

Research Article

Common Coupled Fixed Point Theorems of Single-Valued Mapping for c -Distance in Cone Metric Spaces

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The existence and uniqueness of the common coupled fixed point in cone metric spaces have been studied by considering different types of contractive conditions. A new concept of the c -distance in cone metric space has been recently introduced in 2011. Then, coupled fixed point results for contraction-type mappings in ordered cone metric spaces and cone metric spaces have been considered. In this paper, some common coupled fixed point results on c -distance in cone metric spaces are obtained. Some supporting examples are given.

1. Introduction

In 2007, Huang and Zhang [1] introduced the concept of cone metric space where each pair of points is assigned to a member of a real Banach space with a cone. Subsequently, several authors have studied the existence and uniqueness of the fixed point and common fixed point for self-map f by considering different types of contractive conditions. Some of these works are noted in [2–12].

In [13], Bhaskar and Lakshmikantham introduced the concept of coupled fixed point for a given partially ordered set X . Lakshmikantham and Ćirić [14] proved some more coupled fixed point theorems in partially ordered set.

In [15], Sabetghadam et al. considered the corresponding definition of coupled fixed point for the mapping in complete cone metric space and proved some coupled fixed point theorems. Subsequently, several authors have studied the existence and uniqueness of the coupled fixed point and common coupled fixed point by considering different types of contractive conditions. Some of these works are noted in [16–23].

Recently, Cho et al. [23] introduced a new concept of the c -distance in cone metric spaces (also see [24]) and proved some fixed point theorems in ordered cone metric spaces. This is more general than the classical Banach contraction mapping principle. Sintunavarat et al. [25] extended and developed the Banach contraction theorem on c -distance of Cho et al. [23]. Wang and Guo [24] proved some common fixed point theorems for this new distance. Subsequently, several authors have studied on the generalized distance in cone metric space. Some of this works are noted in [26–31].

In [30], Fadail and Ahmad proved some coupled fixed point theorems in cone metric spaces by using the concept of c -distance.

Recall that an element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.1. An element $(x, y) \in X \times X$ is called

- (1) a coupled coincidence point of mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $gx = F(x, y)$ and $gy = F(y, x)$, and (gx, gy) is called coupled point of coincidence.
- (2) a common coupled fixed point of mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $x = gx = F(x, y)$, $y = gy = F(y, x)$.

Abbas et al. [20] introduced the following definition.

Definition 1.2. The mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are called w -compatible if $g(F(x, y)) = F(gx, gy)$, whenever $gx = F(x, y)$ and $gy = F(y, x)$.

The aim of this paper is to continue the study of common coupled fixed points of mappings but now for c -distance in cone metric space. Our theorems extend and develop some theorems in literature to c -distance in cone metric spaces. In this paper, we do not impose the normality condition for the cones, the only assumption is that cone P has nonempty interior.

2. Preliminaries

Let E be a real Banach space and θ denote the zero element in E . A cone P is a subset of E such that

- (1) P is nonempty set closed and $P \neq \{\theta\}$,
- (2) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$,
- (3) $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subset E$, the partial ordering \leq with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. The notation of $<$ stand for $x \leq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \text{int } P$, where $\text{int } P$ denotes the interior of P . A cone P is called normal if there exists a number K such that

$$\theta \leq x \leq y \implies \|x\| \leq K\|y\|, \quad (2.1)$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P .

Definition 2.1 (see [1]). Let X be a nonempty set and E a real Banach space equipped with the partial ordering \leq with respect to the cone P . Suppose that the mapping $d : X \times X \rightarrow E$ satisfies the following condition:

- (1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- (3) $d(x, y) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 (see [1]). Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X and $x \in X$.

- (1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \rightarrow x$.
- (2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X .
- (3) A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 2.3 (see [8]).

- (1) If E be a real Banach space with a cone P and $a \leq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
- (2) If $c \in \text{int } P$, $\theta \leq a_n$ and $a_n \rightarrow \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next, we give the notation of c -distance on a cone metric space which is a generalization of ω -distance of Kada et al. [32] with some properties.

Definition 2.4 (see [23]). Let (X, d) be a cone metric space. A function $q : X \times X \rightarrow E$ is called a c -distance on X if the following conditions hold:

- (q1) $\theta \leq q(x, y)$ for all $x, y \in X$,
- (q2) $q(x, y) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$,
- (q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \leq u$ for some $u = u_x \in P$, then $q(x, y) \leq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,
- (q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.5 (see [23]). Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \rightarrow E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \rightarrow E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c -distance on X .

Lemma 2.6 (see [23]). Let (X, d) be a cone metric space and q is a c -distance on X . Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that u_n is a sequences in P converging to θ . Then the following hold.

- (1) If $q(x_n, y) \leq u_n$ and $q(x_n, z) \leq u_n$, then $y = z$.
- (2) If $q(x_n, y_n) \leq u_n$ and $q(x_n, z) \leq u_n$, then $\{y_n\}$ converges to z .
- (3) If $q(x_n, x_m) \leq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X .
- (4) If $q(y, x_n) \leq u_n$, then $\{x_n\}$ is a Cauchy sequence in X .

Remark 2.7 (see [23]).

- (1) $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.
- (2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

3. Main Results

In this section, we prove some common coupled fixed point results using c -distance in cone metric space. Also, we generalize the contractive conditions in literature by replacing the constants with functions.

Theorem 3.1. Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings and suppose that there exists mappings $k, l : X \times X \rightarrow [0, 1)$ such that the following hold:

- (a) $k(F(x, y), F(u, v)) \leq k(gx, gy)$ and $l(F(x, y), F(u, v)) \leq l(gx, gy)$ for all $x, y, u, v \in X$,
- (b) $k(x, y) = k(y, x)$ and $l(x, y) = l(y, x)$ for all $x, y \in X$,
- (c) $(k + l)(x, y) < 1$ for all $x, y \in X$,
- (d) $q(F(x, y), F(u, v)) \leq k(gx, gy)q(gx, gu) + l(gx, gy)q(gy, gv)$ for all $x, y, u, v \in X$.

