin{bmatrix} \sum_{n=0}^{\infty} a_1^{-1} (b_1 a_1^{-1})^n & x \\ 0 & \sum_{n=0}^{\infty} a_2^n (b_2^d)^{n+1} \end{bmatrix}_p \\
 &= \left(\sum_{n=0}^{\infty} (a^d b a^d)^n \right)^2 \left[b - b a^\pi \sum_{n=0}^{\infty} a^{n+1} (b^d)^{n+1} \right] \\
 &\quad + \left[\left(\sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 b a^\pi b - \sum_{n=0}^{\infty} (a^d b)^n a^d b a^\pi \right] \sum_{n=0}^{\infty} a^n (b^d)^{n+1} \\
 &\quad + \sum_{n=0}^{\infty} (a^d b)^n a^d + a^\pi \sum_{n=0}^{\infty} a^n (b^d)^{n+1}.
 \end{aligned} \tag{2.31}$$

From (2.31), we derive

$$\begin{aligned}
 (a + b)^\sharp - a^d &= \left(\sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 \left[b - b a^\pi \sum_{n=0}^{\infty} a^{n+1} (b^d)^{n+1} \right] \\
 &\quad + \left[\left(\sum_{n=0}^{\infty} (a^d b)^n a^d \right)^2 b a^\pi b - \sum_{n=0}^{\infty} (a^d b)^n a^d b a^\pi \right] \sum_{n=0}^{\infty} a^n (b^d)^{n+1} \\
 &\quad + a^\pi \sum_{n=0}^{\infty} a^n (b^d)^{n+1} + \sum_{n=1}^{\infty} (a^d b)^n a^d.
 \end{aligned} \tag{2.32}$$

Moreover, by (2.32) we get

$$\begin{aligned}
\|(a+b)^{\sharp} - a^d\| &\leq \left(\frac{\|a^d\|}{1 - \|a^d b\|}\right)^2 \left[\|b\| + \|ba^{\pi}\| \sum_{n=0}^{\infty} \|a^{n+1}\| \|(b^d)^{n+1}\| \right] \\
&\quad + \left[\left(\frac{\|a^d\|}{1 - \|a^d b\|}\right)^2 \|ba^{\pi}b\| + \frac{\|a^d\| \|ba^{\pi}\|}{1 - \|a^d b\|} \right] \sum_{n=0}^{\infty} \|a^n\| \|(b^d)^{n+1}\| \\
&\quad + \|a^{\pi}\| \sum_{n=0}^{\infty} \|a^n\| \|(b^d)^{n+1}\| + \frac{\|a^d\| \|a^d b\|}{1 - \|a^d b\|}.
\end{aligned} \tag{2.33}$$

Finally, from (2.33) we easily finish the proof. \square

Corollary 2.9. Let $a \in \mathcal{A}_g$ and let $b \in \mathcal{A}^d$. If a, b satisfy the conditions

$$\|a^{\sharp}baa^{\sharp}\| < 1, \quad a^{\pi}ba = 0, \tag{2.34}$$

then $(a+b)^{\sharp}$ exists if and only if $a^{\pi}b$ is group invertible. In this case,

$$\begin{aligned}
\frac{\|(a+b)^{\sharp} - a^{\sharp}\|}{\|a^{\sharp}\|} &\leq \|b\| \|(a^{\pi}b)^{\pi}\| \frac{\|a^{\sharp}\|}{(1 - \|a^{\sharp}b\|)^2} \\
&\quad + \frac{\|ba^{\pi}b^{\sharp}\|}{1 - \|a^{\sharp}b\|} + \frac{\|a^{\pi}b^{\sharp}\|}{\|a^{\sharp}\|} + \frac{\|a^{\sharp}b\|}{1 - \|a^{\sharp}b\|}.
\end{aligned} \tag{2.35}$$

The conditions of Theorem 2.8 $\|a^d b a a^d\| < 1$, $a^{\pi}ba = 0$ are weaker than the conditions (\mathcal{W}) (see [12, Theorem 3.2] for finite dimensional cases and [8, Theorem 5.3.2 and Corollary 5.3.3] for Banach algebra). According to $a^{\pi}ba = 0$, we obtain that (2.26) holds. However, in view of (\mathcal{W}) , we have

$$a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 \end{bmatrix}_p. \tag{2.36}$$

Thus, by the conditions (\mathcal{W}) , we know that a and $a + b$ have the same Drazin invertible property (see [12, Theorem 3.1]). Thus, if a is group invertible, then $(a+b)$ is group invertible. It is easy to see that $\|a^d b a a^d\| < 1$, $a^{\pi}ba = 0$ are weaker than the conditions (\mathcal{W}) . From [8, Theorem 5.3.2 and Corollary 5.3.3], we easily state the following remark.

