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Research Article

A Note on Property (gb) and Perturbations

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An operator $T \in \mathcal{B}(X)$ defined on a Banach space X satisfies property (gb) if the complement in the approximate point spectrum $\sigma_a(T)$ of the upper semi-B-Weyl spectrum $\sigma_{SBF_+^-}(T)$ coincides with the set $\Pi(T)$ of all poles of the resolvent of T. In this paper, we continue to study property (gb) and the stability of it, for a bounded linear operator T acting on a Banach space, under perturbations by nilpotent operators, by finite rank operators, and by quasinilpotent operators commuting with T. Two counterexamples show that property (gb) in general is not preserved under commuting quasi-nilpotent perturbations or commuting finite rank perturbations.

1. Introduction

Throughout this paper, let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X, and let $\mathcal{F}(X)$ denote its ideal of finite rank operators on X. For an operator $T \in \mathcal{B}(X)$, let T^* denote its dual, $\mathcal{N}(T)$ its kernel, $\alpha(T)$ its nullity, $\mathcal{R}(T)$ its range, $\beta(T)$ its defect, $\sigma(T)$ its spectrum, and $\sigma_a(T)$ its approximate point spectrum. If the range $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$), then T is said to be *upper semi-Fredholm* (resp., *lower semi-Fredholm*). If $T \in \mathcal{B}(X)$ is both upper and lower semi-Fredholm, then T is said to be *Fredholm*. If $T \in \mathcal{B}(X)$ is either upper or lower semi-Fredholm, then T is said to be *semi-Fredholm*, and its index is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. The *upper semi-Weyl operators* are defined as the class of upper semi-Fredholm operators with index less than or equal to zero, while *Weyl operators* are defined as the class of Fredholm operators of index zero. These classes of operators generate the following spectra: the *Weyl spectrum* defined by

$$\sigma_W(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\}, \tag{1.1}$$

the *upper semi-Weyl spectrum* (in the literature called also *Weyl essential approximate point spectrum*) defined by

$$\sigma_{SF_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Weyl operator}\}. \tag{1.2}$$

Recall that the *descent* and the *ascent* of $T \in \mathcal{B}(X)$ are $\operatorname{dsc}(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$ and $\operatorname{asc}(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, respectively (the infimum of an empty set is defined to be ∞). If $\operatorname{asc}(T) < \infty$ and $\mathcal{R}(T^{\operatorname{asc}(T)+1})$ is closed, then T is said to be *left Drazin invertible*. If $\operatorname{dsc}(T) < \infty$ and $\mathcal{R}(T^{\operatorname{dsc}(T)})$ is closed, then T is said to be *right Drazin invertible*. If $\operatorname{asc}(T) = \operatorname{dsc}(T) < \infty$, then T is said to be *Drazin invertible*. Clearly, $T \in \mathcal{B}(X)$ is both left and right Drazin invertible if and only if T is Drazin invertible. An operator $T \in \mathcal{B}(X)$ is called *upper semi-Browder* if it is an upper semi-Fredholm operator with finite ascent, while T is called *Browder* if it is a Fredholm operator of finite ascent and descent. The *Browder spectrum* of $T \in \mathcal{B}(X)$ is defined by

$$\sigma_B(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Browder operator}\}, \tag{1.3}$$

the *upper semi-Browder spectrum* (in the literature called also *Browder essential approximate point spectrum*) is defined by

$$\sigma_{UB}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Browder operator}\}. \tag{1.4}$$

An operator $T \in \mathcal{B}(X)$ is called *Riesz* if its essential spectrum $\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\} = \{0\}$.

Suppose that $T \in \mathcal{B}(X)$ and that $R \in \mathcal{B}(X)$ is a Riesz operator commuting with T. Then it follows from [1, Proposition 5] and [2, Theorem 1] that

$$\begin{gathered}\sigma_{SF_+^-}(T+R) = \sigma_{SF_+^-}(T),\\ \sigma_W(T+R) = \sigma_W(T),\\ \sigma_{UB}(T+R) = \sigma_{UB}(T),\\ \sigma_B(T+R) = \sigma_B(T).\end{gathered} \tag{1.5}$$

For each integer n, define T_n to be the restriction of T to $\mathcal{R}(T^n)$ viewed as the map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_0 = T$). If there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n is upper semi-Fredholm, then T is called *upper semi-B-Fredholm*. It follows from [3, Proposition 2.1] that if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n is upper semi-Fredholm, then $\mathcal{R}(T^m)$ is closed, T_m is upper semi-Fredholm, and $\operatorname{ind}(T_m) = \operatorname{ind}(T_n)$ for all $m \geq n$. This enables us to define the index of an upper semi-B-Fredholm operator T as the index of the upper semi-Fredholm operator T_n, where n is an integer satisfying that $\mathcal{R}(T^n)$ is closed and T_n is upper semi-Fredholm. An operator $T \in \mathcal{B}(X)$ is called *upper semi-B-Weyl* if T is upper semi-B-Fredholm and $\operatorname{ind}(T) \leq 0$.