If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Proof. Choose $x_0, y_0 \in X$. Set $gx_1 = F(x_0, y_0)$, $gy_1 = F(y_0, x_0)$, this can be done because $F(X \times X) \subseteq g(X)$. Continuing this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that $gx_{n+1} = F(x_n, y_n)$, $gy_{n+1} = F(y_n, x_n)$. Then we have

$$\begin{aligned}
 q(gx_n, gx_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
 &\leq k(gx_{n-1}, gy_{n-1})q(gx_{n-1}, gx_n) \\
 &\quad + l(gx_{n-1}, gy_{n-1})q(gy_{n-1}, gy_n) \\
 &= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_{n-1}, gx_n) \\
 &\quad + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gy_{n-1}, gy_n) \\
 &\leq k(gx_{n-2}, gy_{n-2})q(gx_{n-1}, gx_n) \\
 &\quad + l(gx_{n-2}, gy_{n-2})q(gy_{n-1}, gy_n) \\
 &\vdots \\
 &\leq k(gx_1, gy_1)q(gx_{n-1}, gx_n) + l(gx_1, gy_1)q(gy_{n-1}, gy_n).
 \end{aligned} \tag{3.1}$$

Similarly, we have

$$\begin{aligned}
 q(gy_n, gy_{n+1}) &= q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
 &\leq k(gy_{n-1}, gx_{n-1})q(gy_{n-1}, gy_n) + l(gy_{n-1}, gx_{n-1})q(gx_{n-1}, gx_n) \\
 &= k(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(gy_{n-1}, gy_n) \\
 &\quad + l(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(gx_{n-1}, gx_n) \\
 &\leq k(gy_{n-2}, gx_{n-2})q(gy_{n-1}, gy_n) + l(gy_{n-2}, gx_{n-2})q(gx_{n-1}, gx_n) \\
 &\vdots \\
 &\leq k(gy_1, gx_1)q(gy_{n-1}, gy_n) + l(gy_1, gx_1)q(gx_{n-1}, gx_n) \\
 &= k(gx_1, gy_1)q(gy_{n-1}, gy_n) + l(gx_1, gy_1)q(gx_{n-1}, gx_n).
 \end{aligned} \tag{3.2}$$

Put $q_n = q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1})$. Then we have

$$\begin{aligned}
 q_n &= q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) \\
 &\leq k(gx_1, gy_1)q(gx_{n-1}, gx_n) + l(gx_1, gy_1)q(gy_{n-1}, gy_n) \\
 &\quad + k(gx_1, gy_1)q(gy_{n-1}, gy_n) + l(gx_1, gy_1)q(gx_{n-1}, gx_n)
 \end{aligned}$$

$$\begin{aligned}
&= [k(gx_1, gy_1) + l(gx_1, gy_1)] [q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n)] \\
&= [k(gx_1, gy_1) + l(gx_1, gy_1)] q_{n-1} \\
&= hq_{n-1} \\
&\vdots \\
&\leq h^{n-1}q_1,
\end{aligned} \tag{3.3}$$

where $h = k(gx_1, gy_1) + l(gx_1, gy_1) < 1$.

Let $m > n \geq 1$. It follows that

$$\begin{aligned}
q(gx_n, gx_m) &\leq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m), \\
q(gy_n, gy_m) &\leq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \cdots + q(gy_{m-1}, gy_m).
\end{aligned} \tag{3.4}$$

Then we have

$$\begin{aligned}
q(gx_n, gx_m) + q(gy_n, gy_m) &\leq q_n + q_{n+1} + \cdots + q_{m-1} \\
&\leq h^n q_1 + h^{n+1} q_1 + \cdots + h^{m-1} q_1 \\
&= (h^n + h^{n+1} + \cdots + h^{m-1}) q_1 \\
&\leq \frac{h^n}{1-h} q_1.
\end{aligned} \tag{3.5}$$

Consequently,

$$\begin{aligned}
q(gx_n, gx_m) &\leq \frac{h^n}{1-h} q_1, \\
q(gy_n, gy_m) &\leq \frac{h^n}{1-h} q_1.
\end{aligned} \tag{3.6}$$

Thus, Lemma 2.6 (3) shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exists x^* and y^* in X such that $gx_n \rightarrow gx^*$ and $gy_n \rightarrow gy^*$ as $n \rightarrow \infty$. Using q3, we have

$$\begin{aligned}
q(gx_n, gx^*) &\leq \frac{h^n}{1-h} q_1, \\
q(gy_n, gy^*) &\leq \frac{h^n}{1-h} q_1.
\end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned}
 q(gx_n, F(x^*, y^*)) &= q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \\
 &\leq k(gx_{n-1}, gy_{n-1})q(gx_{n-1}, gx^*) \\
 &\quad + l(gx_{n-1}, gy_{n-1})q(gy_{n-1}, gy^*) \\
 &= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_{n-1}, gx^*) \\
 &\quad + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gy_{n-1}, gy^*) \\
 &\leq k(gx_{n-2}, gy_{n-2})q(gx_{n-1}, gx^*) \\
 &\quad + l(gx_{n-2}, gy_{n-2})q(gy_{n-1}, gy^*) \\
 &\vdots \\
 &\leq k(gx_1, gy_1)q(gx_{n-1}, gx^*) + l(gx_1, gy_1)q(gy_{n-1}, gy^*) \\
 &\leq k(gx_1, gy_1) \frac{h^{n-2}}{1-h} q_1 + l(gx_1, gy_1) \frac{h^{n-2}}{1-h} q_1 \\
 &= [k(gx_1, gy_1) + l(gx_1, gy_1)] \frac{h^{n-2}}{1-h} q_1 \\
 &= h \frac{h^{n-2}}{1-h} q_1 \\
 &= \frac{h^{n-1}}{1-h} q_1.
 \end{aligned} \tag{3.8}$$

Thus, Lemma 2.6 (1), (3.5), and (3.8) show that $gx^* = F(x^*, y^*)$. By similar way, we can prove that $gy^* = F(y^*, x^*)$. Therefore, (x^*, y^*) is a coupled coincidence point of F and g .