Remark 2.10. Let $a \in \mathcal{A}_g$ and let $b \in \mathcal{A}^d$. If a, b satisfy the condition (\mathcal{W})

$$\|a^{\sharp}baa^{\sharp}\| < 1, \quad b = aa^{\sharp}baa^{\sharp}, \tag{2.37}$$

then $a + b$ is group invertible and

$$\frac{\|(a + b)^\# - a^\#\|}{\|a^\#\|} \leq \frac{\|a^\#b\|}{1 - \|a^\#b\|}. \quad (2.38)$$

Theorem 2.11. Let $a, b \in \mathcal{A}$ be generalized Drazin invertible and satisfy the conditions

$$\max\left\{\|a^d b a a^d\|, \|a^\pi a\| \|a^\pi b^d\|\right\} < 1, \quad a^\pi b a = 0. \quad (2.39)$$

Then $(a + b)^\#$ exists if and only if $a^\pi(a + b)$ is group invertible. In this case,

$$\begin{aligned} \frac{\|(a + b)^\# - a^d\|}{\|a^d\|} &\leq \frac{\|a^d\|}{(1 - \|a^d b\|)^2} \left[\|b\| + \|b a^\pi\| \frac{\|a\| \|b^d\|}{1 - \|a\| \|b^d\|} \right] \\ &+ \left[\frac{\|a^d\|}{(1 - \|a^d b\|)^2} \|b a^\pi b\| + \frac{\|b a^\pi\|}{1 - \|a^d b\|} \right] \frac{\|a\| \|b^d\|}{1 - \|a\| \|b^d\|} \\ &+ \frac{\|a^d\|^{-1} \|a^\pi\| \|b^d\|}{1 - \|a\| \|b^d\|} + \frac{\|a^d b\|}{1 - \|a^d b\|}. \end{aligned} \quad (2.40)$$

Proof. The notations are taken as Theorem 2.8, and the rest of proof of theorem is similar to Theorem 2.8. Now, we only consider the perturbation of $a_2 + b_2$. From (2.28) and the first condition of (2.39), we have $\|a_2\| \|b_2^d\| < 1$ and

$$\|(a_2 + b_2)^\#\| = \left\| \sum_{n=0}^{\infty} a_2^n (b_2^d)^{n+1} \right\| \leq \frac{\|a^\pi\| \|b^d\|}{1 - \|a\| \|b^d\|}. \quad (2.41)$$

Thus, from (2.41) we completed the proof. \square

Theorem 2.12. Let $a, b \in \mathcal{A}$ be generalized Drazin invertible and satisfy the conditions

$$\|a^d b a a^d\| < 1, \quad a^\pi b a = a b a^\pi. \quad (2.42)$$

Then $(a + b)^\#$ exists if and only if $a^\pi(a + b)$ is group invertible. In this case,

$$\frac{\|(a + b)^\# - a^d\|}{\|a^d\|} \leq \frac{\|a^d b\|}{1 - \|a^d b\|} + \|a^\pi\| \|a^d\|^{-1} \sum_{n=0}^{\infty} \|(b^d)^{n+1}\| \|a^n\|. \quad (2.43)$$

Proof. Letting $p = aa^d$, and it is similar to Theorem 2.8, we obtain that a , a^d , and p have the matrix forms as (2.1). Here b is taken as (2.23) in the proof of Theorem 2.8. The condition $a^\pi ba = aba^\pi$ implies that

$$\begin{aligned} a^\pi ba &= \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_p \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ b_4 a_1 & b_2 a_2 \end{bmatrix}_p, \\ aba^\pi &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_p = \begin{bmatrix} 0 & a_1 b_3 \\ 0 & a_2 b_2 \end{bmatrix}_p. \end{aligned} \quad (2.44)$$

Thus, according to (2.44), we obtain $b_4 a_1 = 0$, $a_1 b_3 = 0$ and $a_2 b_2 = b_2 a_2$. Because a_1 is invertible in subalgebra $p\mathcal{A}p$, we have $b_3 = b_4 = 0$. Thus, b , $a + b$ have the matrix forms as follows:

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \quad a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix}_p. \quad (2.45)$$

It follows from the condition $\|a^d b a a^d\| < 1$ that $\|a_1^{-1} b_1\| < 1$. Thus, it shows from $\|a_1^{-1} b_1\| < 1$ that $a_1 + b_1$ is invertible in subalgebra $p\mathcal{A}p$. Therefore, easily we observe that $a + b$ is Drazin invertible if and only if $a_2 + b_2 \in (1-p)\mathcal{A}(1-p)$ is Drazin invertible. That is, $(a + b)^\#$ exists if and only if $a^\pi(a + b)$ is group invertible.

In the following, we will consider the perturbation of a_2 .

