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Research Article

Dirichlet Problem for the Schrödinger Operator in a Half Space

Baiyun Su

Department of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450002, China

Correspondence should be addressed to Baiyun Su, baiyun85@163.com

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For continuous boundary data, the modified Poisson integral is used to write solutions to the half space Dirichlet problem for the Schrödinger operator. Meanwhile, a solution of the Poisson integral for any continuous boundary function is also given explicitly by the Poisson integral with the generalized Poisson kernel depending on this boundary function.

1. Introduction and Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n (n \geq 2)$ denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\overline{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H.

For a set E, $E \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H; |x| \in E\}$ and $\{x \in \partial H; |x| \in E\}$ by HE and ∂HE, respectively. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n), y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$, and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j, \qquad |x| = \sqrt{x \cdot x}, \qquad \Theta = \frac{x}{|x|}, \qquad \Phi = \frac{y}{|y|}. \tag{1.1}$$

For $x \in \mathbf{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball with center at x and radius $r (> 0)$ in $\mathbf{R}^n$. We will say that a set $E \subset H$ has a covering $\{r_j, R_j\}$ if there exists a sequence of balls

$\{B_j\}$ with centers in H such that $E \subset \bigcup_{j=1}^{\infty} B_j$, where r_j is the radius of B_j and R_j is the distance between the origin and the center of B_j.

Let $\mathcal{A}_a$ denote the class of nonnegative radial potentials $a(x)$, that is, $0 \leq a(x) = a(|x|)$, $x \in H$, such that $a \in L^b_{\mathrm{loc}}(H)$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

This paper is devoted to the stationary Schrödinger equation

$$\mathrm{SSE}u(x) = -\Delta u(x) + a(x)u(x) = 0, \tag{1.2}$$

where $x \in H$, Δ is the Laplace operator and $a \in \mathcal{A}_a$. These solutions are called a-harmonic functions or generalized harmonic functions associated with the operator SSE. Note that they are (classical) harmonic functions in the case $a = 0$. Under these assumptions the operator SSE can be extended in the usual way from the space $C_0^{\infty}(H)$ to an essentially self-adjoint operator on $L^2(H)$ (see [1–3]). We will denote it by SSE as well. This last one has a Green function $G_a(x,y)$. Here, $G_a(x,y)$ is positive on H and its inner normal derivative $\partial G_a(x,y)/\partial n(y') \geq 0$. We denote this derivative by $P_a(x,y')$, which is called the Poisson a-kernel with respect to H. We remark that $G(x,y)$ and $P(x,y')$ are the Green function and Poisson kernel of the Laplacian in H, respectively.

Let Δ^* be a Laplace-Beltrami operator (spherical part of the Laplace) on the unit sphere. It is known (see, e.g., [4, page 41]) that the eigenvalue problem

$$\begin{gathered}\Delta^*\varphi(\Theta) + \lambda\varphi(\Theta) = 0, \quad \Theta \in \mathbf{S}_+^{n-1}, \\ \varphi(\Theta) = 0, \quad \Theta \in \partial\mathbf{S}_+^{n-1},\end{gathered} \tag{1.3}$$

has the eigenvalues $\lambda_j = j(j+n-2)$ $(j = 0,1,2\ldots)$. Corresponding eigenfunctions are denoted by $\varphi_{jv}(1 \leq v \leq v_j)$, where v_j is the multiplicity of λ_j. We norm the eigenfunctions in $L^2(\mathbf{S}_+^{n-1})$ and $\varphi_1 = \varphi_{11} > 0$.

Hence, well-known estimates (see, e.g., [5, page 14]) imply the following inequality:

$$\sum_{v=1}^{v_j} \varphi_{jv}(\Theta)\frac{\partial\varphi_{jv}(\Phi)}{\partial n_{\Phi}} \leq M(n)j^{2n-1}, \tag{1.4}$$

where the symbol $M(n)$ denotes a constant depending only on n.

Let $V_j(r)$ and $W_j(r)$ stand, respectively, for the increasing and nonincreasing, as $r \to +\infty$, solutions of the equation

$$-y''(r) - \frac{n-1}{r}y'(r) + \left(\frac{\lambda_j}{r^2} + a(r)\right)y(r) = 0, \quad 0 < r < \infty, \tag{1.5}$$

normalized under the condition $V_j(1) = W_j(1) = 1$.

We will also consider the class $\mathcal{B}_a$, consisting of the potentials $a \in \mathcal{A}_a$ such that there exists a finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0,\infty)$. Moreover, $r^{-1}|r^2a(r) - k| \in L(1,\infty)$. If $a \in \mathcal{B}_a$, then solutions of (1.2) are continuous (see [6]).

In the rest of paper, we assume that $a \in \mathcal{B}_a$, and we will suppress this assumption for simplicity. Further, we use the standard notations $u^+ = \max\{u,0\}$, $u^- = -\min\{u,0\}$, $[d]$ is the integer part of d and $d = [d] + \{d\}$, where d is a positive real number.