Suppose that $u = gx^* = F(x^*, y^*)$ and $v = gy^* = F(y^*, x^*)$. Then we have

$$\begin{aligned}
 q(u, u) &= q(gx^*, gx^*) \\
 &= q(F(x^*, y^*), F(x^*, y^*)) \\
 &\leq k(gx^*, gy^*)q(gx^*, gx^*) + l(gx^*, gy^*)q(gy^*, gy^*) \\
 &= k(u, v)q(u, u) + l(u, v)q(v, v),
 \end{aligned} \tag{3.9}$$

$$\begin{aligned}
q(v, v) &= q(gy^*, gy^*) \\
&= q(F(y^*, x^*), F(y^*, x^*)) \\
&\leq k(gy^*, gx^*)q(gy^*, gy^*) + l(gy^*, gx^*)q(gx^*, gx^*) \\
&= k(v, u)q(v, v) + l(v, u)q(u, u) \\
&= k(u, v)q(v, v) + l(u, v)q(u, u).
\end{aligned} \tag{3.10}$$

This implies that

$$\begin{aligned}
q(u, u) + q(v, v) &\leq k(u, v)q(u, u) + l(u, v)q(v, v) + k(u, v)q(v, v) + l(u, v)q(u, u) \\
&= [k(u, v) + l(u, v)][q(u, u) + q(v, v)] \\
&= [(k + l)(u, v)][q(u, u) + q(v, v)].
\end{aligned} \tag{3.11}$$

Since $(k + l)(u, v) < 1$, Lemma 2.3 (1) shows that $q(u, u) + q(v, v) = \theta$. But $q(u, u) \geq \theta$ and $q(v, v) \geq \theta$. Consequently, $q(u, u) = \theta$ and $q(v, v) = \theta$.

Finally, suppose there is another coupled point of coincidence (u_1, v_1) of F and g such that $u_1 = gx' = F(x', y')$ and $v_1 = gy' = F(y', x')$ for some (x', y') in $X \times X$. Then we have

$$\begin{aligned}
q(u, u_1) &= q(gx^*, gx') \\
&= q(F(x^*, y^*), F(x', y')) \\
&\leq k(gx^*, gy^*)q(gx^*, gx') + l(gx^*, gy^*)q(gy^*, gy') \\
&= k(v, u)q(v, v_1) + l(v, u)q(u, u_1) \\
&= k(u, v)q(u, u_1) + l(u, v)q(v, v_1),
\end{aligned} \tag{3.12}$$

and also,

$$\begin{aligned}
q(v, v_1) &= q(gy^*, gy') \\
&= q(F(y^*, x^*), F(y', x')) \\
&\leq k(gy^*, gx^*)q(gy^*, gy') + l(gy^*, gx^*)q(gx^*, gx') \\
&= k(v, u)q(v, v_1) + l(v, u)q(u, u_1) \\
&= k(u, v)q(v, v_1) + l(u, v)q(u, u_1).
\end{aligned} \tag{3.13}$$

This implies that

$$\begin{aligned}
 q(u, u_1) + q(v, v_1) &\leq k(u, v)q(u, u_1) + l(u, v)q(v, v_1) + k(u, v)q(v, v_1) + l(u, v)q(u, u_1) \\
 &= [k(u, v) + l(u, v)][q(u, u_1) + q(v, v_1)] \\
 &= [(k + l)(u, v)][q(u, u_1) + q(v, v_1)].
 \end{aligned} \tag{3.14}$$

Since $(k + l)(u, v) < 1$, Lemma 2.3 (1) shows that $q(u, u_1) + q(v, v_1) = \theta$. But $q(u, u_1) \geq \theta$ and $q(v, v_1) \geq \theta$. Hence $q(u, u_1) = \theta$ and $q(v, v_1) = \theta$. Also we have $q(u, u) = \theta$ and $q(v, v) = \theta$. Thus, Lemma 2.6 (1) shows that $u = u_1$ and $v = v_1$, which implies that $(u, v) = (u_1, v_1)$. Similarly, we can prove that $u = v_1$ and $v = u_1$. Thus, $u = v$. Therefore, (u, u) is the unique coupled point of coincidence. Now, let $u = gx^* = F(x^*, y^*)$. Since F and g are w -compatible, then we have

$$gu = g(gx^*) = gF(x^*, y^*) = F(gx^*, gy^*) = F(gx^*, gx^*) = F(u, u). \tag{3.15}$$

Thus (gu, gu) is a coupled point of coincidence. The uniqueness of the coupled point of coincidence implies that $gu = u$. Therefore, $u = gu = F(u, u)$. Hence (u, u) is the unique common coupled fixed point of F and g . \square

The following corollaries can be obtained as consequences of this theorem.

Corollary 3.2. *Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Suppose the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ satisfy the following contractive condition:*

$$q(F(x, y), F(u, v)) \leq kq(gx, gu) + lq(gy, gv), \tag{3.16}$$

for all $x, y, u, v \in X$, where k, l are nonnegative constants with $k + l < 1$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a coupled coincidence point in X . Further, if $gx_1 = F(x_1, y_1)$ and $gy_1 = F(y_1, x_1)$, then $q(gx_1, gx_1) = \theta$ and $q(gy_1, gy_1) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Corollary 3.3. *Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Suppose the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ satisfy the following contractive condition:*

$$q(F(x, y), F(u, v)) \leq k[q(gx, gu) + q(gy, gv)], \tag{3.17}$$

for all $x, y, u, v \in X$, where $k \in [0, 1/2)$ is a constants. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Theorem 3.4. Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings and suppose that there exists mappings $k, l : X \times X \rightarrow [0, 1)$ such that the following hold:

- (a) $k(F(x, y), F(u, v)) \leq k(gx, gy)$ and $l(F(x, y), F(u, v)) \leq l(gx, gy)$ for all $x, y, u, v \in X$,
- (b) $(k + l)(x, y) < 1$ for all $x, y \in X$,
- (c) $q(F(x, y), F(u, v)) \leq k(gx, gy)q(gx, F(x, y)) + l(gx, gy)q(gu, F(u, v))$ for all $x, y, u, v \in X$.