Let $a_2 + b_2$ be group invertible. The condition $a^\pi ba = aba^\pi$ implies that $a_2 b_2 = b_2 a_2$ holds. Since a_2 is quasinilpotent in subalgebra $(1-p)\mathcal{A}(1-p)$ and by Lemma 2.3, we get

$$(a_2 + b_2)^\# = \sum_{n=0}^{\infty} (b_2^d)^{n+1} (-a_2)^n = a^\pi \sum_{n=0}^{\infty} (b^d)^{n+1} (-a)^n. \quad (2.46)$$

By virtue of $\|a_1^{-1} b_1\| < 1$, we get that

$$\left[a a^d (a + b) \right]^{-1} = (a_1 + b_1)^{-1} = \sum_{n=0}^{\infty} (a_1^{-1} b_1)^n a_1^{-1} = \sum_{n=0}^{\infty} (a^d b)^n a^d. \quad (2.47)$$

It follows from (2.46) and (2.47) that

$$\begin{aligned} \left\| \left[a a^d (a + b) \right]^{-1} - \left[a a^d a \right]_p^{-1} \right\| &= \left\| \sum_{n=1}^{\infty} (a^d b)^n a^d \right\| \leq \frac{\|a^d\| \|a^d b\|}{1 - \|a^d b\|}, \\ \left\| [a^\pi (a + b)]^d \right\| &= \left\| (a_2 + b_2)^\# \right\| \leq \|a^\pi\| \sum_{n=0}^{\infty} \left\| (b^d)^{n+1} \right\| \|a^n\|. \end{aligned} \quad (2.48)$$

Next, according to (2.48), we obtain

$$\left\| (a + b)^\# - a^d \right\| \leq \frac{\|a^d\| \|a^d b\|}{1 - \|a^d b\|} + \|a^\pi\| \sum_{n=1}^{\infty} \left\| (b^d)^{n+1} \right\| \|a^n\|. \quad (2.49)$$

Finally, using (2.49) the proof is finished. \square

Corollary 2.13. Let $a \in \mathcal{A}_g$ and let $b \in \mathcal{A}^d$. If a, b satisfy the conditions

$$\|a^\sharp b a a^\sharp\| < 1, \quad a^\pi b a = a b a^\pi. \quad (2.50)$$

Then $(a + b)^\sharp$ exists if and only if $a^\pi b$ is group invertible. In this case,

$$\frac{\|(a + b)^\sharp - a^\sharp\|}{\|a^\sharp\|} \leq \frac{\|a^\sharp b\|}{1 - \|a^\sharp b\|} + \frac{\|a^\pi\| \|b^d\|}{\|a^\sharp\|}. \quad (2.51)$$

Let $A, E \in C^{n \times n}$ with $B = A + E$, and let

$$A = P^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} P, \quad E = P^{-1} \begin{bmatrix} E_1 & E_{12} \\ E_{21} & E_2 \end{bmatrix} P. \quad (2.52)$$

If $B^\pi = A^\pi$ (see [10, Theorem 2.1]), then

$$B = P^{-1} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} P, \quad A + E = P^{-1} \begin{bmatrix} A_1 + E_1 & 0 \\ 0 & A_2 + E_2 \end{bmatrix} P, \quad (2.53)$$

where B_1 is invertible and $B_2 = A_2$ is quasinilpotent (it follows that $E_2 = 0$). It follows from (2.53) that $A^\pi = B^\pi$ implies that $A^\pi B A = A B A^\pi$, (i.e., $A^\pi E A = A E A^\pi$). If A is group invertible, then B is group invertible and

$$B^\sharp = P^{-1} \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} P, \quad (2.54)$$

where $B_1 = A_1 + E_1$.

By virtue of $A^\pi = B^\pi$ and $\|A^D(B - A)\| < 1$ (see [10]), we give the following remark.

Remark 2.14 (see [10, Theorem 3.1]). Let $A, B \in C^{d \times d}$ with $A^\pi = B^\pi$. Then

$$\frac{\|A^D\|}{1 + \|A^D(B - A)\|} \leq \|B^D\|. \quad (2.55)$$

If $\|A^D(B - A)\| < 1$, then

$$\begin{aligned} \|B^D\| &\leq \frac{\|A^D\|}{1 - \|A^D(B - A)\|}, \\ \frac{\|B^D - A^D\|}{\|A^D\|} &\leq \frac{\|A^D(B - A)\|}{1 - \|A^D(B - A)\|}. \end{aligned} \quad (2.56)$$

Theorem 2.15. Let $a, b \in \mathcal{A}$ be generalized Drazin invertible and satisfy the conditions

$$\max\left\{\left\|a^\pi ab^d\right\|, \left\|a^d b a a^d\right\|\right\} < 1, \quad a^\pi b a = a b a^\pi. \quad (2.57)$$

Then $(a + b)^\sharp$ exists if and only if $a^\pi(a + b)$ is group invertible. In this case,

$$\frac{\left\|(a + b)^\sharp - a^d\right\|}{\left\|a^d\right\|} \leq \frac{\left\|a^d b\right\|}{1 - \left\|a^d b\right\|} + \frac{\left\|a^\pi\right\|\left\|b^d\right\|}{\left\|a^d\right\|(1 - \left\|b^d a\right\|)}. \quad (2.58)$$

Proof. Similarly to Theorem 2.12, we have that the formulas (2.45) hold. The details will be omitted. In the following we only give the simple proof.