Denote

$$\iota^{\pm}_{j,k} = \frac{2-n\pm\sqrt{(n-2)^2+4(k+\lambda_j)}}{2} \quad (j=0,1,2,3\ldots). \tag{1.6}$$

Remark 1.1. $\iota^{+}_{j,0} = j(j=0,1,2,3,\ldots)$ in the case $a=0$.

It is known (see [7]) that in the case under consideration the solutions to (1.5) have the asymptotics

$$V_j(r) \sim d_1 r^{\iota^{+}_{j,k}}, \qquad W_j(r) \sim d_2 r^{\iota^{-}_{j,k}}, \quad \text{as } r \longrightarrow \infty, \tag{1.7}$$

where d_1 and d_2 are some positive constants.

If $a \in \mathcal{A}_a$, it is known that the following expansion for the Green function $G_a(x,y)$ (see [8, Chapter 11], [1, 9])

$$G_a(x,y) = \sum_{j=0}^{\infty}\frac{1}{\chi'(1)}V_j(\min(|x|,|y|))W_j(\max(|x|,|y|))\left(\sum_{v=1}^{v_j}\varphi_{jv}(\Theta)\varphi_{jv}(\Phi)\right), \tag{1.8}$$

where $|x|\neq|y|$ and $\chi'(1) = w(W_1(r),V_1(r))|_{r=1}$ is its Wronskian. The series converges uniformly if either $|x|\le s|y|$ or $|y|\le s|x|(0<s<1)$.

For a nonnegative integer m and two points $x,y\in H$, we put

$$K(a,m)(x,y) = \begin{cases} 0 & \text{if } |y|<1, \\ \widetilde{K}(a,m)(x,y) & \text{if } 1\le|y|<\infty, \end{cases} \tag{1.9}$$

where

$$\widetilde{K}(a,m)(x,y) = \sum_{j=0}^{m}\frac{1}{\chi'(1)}V_j(|x|)W_j(|y|)\left(\sum_{v=1}^{v_j}\varphi_{jv}(\Theta)\varphi_{jv}(\Phi)\right). \tag{1.10}$$

We introduce another function of $x,y\in H$

$$G(a,m)(x,y) = G_a(x,y) - K(a,m)(x,y). \tag{1.11}$$

The generalized Poisson kernel $P(a,m)(x,y')$ with respect to H is defined by

$$P(a,m)(x,y') = \frac{\partial G(a,m)(x,y)}{\partial n(y')}. \tag{1.12}$$

In fact

$$P(a,0)(x,y') = P_a(x,y'). \tag{1.13}$$

We remark that the kernel function $P(0,m)(x,y')$ coincides with ones in Finkelstein and Scheinberg [10] and Siegel and Talvila [11] (see [8, Chapter 11]).

Put

$$U(a,m;u)(x)=\int_{\partial H}P(a,m)(x,y')u(y')dy', \tag{1.14}$$

where $u(y')$ is a continuous function on ∂H.

If γ is a real number and $\gamma\geq 0$, (resp., $\gamma<0$), $\iota^+_{[\gamma],k}+\{\gamma\}>-\iota^+_{1,k}+1$ (resp., $-\iota^+_{[-\gamma],k}-\{-\gamma\}>-\iota^+_{1,k}+1$) and

$$\begin{gathered}\iota^+_{[\gamma],k}+\{\gamma\}-n+1\leq\iota^+_{m+1,k}<\iota^+_{[\gamma],k}+\{\gamma\}-n+2\\ \left(\text{resp., }-\iota^+_{[-\gamma],k}-\{-\gamma\}-n+1\leq\iota^+_{m+1,k}<-\iota^+_{[-\gamma],k}-\{-\gamma\}-n+2\right).\end{gathered} \tag{1.15}$$

If these conditions all hold, we write $\gamma\in\mathcal{C}(k,m,n)$ (resp., $\gamma\in\mathfrak{D}(k,m,n)$).

Let $\gamma\in\mathcal{C}(k,m,n)$ (resp., $\gamma\in\mathfrak{D}(k,m,n)$) and u be functions on ∂H satisfying

$$\int_{\partial H}\frac{|u(y')|}{1+|y'|^{\iota^+_{[\gamma],k}+\{\gamma\}}}dy'<\infty\quad\left(\text{resp.}\int_{\partial H}|u(y')|\left(1+|y'|^{\iota^+_{[-\gamma],k}+\{-\gamma\}}\right)dy'<\infty\right). \tag{1.16}$$

For γ and u, we define the positive measure μ (resp., ν) on $\mathbf{R}^n$ by

$$\begin{gathered}d\mu(y')=\begin{cases}|u(y')||y'|^{-\iota^+_{[\gamma],k}-\{\gamma\}}dy', & y'\in\partial H(1,+\infty),\\ 0, & y'\in\mathbf{R}^n-\partial H(1,+\infty)\end{cases}\\ \left(\text{resp. }d\nu(y')=\begin{cases}|u(y')||y'|^{\iota^+_{[-\gamma],k}+\{-\gamma\}}dy', & y'\in\partial H(1,+\infty),\\ 0, & y'\in\mathbf{R}^n-\partial H(1,+\infty)\end{cases}\right).\end{gathered} \tag{1.17}$$

We remark that the total mass of μ and ν is finite.