If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Proof. Choose $x_0, y_0 \in X$. Set $gx_1 = F(x_0, y_0)$, $gy_1 = F(y_0, x_0)$. This can be done because $F(X \times X) \subseteq g(X)$. Continuing this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that $gx_{n+1} = F(x_n, y_n)$, $gy_{n+1} = F(y_n, x_n)$.

Then we have

$$\begin{aligned}
 q(gx_n, gx_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
 &\leq k(gx_{n-1}, gy_{n-1})q(gx_{n-1}, F(x_{n-1}, y_{n-1})) \\
 &\quad + l(gx_{n-1}, gy_{n-1})q(gx_n, F(x_n, y_n)) \\
 &= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_{n-1}, gx_n) \\
 &\quad + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_n, gx_{n+1}) \\
 &\leq k(gx_{n-2}, gy_{n-2})q(gx_{n-1}, gx_n) \\
 &\quad + l(gx_{n-2}, gy_{n-2})q(gx_n, gx_{n+1}) \\
 &\vdots \\
 &\leq k(gx_1, gy_1)q(gx_{n-1}, gx_n) + l(gx_1, gy_1)q(gx_n, gx_{n+1}).
 \end{aligned} \tag{3.18}$$

Hence

$$\begin{aligned}
 q(gx_n, gx_{n+1}) &\leq \frac{k(gx_1, gy_1)}{1 - l(gx_1, gy_1)} q(gx_{n-1}, gx_n) \\
 &= hq(gx_{n-1}, gx_n) \\
 &\leq h^2 q(gx_{n-2}, gx_{n-1}) \\
 &\vdots \\
 &\leq h^{n-1} q(gx_1, gx_2),
 \end{aligned} \tag{3.19}$$

where $h = k(gx_1, gy_1) / (1 - l(gx_1, gy_1)) < 1$. It follows that

$$\begin{aligned}
 q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) &= q(gx_n, gx_{n+1}) \\
 &\leq hq(gx_{n-1}, gx_n).
 \end{aligned} \tag{3.20}$$

Similarly, we have

$$\begin{aligned}
 q(gy_n, gy_{n+1}) &\leq \frac{k(gy_1, gx_1)}{1 - l(gy_1, gx_1)} q(gy_{n-1}, gy_n) \\
 &= dq(gy_{n-1}, gy_n) \\
 &\leq d^2 q(gy_{n-2}, gy_{n-1}) \\
 &\vdots \\
 &\leq d^{n-1} q(gy_1, gy_2),
 \end{aligned} \tag{3.21}$$

where $d = k(gy_2, gx_2) / (1 - l(gy_0, gx_0)) < 1$. It follows that

$$\begin{aligned}
 q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) &= q(gy_n, gy_{n+1}) \\
 &\leq dq(gy_{n-1}, gy_n).
 \end{aligned} \tag{3.22}$$

Let $m > n \geq 1$. Then, it follows that

$$\begin{aligned}
 q(gx_n, gx_m) &\leq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\
 &\leq (h^{n-1} + h^n + \cdots + h^{m-2})q(gx_1, gx_2) \\
 &\leq \frac{h^{n-1}}{1-h}q(gx_1, gx_2),
 \end{aligned} \tag{3.23}$$

and also,

$$\begin{aligned}
 q(gy_n, gy_m) &\leq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \cdots + q(gy_{m-1}, gy_m) \\
 &\leq (d^{n-1} + d^n + \cdots + d^{m-2})q(gy_1, gy_2) \\
 &\leq \frac{d^n}{1-d}q(gy_1, gy_2).
 \end{aligned} \tag{3.24}$$

Thus, Lemma 2.6 (3) shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exists x^* and y^* in X such that $gx_n \rightarrow gx^*$ and $gy_n \rightarrow gy^*$ as $n \rightarrow \infty$. Using (q3), we have

$$q(gx_n, gx^*) \leq \frac{h^{n-1}}{1-h}q(gx_1, gx_2), \tag{3.25}$$

$$q(gy_n, gy^*) \leq \frac{d^{n-1}}{1-d}q(gy_1, gy_2). \tag{3.26}$$

On the other hand and by using (3.20), we have

$$\begin{aligned}
 q(gx_n, F(x^*, y^*)) &= q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \\
 &\leq hq(gx_{n-1}, gx^*) \\
 &\leq h \frac{h^{n-2}}{1-h}q(gx_1, gx_2) \\
 &= \frac{h^{n-1}}{1-h}q(gx_1, gx_2),
 \end{aligned} \tag{3.27}$$

also by using (3.22), we have

$$\begin{aligned}
 q(gy_n, F(y^*, x^*)) &= q(F(y_{n-1}, x_{n-1}), F(y^*, x^*)) \\
 &\leq dq(gy_{n-1}, gy^*) \\
 &\leq d \frac{d^{n-2}}{1-d} q(gy_1, gy_2) \\
 &= \frac{d^{n-1}}{1-d} q(gy_1, gy_2).
 \end{aligned} \tag{3.28}$$

Thus, Lemma 2.6 (1), (3.25), and (3.27) show that $gx^* = F(x^*, y^*)$. Again, Lemma 2.6 (1), (3.26), and (3.28) show that $gy^* = F(y^*, x^*)$. Therefore, (x^*, y^*) is a coupled coincidence point of F and g .