By the condition

$$\max\left\{\left\|a^\pi ab^d\right\|, \left\|a^d b a a^d\right\|\right\} < 1, \quad (2.59)$$

it shows that $\|a_1^{-1}b_1\| < 1$ and $\|a_2b_2^d\| < 1$. Thus, the first result shows that $a_1 + b_1 \in p\mathcal{A}p$ is invertible. In view of Lemma 2.1, one concludes that $a + b$ is Drazin invertible if and only if $a_2 + b_2$ is Drazin invertible. That is, $(a + b)^\sharp$ exists if and only if $a^\pi(a + b)$ is group invertible.

In the following, we consider the perturbation of a_2 .

After application of the hypothesis $a^\pi b a = a b a^\pi$, we find that $a_2b_2 = b_2a_2$. It follows from Theorem 2.12 and Lemma 2.3 that

$$(a_2 + b_2)^\sharp = \sum_{n=0}^{\infty} (b_2^d)^{n+1} (-a_2)^n = a^\pi \sum_{n=0}^{\infty} (-1)^n (b^d)^{n+1} a^n. \quad (2.60)$$

It follows from the condition $\|a_2b_2^d\| < 1$ and

$$\sigma(a_2b_2^d) \cup \{0\} = \sigma(b_2^da_2) \cup \{0\}. \quad (2.61)$$

It implies that $\|b_2^da_2\| < 1$ and

$$\left\|(a_2 + b_2)^\sharp\right\| \leq \frac{\left\|a^\pi\right\|\left\|b^d\right\|}{1 - \left\|b^d a\right\|}. \quad (2.62)$$

Therefore, combining (2.49) with (2.62), we have

$$\left\|(a + b)^\sharp - a^d\right\| \leq \frac{\left\|a^d\right\|\left\|a^d b\right\|}{1 - \left\|a^d b\right\|} + \frac{\left\|a^\pi\right\|\left\|b^d\right\|}{1 - \left\|b^d a\right\|}. \quad (2.63)$$

Thus, by (2.63), we complete the proof. \square

3. Perturbation Bound of the Generalized Schur Complement

The perturbation bounds of the Schur complement are investigated in [29–31]. In [29] Stewart gave perturbation bounds for the Schur complement of a positive definite matrix in a positive semidefinite matrix. In [30] Wei and Wang generalized the results in [29] and enrich the perturbation theory for the Schur complement. In [31] the authors derived some new norm upper bounds for Schur complements of a positive semidefinite operator matrix. In this section, we consider the perturbation bounds of the generalized Schur complement in Banach algebra.

Some notations of the generalized Schur complement over Banach algebra will be stated in the following.

Let $a \in \mathcal{A}$, and let it be written in the form as follows:

$$a = a_{11} + a_{12} + a_{21} + a_{22}. \quad (3.1)$$

It has the following matrix form:

$$\mathcal{M} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_s, \quad (3.2)$$

where $s \in \mathcal{A}^\bullet$ is idempotent element in \mathcal{A} and a_{ij} is taken as (1.10).

The formulas (1.5) and (1.6) are written in Banach algebra, respectively:

$$\begin{aligned} s_1 &= a_{22} - a_{21} a_{11}^{-1} a_{12}, \\ \overline{\mathcal{M}} &= \begin{bmatrix} a_{11}^{-1} + a_{11}^{-1} a_{12} s_1^{-1} a_{21} a_{11}^{-1} & -a_{11}^{-1} a_{12} s_1^{-1} \\ -s_1^{-1} a_{21} a_{11}^{-1} & s_1^{-1} \end{bmatrix}_s. \end{aligned} \quad (3.3)$$

Similarly, the generalized Schur complement in (1.7) and (1.8) is defined in the following over Banach algebra, respectively:

$$\begin{aligned} s_1 &= a_{22} - a_{21} a_{11}^\# a_{12}, \\ s_1 &= a_{22} - a_{21} a_{11}^d a_{12}, \end{aligned} \quad (3.4)$$

where s_1 denotes the generalized Schur complement of a_{11} in \mathcal{M} .

Theorem 3.1. Let \mathcal{M} be given as (3.2) let and

$$\overline{\mathcal{M}} = \begin{bmatrix} a_{11} + \Delta a_{11} & a_{12} + \Delta a_{12} \\ a_{21} + \Delta a_{21} & a_{22} + \Delta a_{22} \end{bmatrix}_s = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{bmatrix}_s \quad (3.5)$$

be perturbed version of \mathcal{M} , and the following conditions are satisfied:

$$\|\Delta a_{11}\| \leq \epsilon \|a_{11}\|, \quad \|\Delta a_{12}\| \leq \epsilon \|a_{12}\|, \quad \|\Delta a_{21}\| \leq \epsilon \|a_{21}\|, \quad \|\Delta a_{22}\| \leq \epsilon \|a_{22}\|, \quad (3.6)$$

where $\epsilon > 0$. If a_{11} , Δa_{11} and \bar{a}_{11} satisfy the conditions of Theorem 2.8, then