Let $\epsilon>0$ and $\xi\geq 0$, and let μ be any positive measure on $\mathbf{R}^n$ having finite mass. For each $x=(x',x_n)\in\mathbf{R}^n$, the maximal function is defined by

$$M(x;\mu,\xi)=\sup_{0<\rho<|x|/2}\frac{\mu(B(x,\rho))}{\rho^{\xi}}. \tag{1.18}$$

The set $\{x=(x',x_n)\in\mathbf{R}^n;M(x;\mu,\xi)|x|^{\xi}>\epsilon\}$ is denoted by $E(\epsilon;\mu,\xi)$.

About classical solutions of the Dirichlet problem for the Laplacian, Siegel and Talvila (cf. [11, Corollary 2.1]) proved the following result.

Theorem A. *If u is a continuous function on ∂H satisfying*

$$\int_{\partial H}\frac{|u(y')|}{1+|y'|^{n+m}}dy'<\infty, \tag{1.19}$$

then, the function $U(0,m;u)(x)$ *satisfies*

$$\begin{gathered}
U(0,m;u) \in C^2(H) \cap C^0\left(\overline{H}\right), \\
\Delta U(0,m;u) = 0 \quad \text{in } H, \\
U(0,m;u) = u \quad \text{on } \partial H, \\
\lim_{|x| \to \infty, x \in H} U(0,m;u)(x) = o\left(x_n^{1-n} |x|^{n+m}\right).
\end{gathered} \tag{1.20}$$

Our first aim is to give the growth properties at infinity for $U(a,m;u)(x)$.

Theorem 1.2. *If* $0 \le \zeta \le n$, $\gamma \in \mathcal{C}(k,m,n)$ (*resp.*, $\gamma \in \mathfrak{D}(k,m,n)$) *and* u *is a measurable function on* ∂H *satisfying* (1.16), *then there exists a covering* $\{r_j, R_j\}$ *of* $E(\epsilon;\mu,n-\zeta)$ (*resp.*, $E(\epsilon;\nu,n-\zeta)$)$(\subset H)$ *satisfying*

$$\sum_{j=0}^{\infty} \left(\frac{r_j}{R_j}\right)^{2-\zeta} V_j\left(\frac{R_j}{r_j}\right) W_j\left(\frac{R_j}{r_j}\right) < \infty \tag{1.21}$$

such that

$$\lim_{|x| \to \infty, x \in H - E\left(\epsilon;\mu,n-\zeta\right)} |x|^{-\iota^{+}_{[\gamma],k} - \{\gamma\} + n - 1} \varphi_1^{\zeta-1}(\Theta) U(a,m;u)(x) = 0 \tag{1.22}$$

$$\left(\textit{resp.}, \lim_{|x| \to \infty, x \in H - E(\epsilon;\nu,n-\zeta)} |x|^{\iota^{+}_{[-\gamma],k} + \{-\gamma\} + n - 1} \varphi_1^{\zeta-1}(\Theta) U(a,m;u)(x) = 0\right). \tag{1.23}$$

If u *is a measurable function on* ∂H *satisfying*

$$\int_{\partial H} \frac{|u(y')|}{1 + |y'|^{\gamma}} dy' < \infty, \tag{1.24}$$

where γ *is a real number, for this* γ *and* u, *we define*

$$d\mu'(y') = \begin{cases} |u(y')| |y'|^{-\gamma} dy', & y' \in \partial H(1,+\infty), \\ 0, & y' \in \mathbf{R}^n - \partial H(1,+\infty). \end{cases} \tag{1.25}$$

Obviously, the total mass of μ' *is also finite.*

If we take $a = 0$ in Theorem 1.2, then we immediately have the following growth property based on (1.5) and Remark 1.1.

Corollary 1.3. *Let $0 \leq \zeta \leq n$, $\gamma > -(n-1)(p-1)$ and $\gamma - n \leq m < \gamma - n + 1$. If u is defined as previously, then the function $U(0,m;u)(x)$ is a harmonic function on H and there exists a covering $\{r_j, R_j\}$ of $E(\epsilon;\mu',n-\zeta)(\subset H)$ satisfying*

$$\sum_{j=0}^{\infty}\left(\frac{r_j}{R_j}\right)^{n-\zeta} < \infty \tag{1.26}$$

such that

$$\lim_{|x|\to\infty, x\in H-E(\epsilon;\mu',n-\zeta)} |x|^{n-\gamma-1}\varphi_1^{\zeta-1}(\Theta)U(a,m;u)(x) = 0. \tag{1.27}$$

Remark 1.4. In the case $\zeta = n$, (1.26) is a finite sum, and the set $E(\epsilon;\mu',0)$ is a bounded set and (1.27) holds in H.