Suppose that $u = gx^* = F(x^*, y^*)$ and $v = gy^* = F(y^*, x^*)$. Then we have

$$\begin{aligned}
 q(u, u) &= q(gx^*, gx^*) \\
 &= q(F(x^*, y^*), F(x^*, y^*)) \\
 &\leq k(gx^*, gy^*)q(gx^*, F(x^*, y^*)) + l(gx^*, gy^*)q(gx^*, F(x^*, y^*)) \\
 &= k(u, v)q(u, u) + l(u, v)q(u, u) \\
 &= [k(u, v) + l(u, v)]q(u, u) \\
 &= [(k + l)(u, v)]q(u, u).
 \end{aligned} \tag{3.29}$$

Since $(k + l)(u, v) < 1$, Lemma 2.3 (1) shows that $q(u, u) = \theta$. By similar way, we have $q(v, v) = \theta$.

Finally, suppose there is another coupled point of coincidence (u_1, v_1) of F and g such that $u_1 = gx' = F(x', y')$ and $v_1 = gy' = F(y', x')$ for some (x', y') in $X \times X$. Then we have

$$\begin{aligned}
 q(u, u_1) &= q(gx^*, gx') \\
 &= q(F(x^*, y^*), F(x', y')) \\
 &\leq k(gx^*, gy^*)q(gx^*, F(x^*, y^*)) + l(gx^*, gy^*)q(gx', F(x', y')) \\
 &= k(u, v)q(u, u) + l(u, v)q(u_1, u_1) \\
 &= \theta,
 \end{aligned} \tag{3.30}$$

and also,

$$\begin{aligned}
 q(v, v_1) &= q(gy^*, gy') \\
 &= q(F(y^*, x^*), F(y', x')) \\
 &\leq k(gy^*, gx^*)q(gy^*, F(y^*, x^*)) + l(gy^*, gx^*)q(gy', F(y', x')) \quad (3.31) \\
 &= k(v, u)q(v, v) + l(v, u)q(v_1, v_1) \\
 &= \theta.
 \end{aligned}$$

Also, we have $q(u, u) = \theta$ and $q(v, v) = \theta$. Thus, Lemma 2.6 (1) shows that $u = u_1$ and $v = v_1$, which implies that $(u, v) = (u_1, v_1)$. Similarly, we can prove that $u = v_1$ and $v = u_1$. Thus, $u = v$. Therefore, (u, u) is the unique coupled point of coincidence. Now, let $u = gx^* = F(x^*, y^*)$. Since F and g are w -compatible, then we have

$$gu = g(gx^*) = gF(x^*, y^*) = F(gx^*, gy^*) = F(gx^*, gx^*) = F(u, u). \quad (3.32)$$

Then, (gu, gu) is a coupled point of coincidence. The uniqueness of the coupled point of coincidence implies that $gu = u$. Therefore, $u = gu = F(u, u)$. Hence, (u, u) is the unique common coupled fixed point of F and g . \square

The following corollaries can be obtained as consequences of Theorem 3.4.

Corollary 3.5. *Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Suppose the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ satisfy the following contractive condition:*

$$q(F(x, y), F(u, v)) \leq kq(gx, F(x, y)) + lq(gu, F(u, v)), \quad (3.33)$$

for all $x, y, u, v \in X$, where k, l are nonnegative constants with $k + l < 1$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$, and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Corollary 3.6. *Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Suppose the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ satisfy the following contractive condition:*

$$q(F(x, y), F(u, v)) \leq k[q(gx, F(x, y)) + q(gu, F(u, v))], \quad (3.34)$$

for all $x, y, u, v \in X$, where $k \in [0, 1/2)$ is constants. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Finally, we provide another result with another contractive type.

Theorem 3.7. Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings and suppose that there exists mappings $k, l, r : X \times X \rightarrow [0, 1)$ such that the following hold:

- (a) $k(F(x, y), F(u, v)) \leq k(gx, gy), l(F(x, y), F(u, v)) \leq l(gx, gy)$ and $r(F(x, y), F(u, v)) \leq r(gx, gy)$ for all $x, y, u, v \in X$,
- (b) $(k + 2l + r)(x, y) < 1$ for all $x, y \in X$,
- (c) $(1 - r(gx, gy))q(F(x, y), F(u, v)) \leq k(gx, gy)q(gx, F(x, y)) + l(gx, gy)q(gx, F(u, v))$ for all $x, y, u, v \in X$.

If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Proof. Choose $x_0, y_0 \in X$. Set $gx_1 = F(x_0, y_0), gy_1 = F(y_0, x_0)$ this can be done because $F(X \times X) \subseteq g(X)$. Continuing this process, we obtain to sequences $\{x_n\}$ and $\{y_n\}$ such that $gx_{n+1} = F(x_n, y_n), gy_{n+1} = F(y_n, x_n)$. Observe that

$$(1 - r(gx, gy))q(F(x, y), F(u, v)) \leq k(gx, gy)q(gx, F(x, y)) + l(gx, gy)q(gx, F(u, v)), \quad (3.35)$$

equivalently

$$q(F(x, y), F(u, v)) \leq k(gx, gy)q(gx, F(x, y)) + l(gx, gy)q(gx, F(u, v)) + r(gx, gy)q(F(x, y), F(u, v)). \quad (3.36)$$

Then, we have

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq k(gx_{n-1}, gy_{n-1})q(gx_{n-1}, F(x_{n-1}, y_{n-1})) \\ &\quad + l(gx_{n-1}, gy_{n-1})q(gx_{n-1}, F(x_n, y_n)) \\ &\quad + r(gx_{n-1}, gy_{n-1})q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \end{aligned}$$

$$\begin{aligned}
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_{n-1}, gx_n) \\
&\quad + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_{n-1}, gx_{n+1}) \\
&\quad + r(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_n, gx_{n+1}) \\
&\leq k(gx_{n-2}, gy_{n-2})q(gx_{n-1}, gx_n) \\
&\quad + l(gx_{n-2}, gy_{n-2})q(gx_{n-1}, gx_{n+1}) \\
&\quad + r(gx_{n-2}, gy_{n-2})q(gx_n, gx_{n+1}) \\
&\quad \vdots \\
&\leq k(gx_1, gy_1)q(gx_{n-1}, gx_n) + l(gx_1, gy_1)q(gx_{n-1}, gx_{n+1}) \\
&\quad + r(gx_1, gy_1)q(gx_n, gx_{n+1}) \\
&\leq k(gx_1, gy_1)q(x_{n-1}, x_n) + l(gx_1, gy_1)q(x_{n-1}, x_n) \\
&\quad + l(gx_1, gy_1)q(gx_n, gx_{n+1}) + r(gx_1, gy_1)q(gx_n, gx_{n+1}).
\end{aligned} \tag{3.37}$$