$$\begin{aligned} \|\bar{s}_1 - s_1\| &\leq \epsilon \|a_{22}\| + \|a_{21}\| \|a_{12}\| \left(\theta (2\epsilon + \epsilon^2) + \|a_{11}^\pi\| \eta_2 + \epsilon \|a_{11}^d\| \|a_{11}\| \eta_1 \right) \\ &\quad + \epsilon \|a_{21}\| \|a_{11}\| \|a_{12}\| \left(\eta_1^2 + \|a_{11}^\pi\| \eta_1 \eta_2 + (\epsilon + \epsilon^2) \|a_{11}^\pi\| \|a_{11}\| \eta_1^2 \eta_2 \right), \end{aligned} \quad (3.7)$$

where

$$\eta_1 = \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|}, \quad \eta_2 = \sum_{n=0}^{\infty} \|a_{11}^{n+1}\| \left\| \left[(\Delta a_{11})^d \right]^{n+1} \right\|, \quad (3.8)$$

$$\begin{aligned} \theta &= \epsilon \|a_{11}\| \eta_1^2 (1 + \|a_{11}^\pi\| \eta_2) + \epsilon \|a_{11}\| \|a_{11}^\pi\| \eta_1 \eta_2 (1 + \epsilon \eta_1 \|a_{11}\|) \\ &\quad + \|a_{11}^\pi\| \eta_2 + \epsilon \|a_{11}^d\| \|a_{11}\| \eta_1, \end{aligned} \quad (3.9)$$

and \bar{s}_1 and s_1 are Schur complement of a_{11} in \mathcal{M} and Schur complement of \bar{a}_{11} in $\bar{\mathcal{M}}$, respectively.

Proof. Since a_{11} , Δa_{11} and \bar{a}_{11} satisfy Theorem 2.8, according to (2.31), we obtain

$$\begin{aligned} \bar{a}_{11}^\# &= \left(\sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \right)^2 \left[\Delta a_{11} - \Delta a_{11} a_{11}^\pi \sum_{n=0}^{\infty} a_{11}^{n+1} \left[(\Delta a_{11})^d \right]^{n+1} \right] \\ &\quad + \left[\left(\sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \right)^2 \Delta a_{11} a_{11}^\pi \Delta a_{11} - \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \Delta a_{11} a_{11}^\pi \right] \sum_{n=0}^{\infty} a_{11}^n \left[(\Delta a_{11})^d \right]^{n+1} \\ &\quad + \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d + a_{11}^\pi \sum_{n=0}^{\infty} a_{11}^n \left[(\Delta a_{11})^d \right]^{n+1}. \end{aligned} \quad (3.10)$$

Therefore, it is easy to see that

$$\begin{aligned} \bar{s}_1 &= a_{22} + \Delta a_{22} - (a_{21} + \Delta a_{21}) \bar{a}_{11}^\# (a_{12} + \Delta a_{12}) \\ &= a_{22} + \Delta a_{22} - a_{21} \bar{a}_{11}^\# a_{12} - \Delta a_{21} \bar{a}_{11}^\# a_{12} - a_{21} \bar{a}_{11}^\# \Delta a_{12} - \Delta a_{21} \bar{a}_{11}^\# \Delta a_{12} \\ &= s_1 + \Delta a_{22} - \Delta a_{21} \bar{a}_{11}^\# a_{12} - a_{21} \bar{a}_{11}^\# \Delta a_{12} - \Delta a_{21} \bar{a}_{11}^\# \Delta a_{12} \\ &\quad - a_{21} a_{11}^\pi \sum_{n=0}^{\infty} a_{11}^n \left[(\Delta a_{11})^d \right]^{n+1} a_{12} + a_{21} \sum_{n=1}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d a_{12} \\ &\quad + a_{21} \left(\sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \right)^2 \Delta a_{11} \left[1 - a_{11}^\pi (a_{11} + \Delta a_{11}) \sum_{n=0}^{\infty} a_{11}^n \left[(\Delta a_{11})^d \right]^{n+1} \right] a_{12} \\ &\quad - a_{21} \sum_{n=0}^{\infty} (a_{11}^d \Delta a_{11})^n a_{11}^d \Delta a_{11} a_{11}^\pi \sum_{n=0}^{\infty} a_{11}^n \left[(\Delta a_{11})^d \right]^{n+1} a_{12}, \end{aligned}$$

$$\begin{aligned}
\|\bar{a}_{11}^\# \| &\leq \epsilon \|a_{11}\| \left(\frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right)^2 \left[1 + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^{n+1}\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \right] \\
&+ \left[\epsilon^2 \left(\frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right)^2 \|a_{11}^\pi\| \|a_{11}\|^2 + \frac{\epsilon \|a_{11}^d\| \|a_{11}\| \|a_{11}^\pi\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right] \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \\
&+ \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| + \frac{\epsilon \|a_{11}^d\|^2 \|a_{11}\|}{1 - \|a_{11}^d \Delta a_{11}\|}.
\end{aligned} \tag{3.11}$$