Next we are concerned with solutions of the Dirichlet problem for the Schrödinger operator on H. For related results, we refer the readers to the paper by Kheyfits [1].

Theorem 1.5. *If $\gamma \in \mathcal{C}(k,m,n)$ (resp., $\gamma \in \mathfrak{D}(k,m,n)$) and u is a continuous function on ∂H satisfying* (1.16)*, then*

$$\begin{gathered} U(a,m;u) \in C^2(H)\cap C^0\left(\overline{H}\right), \\ \mathrm{SSE}\, U(a,m;u) = 0 \quad \text{in } H, \\ U(a,m;u) = u \quad \text{on } \partial H, \end{gathered} \tag{1.28}$$

$$\lim_{|x|\to\infty, x\in H} |x|^{-\iota^+_{[\gamma],k}-\{\gamma\}+n-1}\varphi_1^{n-1}(\Theta)U(a,m;u)(x) = 0 \tag{1.29}$$

$$\left(\textit{resp.}, \lim_{|x|\to\infty, x\in H} |x|^{\iota^+_{[-\gamma],k}+\{-\gamma\}+n-1}\varphi_1^{n-1}(\Theta)U(a,m;u)(x) = 0\right). \tag{1.30}$$

If we take $\iota^+_{[\gamma],k} + \{\gamma\} = \iota^+_{m+1,k} + n - 1$, then we immediately have the following corollary, which is just Theorem A in the case $a = 0$.

Corollary 1.6. *If u is a continuous function on ∂H satisfying*

$$\int_{\partial H} \frac{|u(y')|}{1+|y'|^{\iota^+_{m+1,k}+n-1}}dy' < \infty, \tag{1.31}$$

then (1.28) *hold and*

$$\lim_{|x|\to\infty, x\in H} |x|^{-\iota^+_{m+1,k}}\varphi_1^{n-1}(\Theta)U(a,m;u)(x) = 0. \tag{1.32}$$

As an application of Corollary 1.6, we can give a solution of the Dirichlet problem for any continuous function on ∂H.

Theorem 1.7. *If u is a continuous function on ∂H satisfying* (1.31) *and $h(x)$ is a solution of the Dirichlet problem for the Schrödinger operator on H with u satisfying*

$$\lim_{|x|\to\infty, x\in H} |x|^{-\iota^+_{m+1,k}} h^+(x) = 0, \tag{1.33}$$

then

$$h(x) = U(a,m;u)(x) + \sum_{j=0}^{m}\left(\sum_{v=1}^{v_j} d_{jv}\varphi_{jv}(\Theta)\right)V_j(|x|), \tag{1.34}$$

where $x \in H$ and d_{jv} are constants.

2. Lemmas

Throughout this paper, let M denote various constants independent of the variables in questions, which may be different from line to line.

Lemma 2.1. *If $1 \le |y'| < (1/2)|x|$, then*

$$\left|P_a(x,y')\right| \le M|x|^{\iota^-_{1,k}}\left|y'\right|^{\iota^+_{1,k}-1}\varphi_1(\Theta). \tag{2.1}$$

If $|y'| \ge 1$ and $|y'| \ge 2|x|$, then

$$\left|P(a,m)(x,y')\right| \le MV_{m+1}(|x|)\frac{W_{m+1}(|y'|)}{|y'|}\varphi_1(\Theta)\frac{\partial\varphi_1(\Phi)}{\partial n_\Phi}. \tag{2.2}$$

If $(1/2)|x| < |y'| < 2|x|$, then

$$\left|P(x,y')\right| \le M\left|x-y'\right|^{-n}|x|\varphi_1(\Theta). \tag{2.3}$$

Proof. Equations (2.1) and (2.2) are obtained by Kheyfits (see [8, Chapter 11] or [1, Lemma 1]). Equation (2.3) follows from Hayman and Kennedy (see [12, Lemma 4.2]). □

Lemma 2.2 (see [2, Theorem 1]). *If $u(x)$ is a solution of* (1.2) *on H satisfying*

$$\lim_{|x|\to\infty, x\in H} |x|^{-\iota^+_{m+1,k}} u^+(x) = 0, \tag{2.4}$$

then

$$u(x) = \sum_{j=0}^{m}\left(\sum_{v=1}^{v_j} d_{jv}\varphi_{jv}(\Theta)\right)V_j(|x|). \tag{2.5}$$

Lemma 2.3. *Let $\epsilon > 0$ and $\xi \geq 0$, and let μ be any positive measure on $\mathbf{R}^n$ having finite total mass. Then, $E(\epsilon;\mu,\xi)$ has a covering $\{r_j, R_j\}$ $(j = 1,2,\ldots)$ satisfying*

$$\sum_{j=1}^{\infty}\left(\frac{r_j}{R_j}\right)^{2-n+\xi} V_j\left(\frac{R_j}{r_j}\right)W_j\left(\frac{R_j}{r_j}\right) < \infty. \tag{2.6}$$