Hence

$$\begin{aligned}
q(gx_n, gx_{n+1}) &\leq \frac{k(gx_1, gy_1) + l(gx_1, gy_1)}{1 - l(gx_1, gy_1) - r(gx_1, gy_1)} q(gx_{n-1}, gx_n) \\
&= hq(gx_{n-1}, gx_n) \\
&\leq h^2 q(gx_{n-2}, gx_{n-1}) \\
&\quad \vdots \\
&\leq h^{n-1} q(gx_1, gx_2),
\end{aligned} \tag{3.38}$$

where $h = (k(gx_1, gy_1) + l(gx_1, gy_1)) / (1 - l(gx_1, gy_1) - r(gx_1, gy_1)) < 1$.

Similarly, we have

$$\begin{aligned}
q(gy_n, gy_{n+1}) &\leq \frac{k(gy_1, gx_1) + l(gy_1, gx_1)}{1 - l(gy_1, gx_1) - r(gy_1, gx_1)} q(gy_{n-1}, gy_n) \\
&= dq(gy_{n-1}, gy_n)
\end{aligned}$$

$$\begin{aligned}
&\leq d^2 q(gy_{n-2}, gy_{n-1}) \\
&\vdots \\
&\leq d^{n-1} q(gy_1, gy_2),
\end{aligned} \tag{3.39}$$

where $d = (k(gy_1, gx_1) + l(gy_1, gx_1)) / (1 - l(gy_1, gx_1) - r(gy_1, gx_1)) < 1$.

Let $m > n \geq 1$. Then, it follows that

$$\begin{aligned}
q(gx_n, gx_m) &\leq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \cdots + q(gx_{m-1}, gx_m) \\
&\leq (h^{n-1} + h^n + \cdots + h^{m-2}) q(gx_1, gx_2) \\
&\leq \frac{h^n}{1-h} q(gx_1, gx_2), \\
q(gy_n, gy_m) &\leq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \cdots + q(gy_{m-1}, gy_m) \\
&\leq (d^{n-1} + d^n + \cdots + d^{m-2}) q(gy_1, gy_2) \\
&\leq \frac{d^n}{1-d} q(gy_1, gy_2).
\end{aligned} \tag{3.40}$$

Thus, Lemma 2.6 (3) shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exists $x^*, y^* \in X$ such that $gx_n \rightarrow gx^*$ and $gy_n \rightarrow gy^*$ as $n \rightarrow \infty$. Using (q3), we have

$$q(gx_n, gx^*) \leq \frac{h^n}{1-h} q(gx_1, gx_2), \tag{3.41}$$

$$q(gy_n, gy^*) \leq \frac{d^n}{1-d} q(gy_1, gy_2). \tag{3.42}$$

On the other hand,

$$\begin{aligned}
q(gx_n, F(x^*, y^*)) &= q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \\
&\leq k(gx_{n-1}, gy_{n-1}) q(gx_{n-1}, F(x_{n-1}, y_{n-1})) \\
&\quad + l(gx_{n-1}, gy_{n-1}) q(gx_{n-1}, F(x^*, y^*)) \\
&\quad + r(gx_{n-1}, gy_{n-1}) q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \\
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(gx_{n-1}, x_n) \\
&\quad + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(gx_{n-1}, F(x^*, y^*))
\end{aligned}$$

$$\begin{aligned}
& + r(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2}))q(gx_n, F(x^*, y^*)) \\
& \leq k(gx_{n-2}, gy_{n-2})q(gx_{n-1}, gx_n) \\
& \quad + l(gx_{n-2}, gy_{n-2})q(gx_{n-1}, F(x^*, y^*)) \\
& \quad + r(gx_{n-2}, gy_{n-2})q(gx_n, F(x^*, y^*)) \\
& \quad \vdots \\
& \leq k(gx_1, gy_1)q(gx_{n-1}, gx_n) \\
& \quad + l(gx_1, gy_1)q(gx_{n-1}, F(x^*, y^*)) \\
& \quad + r(gx_1, gy_1)q(gx_n, F(x^*, y^*)) \\
& \leq k(gx_1, gy_1)q(gx_{n-1}, gx_n) + l(gx_1, gy_1)q(gx_{n-1}, gx_n) \\
& \quad + l(gx_1, gy_1)q(gx_n, F(x^*, y^*)) \\
& \quad + r(gx_1, gy_1)q(gx_n, F(x^*, y^*)).
\end{aligned} \tag{3.43}$$

Then,

$$\begin{aligned}
q(gx_n, F(x^*, y^*)) & \leq \frac{k(gx_1, gy_1) + l(gx_1, gy_1)}{1 - l(gx_1, gy_1) - r(gx_1, gy_1)} q(gx_{n-1}, gx_n) \\
& = hq(gx_{n-1}, gx_n) \\
& \leq hh^{n-2}q(gx_1, gx_2) \\
& = h^{n-1}q(gx_1, gx_2) \\
& \leq \frac{h^{n-1}}{1-h}q(gx_1, gx_2).
\end{aligned} \tag{3.44}$$

Thus, Lemma 2.6 (1), (3.41), and (3.44) show that $gx^* = F(x^*, y^*)$. Similarly, we can prove that $gy^* = F(y^*, x^*)$. Therefore, (x^*, y^*) is a coupled coincidence point of F and g . Hence, (gx^*, gy^*) is the coupled point of coincidence.