From (3.11) and by the conditions (3.6), we obtain

$$\begin{aligned}
\|\bar{s}_1 - s_1\| &\leq \|\Delta a_{22}\| + \theta (2\epsilon + \epsilon^2) \|a_{21}\| \|a_{12}\| \\
&+ \|a_{21}\| \|a_{12}\| \left(\|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| + \frac{\epsilon \|a_{11}^d\|^2 \|a_{11}\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right) \\
&+ \epsilon \|a_{21}\| \|a_{11}\| \|a_{12}\| \left(\frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right)^2 \left[1 + (1 + \epsilon) \|a_{11}\| \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \right] \\
&+ \epsilon \|a_{21}\| \|a_{12}\| \frac{\|a_{11}^d\| \|a_{11}\| \|a_{11}^\pi\|}{1 - \|a_{11}^d \Delta a_{11}\|} \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\|,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\theta &= \epsilon \|a_{11}\| \left(\frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right)^2 \left[1 + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^{n+1}\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \right] \\
&+ \frac{\epsilon \|a_{11}^d\| \|a_{11}\| \|a_{11}^\pi\|}{1 - \|a_{11}^d \Delta a_{11}\|} \left[1 + \frac{\epsilon \|a_{11}^d\| \|a_{11}\| \|a_{11}^\pi\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right] \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| \\
&+ \|a_{11}^\pi\| \sum_{n=0}^{\infty} \|a_{11}^n\| \left\| [(\Delta a_{11})^d]^{n+1} \right\| + \frac{\epsilon \|a_{11}^d\|^2 \|a_{11}\|}{1 - \|a_{11}^d \Delta a_{11}\|}.
\end{aligned} \tag{3.13}$$

Thus, we finish the proof. \square

Similar to Theorem 3.1. It follows from the proof of Theorem 2.11 that the results are given as follow.

Theorem 3.2. Let \mathcal{M} and $\overline{\mathcal{M}}$ be taken as Theorem 3.1, and let the relations in (3.6) be satisfied, where $\epsilon > 0$. If a_{11} , Δa_{11} and \overline{a}_{11} satisfy the conditions of Theorem 2.11, then

$$\begin{aligned} \|\overline{s}_1 - s\| &\leq \epsilon \|a_{22}\| + \|a_{21}\| \|a_{12}\| \left(\theta (2\epsilon + \epsilon^2) + \|a_{11}^\pi\| \eta_2 + \epsilon \|a_{11}^d\| \|a_{11}\| \eta_1 \right) \\ &\quad + \epsilon \|a_{21}\| \|a_{11}\| \|a_{12}\| \left(\eta_1^2 + \|a_{11}^\pi\| \eta_1 \eta_2 + (\epsilon + \epsilon^2) \|a_{11}^\pi\| \|a_{11}\| \eta_1^2 \eta_2 \right), \end{aligned} \quad (3.14)$$

where $\eta_1, \theta, \overline{s}_1$ and s_1 are taken as Theorem 3.1.

Theorem 3.3. Let \mathcal{M} and $\overline{\mathcal{M}}$ be taken as Theorem 3.1, and let the relations in (3.6) be satisfied, where $\epsilon > 0$. If a_{11} , Δa_{11} and \overline{a}_{11} satisfy the conditions of Theorem 2.12, then

$$\|\overline{s}_1 - s\| \leq \epsilon \|a_{22}\| + \left[(1 + 2\epsilon + \epsilon^2) \delta_1 + \epsilon \|a_{11}^d\|^2 \|a_{11}\| \right] \|a_{12}\| \|a_{21}\|, \quad (3.15)$$

where

$$\delta_1 = \left\{ \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[(\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| \right\}, \quad (3.16)$$

and \overline{s}_1 and s_1 are taken as Theorem 3.1.

Proof. Similar to the proof of Theorem 3.1 the details are omitted. A simple proof is given as follows.

By (2.45), (2.46), and (2.47), we obtain

$$\overline{a}_{11}^\sharp = \sum_{n=0}^{\infty} \left(a_{11}^d \Delta a_{11} \right)^n a_{11}^d + a_{11}^\pi \sum_{n=0}^{\infty} \left[(\Delta a_{11})^d \right]^{n+1} (-a_{11})^n. \quad (3.17)$$