Proof. Set

$$E_j(\epsilon;\mu,\xi) = \left\{x \in E(\epsilon;\mu,\xi) : 2^j \leq |x| < 2^{j+1}\right\} \quad (j = 2,3,4,\ldots). \tag{2.7}$$

If $x \in E_j(\epsilon;\mu,\xi)$, then there exists a positive number $\rho(x)$ such that

$$\left(\frac{\rho(x)}{|x|}\right)^{2-n+\xi} V_j\left(\frac{|x|}{\rho(x)}\right)W_j\left(\frac{|x|}{\rho(x)}\right) \sim \left(\frac{\rho(x)}{|x|}\right)^{\xi} \leq \frac{\mu(B(x,\rho(x)))}{\epsilon}. \tag{2.8}$$

Here, $E_j(\epsilon;\mu,\xi)$ can be covered by the union of a family of balls $\{B(x_{j,i},\rho_{j,i}) : x_{j,i} \in E_j(\epsilon;\mu,\xi)\}$ $(\rho_{j,i} = \rho(x_{j,i}))$. By the Vitali lemma (see [13]), there exists $\Lambda_j \subset E_j(\epsilon;\mu,\xi)$, which is at most countable, such that $\{B(x_{j,i},\rho_{j,i}) : x_{j,i} \in \Lambda_j\}$ are disjoint and $E_j(\epsilon;\mu,\xi) \subset \bigcup_{x_{j,i}\in\Lambda_j} B(x_{j,i},5\rho_{j,i})$.

So

$$\bigcup_{j=2}^{\infty} E_j(\epsilon;\mu,\xi) \subset \bigcup_{j=2}^{\infty}\bigcup_{x_{j,i}\in\Lambda_j} B(x_{j,i},5\rho_{j,i}). \tag{2.9}$$

□

On the other hand, note that $\bigcup_{x_{j,i}\in\Lambda_j} B(x_{j,i},\rho_{j,i}) \subset \{x : 2^{j-1} \leq |x| < 2^{j+2}\}$, so that

$$\begin{aligned}
\sum_{P_{j,i}\in\Lambda_j}\left(\frac{5\rho_{j,i}}{|x_{j,i}|}\right)^{2-n+\xi} V_j\left(\frac{|x_{j,i}|}{5\rho_{j,i}}\right)W_j\left(\frac{|x_{j,i}|}{5\rho_{j,i}}\right) &\sim \sum_{x_{j,i}\in\Lambda_j}\left(\frac{5\rho_{j,i}}{|x_{j,i}|}\right)^{\xi} \\
&\leq 5^{\xi}\sum_{x_{j,i}\in\Lambda_j}\frac{\mu(B(x_{j,i},\rho_{j,i}))}{\epsilon} \\
&\leq \frac{5^{\xi}}{\epsilon}\mu\left(H\left[2^{j-1},2^{j+2}\right)\right).
\end{aligned} \tag{2.10}$$

Hence, we obtain

$$\begin{aligned}
\sum_{j=1}^{\infty}\sum_{x_{j,i}\in\Lambda_j}\left(\frac{\rho_{j,i}}{|x_{j,i}|}\right)^{2-n+\xi} V_j\left(\frac{|x_{j,i}|}{\rho_{j,i}}\right)W_j\left(\frac{|x_{j,i}|}{\rho_{j,i}}\right) &\sim \sum_{j=1}^{\infty}\sum_{x_{j,i}\in\Lambda_j}\left(\frac{\rho_{j,i}}{|x_{j,i}|}\right)^{\xi} \\
&\leq \sum_{j=1}^{\infty}\frac{\mu(H[2^{j-1},2^{j+2}))}{\epsilon} \\
&\leq \frac{3\mu(\mathbf{R}^n)}{\epsilon}.
\end{aligned} \tag{2.11}$$

Since $E(\epsilon;\mu,\xi)\cap\{x\in\mathbf{R}^n;|x|\geq 4\}=\bigcup_{j=2}^{\infty}E_j(\epsilon;\mu,\xi)$, then $E(\epsilon;\mu,\xi)$ is finally covered by a sequence of balls $(B(x_{j,i},\rho_{j,i}),B(x_1,6))(j=2,3,\ldots;i=1,2,\ldots)$ satisfying

$$\sum_{j,i}\left(\frac{\rho_{j,i}}{|x_{j,i}|}\right)^{2-n+\xi}V_j\left(\frac{|x_{j,i}|}{\rho_{j,i}}\right)W_j\left(\frac{|x_{j,i}|}{\rho_{j,i}}\right)\sim\sum_{j,i}\left(\frac{\rho_{j,i}}{|x_{j,i}|}\right)^{\xi}\leq\frac{3\mu(\mathbf{R}^n)}{\epsilon}+6^{\xi}<+\infty, \tag{2.12}$$

where $B(x_1,6)(x_1=(1,0,\ldots,0)\in\mathbf{R}^n)$ is the ball that covers $\{x\in\mathbf{R}^n;|x|<4\}$.