Suppose that $u = gx^* = F(x^*, y^*)$ and $v = gy^* = F(y^*, x^*)$. Then, we have

$$\begin{aligned}
q(u, u) & = q(gx^*, gx^*) \\
& = q(F(x^*, y^*), F(x^*, y^*)) \\
& \leq k(gx^*, gy^*)q(gx^*, F(x^*, y^*)) + l(gx^*, gy^*)q(gx^*, F(x^*, y^*)) \\
& \quad + r(gx^*, gy^*)q(F(x^*, y^*), F(x^*, y^*))
\end{aligned}$$

$$\begin{aligned}
 &= k(u, v)q(u, u) + l(u, v)q(u, u) + r(u, v)q(u, u) \\
 &= k(u, v)q(u, u) + l(u, v)q(u, u) + l(u, v)q(u, u) + r(u, v)q(u, u) \\
 &= [k(u, v) + l(u, v) + l(u, v) + r(u, v)]q(u, u) \\
 &= [(k + 2l + r)(u, v)]q(u, u).
 \end{aligned} \tag{3.45}$$

Since $(k + 2l + r)(u, v) < 1$, Lemma 2.3(1) shows that $q(u, u) = \theta$. Similarly, we have $q(v, v) = \theta$.

Finally, suppose there is another coupled point of coincidence (u_1, v_1) of F and g such that $u_1 = gx' = F(x', y')$ and $v_1 = gy' = F(y', x')$ for some (x', y') in $X \times X$. Then, we have

$$\begin{aligned}
 q(u, u_1) &= q(gx^*, gx') \\
 &= q(F(x^*, y^*), F(x', y')) \\
 &\leq k(gx^*, gy^*)q(gx^*, F(x^*, y^*)) + l(gx^*, gy^*)q(gx^*, F(x', y')) \\
 &\quad + r(gx^*, gy^*)q(F(x^*, y^*), F(x', y')) \\
 &= k(u, v)q(u, u) + l(u, v)q(u, u_1) + r(u, v)q(u, u_1) \\
 &= l(u, v)q(u, u_1) + r(u, v)q(u, u_1) \\
 &\leq k(uv)q(u, u_1) + l(u, v)q(u, u_1) + l(u, v)q(u, u_1) + r(u, v)q(u, u_1) \\
 &= [k(u, v) + l(u, v) + l(u, v) + r(u, v)]q(u, u_1) \\
 &= [(k + 2l + r)(u, v)]q(u, u_1).
 \end{aligned} \tag{3.46}$$

Since $(k + 2l + r)(u, v) < 1$, Lemma 2.3 (1) shows that $q(u, u_1) = \theta$. By similar way, $q(v, v_1) = \theta$. Also, we have $q(u, u) = \theta$ and $q(v, v) = \theta$. Thus, Lemma 2.6 (1) shows that $u = u_1$ and $v = v_1$, which implies that $(u, v) = (u_1, v_1)$. Similarly, we can prove that $u = v_1$ and $v = u_1$. Thus, $u = v$. Therefore, (u, u) is the unique coupled point of coincidence. Now, let $u = gx^* = F(x^*, y^*)$. Since F and g are w -compatible, then we have

$$gu = g(gx^*) = gF(x^*, y^*) = F(gx^*, gy^*) = F(gx^*, gx^*) = F(u, u). \tag{3.47}$$

Then, (gu, gu) is a coupled point of coincidence. The uniqueness of the coupled point of coincidence implies that $gu = u$. Therefore, $u = gu = F(u, u)$. Hence, (u, u) is the unique common coupled fixed point of F and g . \square

The following corollaries can be obtained as consequences of Theorem 3.7.

Corollary 3.8. *Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings and suppose that there exists mappings $k, l : X \times X \rightarrow [0, 1)$ such that the following hold:*

$$(a) \quad k(F(x, y), F(u, v)) \leq k(gx, gy) \text{ and } l(F(x, y), F(u, v)) \leq l(gx, gy),$$

$$(b) (k + 2l)(x, y) < 1,$$

$$(c) q(F(x, y), F(u, v)) \leq k(gx, gy)q(gx, F(x, y)) + l(gx, gy)q(gx, F(u, v)).$$

for all $x, y, u, v \in X$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Corollary 3.9. Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Suppose the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ satisfy the following contractive condition:

$$(1 - r)q(F(x, y), F(u, v)) \leq kq(gx, F(x, y)) + lq(gx, F(u, v)), \quad (3.48)$$

for all $x, y, u, v \in X$, where k, l, r are nonnegative constants with $k + 2l + r < 1$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$ then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Corollary 3.10. Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Suppose the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ satisfy the following contractive condition:

$$q(F(x, y), F(u, v)) \leq kq(gx, F(x, y)) + lq(gx, F(u, v)), \quad (3.49)$$

for all $x, y, u, v \in X$, where k, l are nonnegative constants with $k + 2l < 1$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Corollary 3.11. Let (X, d) be a cone metric space with a cone P having nonempty interior and q is a c -distance on X . Suppose the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ satisfy the following contractive condition:

$$q(F(x, y), F(u, v)) \leq k[q(gx, F(x, y)) + q(gx, F(u, v))], \quad (3.50)$$

for all $x, y, u, v \in X$, where $k \in [0, 1/3]$ is a constants. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X , then F and g have a unique coupled point of coincidence (u, v) in $X \times X$. Further, if $u = gx_1 = F(x_1, y_1)$ and $v = gy_1 = F(y_1, x_1)$, then $q(u, u) = \theta$ and $q(v, v) = \theta$. Moreover, if F and g are w -compatible, then F and g have a unique common coupled fixed point and the common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