In view of (3.17), we easily have

$$\begin{aligned} \overline{s}_1 &= a_{22} + \Delta a_{22} - (a_{21} + \Delta a_{21}) \overline{a}_{11}^\sharp (a_{12} + \Delta a_{12}) \\ &= s_1 + \Delta a_{22} - a_{21} \sum_{n=0}^{\infty} \left(a_{11}^d \Delta a_{11} \right)^n a_{11}^d a_{12} + a_{21} a_{11}^\pi \sum_{n=0}^{\infty} \left[(\Delta a_{11})^d \right]^{n+1} (-a_{11})^n a_{12} \\ &\quad - a_{21} a_{11}^d \Delta a_{11} a_{11}^d a_{12} - \Delta a_{21} \overline{a}_{11}^\sharp a_{12} - a_{21} \overline{a}_{11}^\sharp \Delta a_{12} - \Delta a_{21} \overline{a}_{11}^\sharp \Delta a_{12}, \\ \|\overline{a}_{11}^\sharp\| &\leq \left\| \sum_{n=0}^{\infty} \left(a_{11}^d \Delta a_{11} \right)^n a_{11}^d \right\| + \left\| a_{11}^\pi \sum_{n=0}^{\infty} \left[(\Delta a_{11})^d \right]^{n+1} (-a_{11})^n \right\| \\ &\leq \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[(\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\|. \end{aligned} \quad (3.18)$$

It shows from (3.18) that

$$\begin{aligned}
\|\bar{s}_1 - s_1\| &\leq \epsilon \|a_{22}\| + \frac{\|a_{11}^d\| \|a_{21}\| \|a_{12}\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{21}\| \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[(\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| \|a_{12}\| \\
&\quad + (2\epsilon + \epsilon^2) \|a_{21}\| \|\bar{a}_{11}^\# \| \|a_{12}\| + \epsilon \|a_{11}^d\| \|a_{11}\| \|a_{12}\| \|a_{21}\| \\
&\leq \epsilon \|a_{22}\| + \left\{ \epsilon \|a_{11}^d\| \|a_{11}\| + \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[(\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| \right\} \|a_{21}\| \|a_{12}\| \\
&\quad + (2\epsilon + \epsilon^2) \|a_{21}\| \|a_{12}\| \left\{ \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[(\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| + \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} \right\} \\
&= \epsilon \|a_{22}\| + \|a_{21}\| \|a_{12}\| \left[(1 + 2\epsilon + \epsilon^2) \delta_1 + \epsilon \|a_{11}^d\| \|a_{11}\| \right],
\end{aligned} \tag{3.19}$$

where

$$\delta_1 = \left\{ \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \|a_{11}^\pi\| \sum_{n=0}^{\infty} \left\| \left[(\Delta a_{11})^d \right]^{n+1} a_{11}^n \right\| \right\}. \tag{3.20}$$

Therefore, we complete the proof. \square

Similar to Theorems 3.1 and 3.3. The proof of the following theorem follows from Theorem 2.15.

Theorem 3.4. Let \mathcal{M} and $\overline{\mathcal{M}}$ be taken as Theorem 3.1, and let the relations in (3.6) be satisfied, where $\epsilon > 0$. If a_{11} , Δa_{11} and \bar{a}_{11} satisfy the conditions of Theorem 2.15, then

$$\|\bar{s}_1 - s_1\| \leq \epsilon \|a_{22}\| + (1 + 2\epsilon + \epsilon^2) \|a_{21}\| \|a_{12}\| \delta_1, \tag{3.21}$$

where

$$\delta_1 = \left\{ \frac{\|a_{11}^d\|}{1 - \|a_{11}^d \Delta a_{11}\|} + \frac{\|a_{11}^\pi\| \|(\Delta a_{11})^d\|}{1 - \|(\Delta a_{11})^d a_{11}\|} \right\}, \tag{3.22}$$

and \bar{s}_1 and s_1 are taken as Theorem 3.1.

Acknowledgments

X. Liu is supported by the NSFC grant (11061005, 61165015), the Ministry of Education Science and Technology Key Project under Grant 210164 and Grants (HCIC201103) of

Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis Open Fund and Key issues for Department of Education of Guangxi (201202ZD031). H. Wei is supported by 973 Program (Project no. 2010CB327900).