3. Proof of Theorem 1.2

We only prove the case $\gamma\geq 0$, the remaining case $\gamma<0$ can be proved similarly.

For any $\epsilon>0$, there exists $R_\epsilon>1$ such that

$$\int_{\partial H(R_\epsilon,\infty)}\frac{|u(y')|}{1+|y'|^{\iota^+_{[\gamma],k}+\{\gamma\}}}dy'<\epsilon. \tag{3.1}$$

The relation $G_a(x,y)\leq G(x,y)$ implies this inequality (see [14])

$$P_a(x,y')\leq P(x,y'). \tag{3.2}$$

For any fixed point $x\in H(R_\epsilon,+\infty)-E(\epsilon;\mu,n-\zeta)$ satisfying $|x|>2R_\epsilon$, letting $I_1=\partial H[0,1)$, $I_2=\partial H[1,R_\epsilon]$, $I_3=\partial H(R_\epsilon,(1/2)|x|]$, $I_4=\partial H((1/2)|x|,2|x|)$, $I_5=\partial H[2|x|,\infty)$ and $I_6=\partial H[1,2|x|)$, we write

$$|U(a,m;u)(x)|\leq\sum_{i=1}^{6}U_{a,i}(x), \tag{3.3}$$

where

$$\begin{aligned}U_{a,i}(x)&=\int_{I_i}|P_a(x,y')||u(y')|dy'\quad(i=1,2,3,4),\\U_{a,5}(x)&=\int_{I_5}|P(a,m)(x,y')||u(y')|dy',\\U_{a,6}(x)&=\int_{I_6}\left|\frac{\partial\widetilde{K}(\Omega,a,m)(x,y)}{\partial n(y')}\right||u(y')|dy'.\end{aligned} \tag{3.4}$$

By $\iota^{+}_{[\gamma],k}+\{\gamma\} > -\iota^{+}_{1,k}+1$, (1.16), (2.1), and (3.1), we have the following growth estimates

$$\begin{aligned} U_{a,2}(x) &\le M|x|^{\iota^{-}_{1,k}}\varphi_1(\Theta)\int_{I_2}|y'|^{\iota^{+}_{1,k}-1}|u(y')|dy' \\ &\le M|x|^{\iota^{-}_{1,k}}R_{\epsilon}^{\iota^{+}_{[\gamma],k}+\{\gamma\}+\iota^{+}_{1,k}-1}\varphi_1(\Theta), \\ U_{a,1}(x) &\le M|x|^{\iota^{-}_{1,k}}\varphi_1(\Theta), \\ U_{a,3}(x) &\le M\epsilon|x|^{\iota^{+}_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta). \end{aligned} \tag{3.5}$$

Next, we will estimate $U_{a,4}(x)$.

Take a sufficiently small positive number d_3 such that $I_4 \subset B(x,(1/2)|x|)$ for any $x \in \Pi(d_3)$, where

$$\Pi(d_3) = \left\{x \in H;\ \inf_{z\in\partial\mathbf{S}^{n-1}_{+}}\left|\frac{x}{|x|}-\frac{z}{|z|}\right| < d_3,\ 0<|x|<\infty\right\}, \tag{3.6}$$

and divide H into two sets $\Pi(d_3)$ and $H-\Pi(d_3)$.

If $x \in H-\Pi(d_3)$, then there exists a positive d'_3 such that $|x-y'| \ge d'_3|x|$ for any $y' \in \partial H$, and hence

$$\begin{aligned} U_{a,4}(x) &\le M|x|^{1-n}\varphi_1(\Theta)\int_{I_4}|u(y')|dy' \\ &\le M\epsilon|x|^{\iota^{+}_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta). \end{aligned} \tag{3.7}$$

We will consider the case $x \in \Pi(d_3)$. Now put

$$\Xi_i(x) = \left\{y \in I_4; 2^{i-1}\delta(x) \le |x-y'| < 2^{i}\delta(x)\right\}, \tag{3.8}$$

where $\delta(x) = \inf_{y'\in\partial H}|x-y'|$.

Since $\partial H \cap \{y \in \mathbf{R}^n : |x-y| < \delta(x)\} = \emptyset$, we have

$$U_{a,4}(x) = M\sum_{i=1}^{i(x)}\int_{\Xi_i(x)}|x|\varphi_1(\Theta)\frac{|u(y')|}{|x-y'|^{n}}dy', \tag{3.9}$$

where $i(x)$ is a positive integer satisfying $2^{i(x)-1}\delta(x) \le |x|/2 < 2^{i(x)}\delta(x)$.