For $T \in \mathcal{B}(X)$, let us define the *left Drazin spectrum*, the *Drazin spectrum*, and the *upper semi-B-Weyl spectrum* of T as follows, respectively:

$$\begin{aligned}\sigma_{LD}(T) &:= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a left Drazin invertible operator}\};\\ \sigma_D(T) &:= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Drazin invertible operator}\};\\ \sigma_{SBF_+^-}(T) &:= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-B-Weyl operator}\}.\end{aligned} \tag{1.6}$$

Let $\Pi(T)$ denote the set of all poles of T. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I$ is left Drazin invertible. Let $\Pi_a(T)$ denote the set of all left poles of T. It is well know that $\Pi(T) = \sigma(T) \setminus \sigma_D(T) = \mathrm{iso}\sigma(T) \setminus \sigma_D(T)$ and $\Pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T) = \mathrm{iso}\sigma_a(T) \setminus \sigma_{LD}(T)$. Here and henceforth, for $A \subseteq \mathbb{C}$, $\mathrm{iso}A$ is the set of isolated points of A. An operator $T \in \mathcal{B}(X)$ is called *a-polaroid* if $\mathrm{iso}\sigma_a(T) = \emptyset$ or every isolated point of $\sigma_a(T)$ is a left pole of T.

Following Harte and Lee [4], we say that $T \in \mathcal{B}(X)$ satisfies Browder's theorem if $\sigma_W(T) = \sigma_B(T)$, while, according to Djordjević and Han [5], we say that T satisfies a-Browder's theorem if $\sigma_{\mathrm{SF}_+^-}(T) = \sigma_{UB}(T)$.

The following two variants of Browder's theorem have been introduced by Berkani and Zariouh [6] and Berkani and Koliha [7], respectively.

Definition 1.1. An operator $T \in \mathcal{B}(X)$ is said to possess property (gb) if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T), \tag{1.7}$$

while $T \in \mathcal{B}(X)$ is said to satisfy generalized a-Browder's theorem if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T). \tag{1.8}$$

From formulas (1.5), it follows immediately that Browder's theorem and a-Browder's theorem are preserved under commuting Riesz perturbations. It is proved in [8, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem. Hence, generalized a-Browder's theorem is stable under commuting Riesz perturbations. That is, if $T \in \mathcal{B}(X)$ satisfies generalized a-Browder's theorem and R is a Riesz operator commuting with T, then $T + R$ satisfies generalized a-Browder's theorem.

The single-valued extension property was introduced by Dunford in [9, 10] and has an important role in local spectral theory and Fredholm theory, see the recent monographs [11] by Aiena and [12] by Laursen and Neumann.

Definition 1.2. An operator $T \in \mathcal{B}(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for brevity) if for every open neighborhood U of λ_0 the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f(\lambda) \equiv 0$.

Let $S(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$. An operator $T \in \mathcal{B}(X)$ is said to have SVEP if $S(T) = \emptyset$.

In this paper, we continue the study of property (gb) which is studied in some recent papers [6, 13–15]. We show that property (gb) is satisfied by an operator T satisfying $S(T^*) \subseteq \sigma_{SBF_+^-}(T)$. We give a revised proof of [15, Theorem 3.10] to prove that property (gb) is preserved under commuting nilpotent perturbations. We show also that if $T \in \mathcal{B}(X)$ satisfies $S(T^*) \subseteq \sigma_{SBF_+^-}(T)$ and F is a finite rank operator commuting with T, then $T + F$ satisfies property (gb). We show that if $T \in \mathcal{B}(X)$ is an a-polaroid operator satisfying property (gb) and Q is a quasinilpotent operator commuting with T, then $T + Q$ satisfies property (gb). Two counterexamples are also given to show that property (gb) in general is not preserved under commuting quasinilpotent perturbations or commuting finite rank perturbations. These results improve and revise some recent results of Rashid in [15].

2. Main Results

We begin with the following lemmas.