References

- [1] I. Schur, "Über potenzreihen die im innern des einheitskreises sind," *Journal Fur Die Reine Und Angewandte Mathematik*, vol. 147, pp. 205–234, 1917.
- [2] J. Benítez and N. Thome, "The generalized Schur complement in group inverses and $(k + 1)$ -potent matrices," *Linear and Multilinear Algebra*, vol. 54, no. 6, pp. 405–413, 2006.
- [3] X. Sheng and G. Chen, "Some generalized inverses of partition matrix and quotient identity of generalized Schur complement," *Applied Mathematics and Computation*, vol. 196, no. 1, pp. 174–184, 2008.
- [4] R. E. Hartwig, X. Li, and Y. Wei, "Representations for the Drazin inverse of a 2×2 block matrix," *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 3, pp. 757–771, 2005.
- [5] X. Li, "A representation for the Drazin inverse of block matrices with a singular generalized Schur complement," *Applied Mathematics and Computation*, vol. 217, no. 18, pp. 7531–7536, 2011.
- [6] Y. Wei, "Expressions for the Drazin inverse of a 2×2 block matrix," *Linear and Multilinear Algebra*, vol. 45, no. 2-3, pp. 131–146, 1998.
- [7] C. Deng and Y. Wei, "Representations for the Drazin inverse of 2×2 block-operator matrix with singular Schur complement," *Linear Algebra and Its Applications*, vol. 435, no. 11, pp. 2766–2783, 2011.
- [8] D. S. Djordjević and V. Rakočević, *Lectures on Generalized Inverses*, Faculty of Sciences and Mathematics, University of Niš, Niš, Serbia, 2008.
- [9] N. C. González, J. J. Koliha, and I. Straškraba, "Perturbation of the Drazin inverse," *Soochow Journal of Mathematics*, vol. 27, no. 2, pp. 201–211, 2001.
- [10] N. C. González, J. J. Koliha, and Y. Wei, "Perturbation of the Drazin inverse for matrices with equal eigenprojections at zero," *Linear Algebra and Its Applications*, vol. 312, no. 1–3, pp. 181–189, 2000.
- [11] X. Li and Y. Wei, "A note on the perturbation bound of the Drazin inverse," *Applied Mathematics and Computation*, vol. 140, no. 2-3, pp. 329–340, 2003.
- [12] W. Yimin and W. Guorong, "The perturbation theory for the Drazin inverse and its application," *Linear Algebra and Its Applications*, vol. 258, no. 1–3, pp. 179–186, 1997.
- [13] Y. Wei, "On the perturbation of the group inverse and oblique projection," *Applied Mathematics and Computation*, vol. 98, no. 1, pp. 29–42, 1999.
- [14] Y. Wei and H. Wu, "The perturbation of the Drazin inverse and oblique projection," *Applied Mathematics Letters*, vol. 13, no. 3, pp. 77–83, 2000.
- [15] Y. Wei and X. Li, "An improvement on perturbation bounds for the Drazin inverse," *Numerical Linear Algebra with Applications*, vol. 10, no. 7, pp. 563–575, 2003.
- [16] Y. Wei, "Perturbation bound of the Drazin inverse," *Applied Mathematics and Computation*, vol. 125, no. 2-3, pp. 231–244, 2002.
- [17] Y. Wei, X. Li, and F. Bu, "A perturbation bound of the Drazin inverse of a matrix by separation of simple invariant subspaces," *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 1, pp. 72–81, 2006.
- [18] Q. Xu, C. Song, and Y. Wei, "The stable perturbation of the Drazin inverse of the square matrices," *SIAM Journal on Matrix Analysis and Applications*, vol. 31, no. 3, pp. 1507–1520, 2009.
- [19] Y. Xue and G. Chen, "Perturbation analysis for the Drazin inverse under stable perturbation in Banach space," *Missouri Journal of Mathematical Sciences*, vol. 19, no. 2, pp. 106–120, 2007.
- [20] Y. Xue, "Stable perturbation in Banach algebras," *Journal of the Australian Mathematical Society*, vol. 83, no. 2, pp. 271–284, 2007.
- [21] S. L. Campbell and C. D. Meyer, "Continuity properties of the Drazin pseudoinverse," *Linear Algebra and Its Applications*, vol. 10, no. 1, pp. 77–83, 1975.
- [22] N. C. González and J. J. Koliha, "Perturbation of the Drazin inverse for closed linear operators," *Integral Equations and Operator Theory*, vol. 36, no. 1, pp. 92–106, 2000.
- [23] N. C. González and J. Y. Vélez-Cerrada, "On the perturbation of the group generalized inverse for a class of bounded operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1213–1223, 2008.
- [24] N. C. González and J. J. Koliha, "New additive results for the g -Drazin inverse," *Proceedings of the Royal Society of Edinburgh A*, vol. 134, no. 6, pp. 1085–1097, 2004.

- [25] C. Y. Deng and Y. M. Wei, "New additive results for the generalized Drazin inverse," *Journal of Mathematical Analysis and Applications*, vol. 370, no. 2, pp. 313–321, 2010.
- [26] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*, SIAM, Philadelphia, Pa, USA, 2009.
- [27] C. Deng and Y. Wei, "A note on the Drazin inverse of an anti-triangular matrix," *Linear Algebra and Its Applications*, vol. 431, no. 10, pp. 1910–1922, 2009.
- [28] J. Chen, Z. Xu, and Y. Wei, "Representations for the Drazin inverse of the sum $P + Q + R + S$ and its applications," *Linear Algebra and Its Applications*, vol. 430, no. 1, pp. 438–454, 2009.
- [29] G. W. Stewart, "On the perturbation of Schur complement in positive semi-definite matrix," Tech. Rep. TR 95-38, CS-TR-3441, University of Maryland, 1995.
- [30] M. Wei and M. Wang, "Perturbation analysis for the generalized Schur complement of a positive semi-definite matrix," *Numerical Linear Algebra with Applications*, vol. 15, no. 1, pp. 1–11, 2008.
- [31] Q. X. Xu, Y. M. Wei, and Y. Y. Gu, "Sharp norm-estimations for Moore-Penrose inverses of stable perturbations of Hilbert C^* -module operators," *SIAM Journal on Numerical Analysis*, vol. 47, no. 6, pp. 4735–4758, 2010.