Since $|x|\varphi_1(\Theta) \le M\delta(x)$ $(x \in H)$, we obtain

$$\begin{aligned}
\int_{\Xi_i(x)} |x|\varphi_1(\Theta)\frac{|u(y')|}{|x-y'|^n}dy' &\le 2^{(1-i)n}\varphi_1(\Theta)\delta(x)^{\zeta-n}\int_{\Xi_i(x)} |x|\delta(x)^{-\zeta}|u(y')|dy' \\
&\le M\varphi_1^{1-\zeta}(\Theta)\delta(x)^{\zeta-n}\int_{\Xi_i(x)} |x|^{1-\zeta}|u(y')|dy' \\
&\le M|x|^{n-\zeta}\varphi_1^{1-\zeta}(\Theta)\delta(x)^{\zeta-n}\int_{\Xi_i(x)} |y'|^{1-n}|u(y')|dy' \\
&\le M\epsilon|x|^{\iota^+_{[\gamma],k}+\{\gamma\}-\zeta+1}\varphi_1^{1-\zeta}(\Theta)\left(\frac{\mu(\Xi_i(x))}{(2^i\delta(x))^{n-\zeta}}\right)
\end{aligned} \tag{3.10}$$

for $i = 0, 1, 2, \ldots, i(x)$.

Since $x \notin E(\epsilon; \mu, n-\zeta)$, we have

$$\begin{gathered}
\frac{\mu(\Xi_i(x))}{(2^i\delta(x))^{n-\zeta}} \le \frac{\mu(B(x, 2^i\delta(x)))}{(2^i\delta(x))^{n-\zeta}} \le M(x; \mu, n-\zeta) \le \epsilon|x|^{\zeta-n} \quad (i = 0, 1, 2, \ldots, i(x)-1), \\
\frac{\mu(\Lambda_{i(x)}(x))}{(2^i\delta(x))^{n-\zeta}} \le \frac{\mu(B(x, |x|/2))}{(|x|/2)^{n-\zeta}} \le \epsilon|x|^{\zeta-n}.
\end{gathered} \tag{3.11}$$

So

$$U_{a,4}(x) \le M\epsilon|x|^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1^{1-\zeta}(\Theta). \tag{3.12}$$

By $\iota^+_{m+1,k} \ge \iota^+_{[\gamma],k} + \{\gamma\} - n + 1$, (1.7), (2.2), and (3.1), we have

$$\begin{aligned}
U_{a,5}(x) &\le MV_{m+1}(|x|)\int_{I_5} \frac{|u(y')|}{V_{m+1}(|y'|)|y'|^{n-1}}dy' \\
&\le M\epsilon|x|^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta).
\end{aligned} \tag{3.13}$$

We only consider $U_{a,6}(x)$ in the case $m \ge 1$, since $U_{a,6}(x) \equiv 0$ for $m = 0$. By the definition of $\widetilde{K}(a, m)$, (1.4), and (2.2), we see that

$$U_{a,6}(x) \le \frac{M}{\chi'(1)}\sum_{j=0}^{m} j^{2n-1}q_j(|x|), \tag{3.14}$$

where

$$q_j(|x|) = V_j(|x|)\int_{I_6} \frac{W_j(|y'|)|u(y')|}{|y'|}dy'. \tag{3.15}$$

To estimate $q_j(|x|)$, we write

$$q_j(|x|) \le q_j'(|x|) + q_j''(|x|), \tag{3.16}$$

where

$$\begin{aligned} q_j'(|x|) &= V_j(|x|)\varphi_1(\Theta)\int_{I_2} \frac{W_j(|y'|)|u(y')|}{|y'|}dy', \\ q_j''(|x|) &= V_j(|x|)\varphi_1(\Theta)\int_{\{y'\in\partial H: R_\epsilon<|y'|<2|x|\}} \frac{W_j(|y'|)|u(y')|}{|y'|}dy'. \end{aligned} \tag{3.17}$$

Notice that

$$V_j(|x|)\frac{V_{m+1}(|y'|)}{V_j(|y'|)|y'|} \le M\frac{V_{m+1}(|x|)}{|x|} \le M|x|^{\iota^+_{m+1,k}-1} \quad (|y'| \ge 1,\ R_\epsilon < 2|x|). \tag{3.18}$$

Thus, by $\iota^+_{m+1,k} < \iota^+_{[\gamma],k} + \{\gamma\} - n + 2$, (1.7), and (1.16), we conclude

$$\begin{aligned} q_j'(|x|) &= V_j(|x|)\varphi_1(\Theta)\int_{I_2} \frac{|u(y')|}{V_j(|y'|)|y'|^{n-1}}dy' \\ &\le MV_j(|x|)\varphi_1(\Theta)\int_{I_2} \frac{V_{m+1}(|y'|)}{|y'|^{\iota^+_{m+1,k}}}\frac{|u(y')|}{V_j(|y'|)|y'|^{n-1}}dy' \\ &\le M|x|^{\iota^+_{m+1,k}-1}R_\epsilon^{\iota^+_{[\gamma],k}+\{\gamma\}-\iota^+_{m+1,k}-n+2}\varphi_1(\Theta). \end{aligned} \tag{3.19}$$