4. Some Examples

Example 4.1. Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, 1]$ and define a mapping $d : X \times X \rightarrow E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Clearly, (X, d) is a cone metric space. Define a mapping $q : X \times X \rightarrow E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c -distance on X . Define the mappings $F : X \times X \rightarrow X$ by $F(x, y) = (xy)^2/16$ and $g : X \rightarrow X$ by $gx = x/2$ for all $x, y \in X$. Obviously that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X . Take $k(x, y) = l(x, y) = (xy + 1)/8$, where $x, y \in X$. Observe the following:

- (a) $k(F(x, y), F(u, v)) = l(F(x, y), F(u, v)) = (((xy)^2(uv)^2/16^2) + 1)/8 = (1/8)((xy)^2(uv)^2/16^2) + 1 \leq (1/8)((xy)^2/16 + 1) \leq (1/8)(xy/16 + 1) \leq (1/8)(xy/4 + 1) = k(gx, gy)$ for all $x, y, u, v \in X$.
- (b) $k(x, y) = k(y, x)$ and $l(x, y) = l(y, x)$ for all $x, y \in X$.
- (c) $k(x, y) + l(x, y) = (xy + 1)/4 < 1$ for all $x, y \in X$.
- (d) For all $x, y, u, v \in X$, we have

$$\begin{aligned}
 q(F(x, y), F(u, v)) &= F(u, v) \\
 &= \frac{(uv)^2}{16} \\
 &\leq \frac{uv}{16} \\
 &\leq \frac{u+v}{16} \\
 &= \frac{u}{16} + \frac{v}{16} \\
 &= \frac{1}{8} \frac{u}{2} + \frac{1}{8} \frac{v}{2} \\
 &\leq \left(\frac{xy}{32} + \frac{1}{8} \right) \frac{u}{2} + \left(\frac{xy}{32} + \frac{1}{8} \right) \frac{v}{2} \\
 &= \left(\frac{xy/4}{8} + \frac{1}{8} \right) \frac{u}{2} + \left(\frac{xy/4}{8} + \frac{1}{8} \right) \frac{v}{2} \\
 &= \frac{(xy/4) + 1}{8} \frac{u}{2} + \frac{(xy/4) + 1}{8} \frac{v}{2} \\
 &= k(gx, gy)q(gx, gu) + l(gx, gy)q(gy, gv).
 \end{aligned} \tag{4.1}$$

Clearly that F and g are w -compatible. Therefore, F and g satisfy all the conditions of Theorem 3.1. Hence, F and g have a unique common coupled fixed point $(u, u) = (0, 0)$ and $g(0) = F(0, 0) = 0$ with $q(0, 0) = 0$.

Example 4.2. Consider Example 2.5. Define the mappings $F : X \times X \rightarrow X$ by $F(x, y) = (x + y)/4$ and $g : X \rightarrow X$ by $gx = x$ for all $x, y \in X$. Obviously that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X . Then, we have $q(F(x, y), F(u, v)) = F(u, v) = (u + v)/4 \leq (3/7)(u + v) = k[q(gx, gu) + q(gy, gv)]$ with $k = 3/7 \in [0, 1/2)$. Clearly that F and g are

w -compatible. Therefore, F and g satisfy all the conditions of Corollary 3.3. Hence, F and g have a unique common coupled fixed point $(u, u) = (0, 0)$ and $g(0) = F(0, 0) = 0$ with $q(0, 0) = 0$.

Example 4.3. Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $X = [0, 1]$ and define a mapping $d : X \times X \rightarrow E$ by $d(x, y) = (|x - y|, |x - y|)$ for all $x, y \in X$. Then (X, d) is a complete cone metric space (see [15]). Define a mapping $q : X \times X \rightarrow E$ by $q(x, y) = (y, y)$ for all $x, y \in X$. Then q is a c -distance on X . In fact (q1)–(q3) are immediate. Let $c \in E$ with $\theta \ll c$ and put $e = c/2$. If $q(z, x) \ll e$ and $q(z, y) \ll e$, then we have

$$\begin{aligned}
 d(x, y) &= (|x - y|, |x - y|) \\
 &\leq (x + y, x + y) \\
 &= (x, x) + (y, y) \\
 &= q(z, x) + q(z, y) \\
 &\ll e + e \\
 &= c.
 \end{aligned} \tag{4.2}$$

This show, that (q4) holds. Therefore, q is a c -distance on X . Define the mappings $F : X \times X \rightarrow X$ by $F(x, y) = (x + y)/16$ and $g : X \rightarrow X$ by $gx = x/2$ for all $x, y \in X$. Obviously that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X . Then we have

$$\begin{aligned}
 q(F(x, y), F(u, v)) &= (F(u, v), F(u, v)) \\
 &= \left(\frac{u + v}{16}, \frac{u + v}{16} \right) \\
 &= \left(\frac{u}{16}, \frac{u}{16} \right) + \left(\frac{v}{16}, \frac{v}{16} \right) \\
 &= \left(\frac{u/2}{8}, \frac{u/2}{8} \right) + \left(\frac{v/2}{8}, \frac{v/2}{8} \right) \\
 &= \frac{1}{8} \left(\frac{u}{2}, \frac{u}{2} \right) + \frac{1}{8} \left(\frac{v}{2}, \frac{v}{2} \right) \\
 &\leq \frac{1}{4} \left(\frac{u}{2}, \frac{u}{2} \right) + \frac{1}{4} \left(\frac{v}{2}, \frac{v}{2} \right) \\
 &= \frac{1}{4} \left[\left(\frac{u}{2}, \frac{u}{2} \right) + \left(\frac{v}{2}, \frac{v}{2} \right) \right] \\
 &= k[q(gx, gu) + q(gy, gv)],
 \end{aligned} \tag{4.3}$$

with $k = 1/4 \in [0, 1/2)$. Clearly that F and g are w -compatible. Therefore, F and g satisfy all the conditions of Corollary 3.3. Hence, F and g have a unique common coupled fixed point $(u, u) = (0, 0)$ and $g(0) = F(0, 0) = 0$ with $q(0, 0) = 0$.

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