Analogous to the estimate of $q_j'(|x|)$, we have

$$q_j''(|x|) \le M\epsilon|x|^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta). \tag{3.20}$$

Thus, we can conclude that

$$q_j(|x|) \le M\epsilon|x|^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta), \tag{3.21}$$

which yields

$$U_{a,6}(x) \le M\epsilon|x|^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta). \tag{3.22}$$

Combining (3.5)–(3.22), we obtain that if R_ϵ is sufficiently large and ϵ is sufficiently small, then $U(a,m;u)(x) = o(|x|^{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1^{1-\zeta}(\Theta))$ as $|x| \to \infty$, where $x \in H(R_\epsilon,+\infty) - E(\epsilon;\mu,n-\zeta)$. Finally, there exists an additional finite ball B_0 covering $H[0,R_\epsilon]$, which together with Lemma 2.3 gives the conclusion of Theorem 1.2.

4. Proof of Theorem 1.5

For any fixed $x \in H$, take a number satisfying $R > \max\{1, 2|x|\}$. By $\iota^{+}_{m+1,k} \geq \iota^{+}_{[\gamma],k} + \{\gamma\} - n + 1$, (1.5), (1.16), and (2.2), we have

$$\begin{aligned}
\int_{\partial H(R,\infty)} \left|P(a,m)(x,y')\right|\left|u(y')\right|dy' &\leq MV_{m+1}(|x|)\varphi_1(\Theta)\int_{\partial H(R,\infty)} \frac{|u(y')|}{|y'|^{\iota^{+}_{m+1,k}+n-1}}dy' \\
&\leq M|x|^{\iota^{+}_{m+1,k}}\varphi_1(\Theta)\int_{\partial H(2|x|,\infty)} |y'|^{\iota^{+}_{[\gamma],k}+\{\gamma\}-\iota^{+}_{m+1,k}-n+1}dy' \\
&\leq M|x|^{\iota^{+}_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta) \\
&< \infty.
\end{aligned} \tag{4.1}$$

Then, $U(a,m;u)(x)$ is absolutely convergent and finite for any $x \in H$. Thus $U(a,m;u)(x)$ is a solution of (1.2) on H.

Now we study the boundary behavior of $U(a,m;u)(x)$. Let $y' \in \partial H$ be any fixed point and l any positive number satisfying $l > \max\{|y'| + 1, (1/2)R\}$.

Set $\chi_{S(l)}$ as the characteristic function of $S(l) = \{y' \in \partial H, |y'| \leq l\}$, and write

$$U(a,m;u)(x) = U'(x) - U''(x) + U'''(x), \tag{4.2}$$

where

$$\begin{aligned}
U'(x) &= \int_{\partial H[0,2l]} P_a(x,y')u(y')dy', \\
U''(x) &= \int_{\partial H(1,2l]} \frac{\partial K(a,m)(x,y)}{\partial n(y')}u(y')dy', \\
U'''(x) &= \int_{\partial H(2l,\infty)} P(a,m)(x,y')u(y')dy'.
\end{aligned} \tag{4.3}$$

Notice that $U'(x)$ is the Poisson a-integral of $u(y')\chi_{S(2l)}$, We have $\lim_{x\to y', x\in H}U'(x) = u(y')$. Since $\lim_{\Theta\to\Phi'}\varphi_{jv}(\Theta) = 0 (j = 1,2,3\ldots; 1 \leq v \leq v_j)$ as $x \to y' \in \partial H$, we have $\lim_{x\to y', x\in H}U''(x) = 0$ from the definition of the kernel function $K(a,m)(x,y)$. $U'''(x) = O(|x|^{\iota^{+}_{[\gamma],k}+\{\gamma\}-n+1}\varphi_1(\Theta))$ and therefore tends to zero.

So the function $U(a,m;u)(x)$ can be continuously extended to $\overline{H}$ such that

$$\lim_{x\to y', x\in H} U(a,m;u)(x) = u(y') \tag{4.4}$$

for any $y' \in \partial H$ from the arbitrariness of l.

Finally, (1.29) and (1.30) follow from (1.22) and (1.23), respectively, in the case $\zeta = n$. Thus, we complete the proof of Theorem 1.5.

5. Proof of Theorem 1.7

From Corollary 1.6, we have the solution $U(a,m;u)(x)$ of the Dirichlet problem on H with u satisfying (1.31). Consider the function $h(x) - U(a,m;u)(x)$. Then, it follows that this is a solution of (1.2) in H and vanishes continuously on ∂H.

Since

$$0 \leq (h - U(\Omega,a,m;u))^{+}(x) \leq h^{+}(x) + (U(a,m;u))^{-}(x) \tag{5.1}$$

for any $x \in H$, we have

$$\lim_{|x|\to\infty, x\in H} |x|^{-\iota^{+}_{m+1,k}} (h - U(\Omega,a,m;u))^{+}(x) \ = 0 \tag{5.2}$$

from (1.32) and (1.33). Then, the conclusion of Theorem 1.7 follows immediately from Lemma 2.2.

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