Lemma 2.1 (See [6], Corollary 2.9). *An operator $T \in \mathcal{B}(X)$ possesses property (gb) if and only if T satisfies generalized a-Browder's theorem and $\Pi(T) = \Pi_a(T)$.*

Lemma 2.2. *If the equality $\sigma_{SBF_+^-}(T) = \sigma_D(T)$ holds for $T \in \mathcal{B}(X)$, then T possesses property (gb).*

Proof. Suppose that $\sigma_{SBF_+^-}(T) = \sigma_D(T)$. If $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, then $\lambda \in \sigma_a(T) \setminus \sigma_D(T) \subseteq \Pi(T)$. This implies that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$. Since $\Pi(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ is always true, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$, that is, T possesses property (gb). □

Lemma 2.3. *If $T \in \mathcal{B}(X)$, then $\sigma_{SBF_+^-}(T) \cup S(T^*) = \sigma_D(T)$.*

Proof. Let $\lambda \notin \sigma_{SBF_+^-}(T) \cup S(T^*)$. Then $T - \lambda$ is an upper semi-Weyl operator and T^* has SVEP at λ. Thus, $T - \lambda$ is an upper semi-B-Fredholm operator and $\operatorname{ind}(T - \lambda) \leq 0$. Hence, there exists $n \in \mathbb{N}$ such that $\mathcal{R}((T - \lambda)^n)$ is closed, $(T - \lambda)_n$ is an upper semi-Fredholm operator, and $\operatorname{ind}(T - \lambda)_n \leq 0$. By [16, Theorem 2.11], $\operatorname{dsc}(T - \lambda) < \infty$. Thus, $\operatorname{dsc}(T - \lambda)_n < \infty$, by [11, Theorem 3.4(ii)], $\operatorname{ind}(T - \lambda)_n \geq 0$. By [11, Theorem 3.4(iv)], $\operatorname{asc}(T - \lambda)_n = \operatorname{dsc}(T - \lambda)_n < \infty$. Consequently, $(T - \lambda)_n$ is a Browder operator. Thus, by [17, Theorem 2.9], we then conclude that $T - \lambda$ is Drazin invertible, that is, $\lambda \notin \sigma_D(T)$. Hence, $\sigma_D(T) \subseteq \sigma_{SBF_+^-}(T) \cup S(T^*)$. Since the reverse inclusion obviously holds, we get $\sigma_{SBF_+^-}(T) \cup S(T^*) = \sigma_D(T)$. □

Theorem 2.4. *If $T \in \mathcal{B}(X)$ satisfies $S(T^*) \subseteq \sigma_{SBF_+^-}(T)$, then T possesses property (gb). In particular, if T^* has SVEP, then T possesses property (gb).*

Proof. Suppose that $S(T^*) \subseteq \sigma_{SBF_+^-}(T)$. Then by Lemma 2.3, we get $\sigma_{SBF_+^-}(T) = \sigma_D(T)$. Consequently, by Lemma 2.2, T possesses property (gb). If T^* has SVEP, then $S(T^*) = \emptyset$; the conclusion follows immediately. □

The following example shows that the converse of Theorem 2.4 is not true.

Example 2.5. Let X be the Hilbert space $l_2(\mathbb{N})$, and let $T : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be the unilateral right shift operator defined by

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots) \quad \forall (x_n) \in l_2(\mathbb{N}). \tag{2.1}$$

Then,

$$\begin{gathered}\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \\ \sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \\ \Pi(T) = \emptyset.\end{gathered} \tag{2.2}$$

Hence $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$, that is, T possesses property (gb), but $S(T^*) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < 1\} \nsubseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \sigma_{SBF_+^-}(T)$.

The next theorem improves a recent result of Berkani and Zariouh [14, Theorem 2.5] by removing the extra assumption that T is an a-polaroid operator. It also improves [14, Theorem 2.7]. We mention that it had been established in [15, Theorem 3.10], but its proof was not so clear. Hence, we give a revised proof of it.

Theorem 2.6. *If $T \in \mathcal{B}(X)$ satisfies property (gb) and N is a nilpotent operator that commutes with T, then $T + N$ satisfies property (gb).*

Proof. Suppose that $T \in \mathcal{B}(X)$ satisfies property (gb) and N is a nilpotent operator that commutes with T. By Lemma 2.1, T satisfies generalized a-Browder's theorem and $\Pi(T) = \Pi_a(T)$. Hence, $T + N$ satisfies generalized a-Browder's theorem. By [18], $\sigma(T + N) = \sigma(T)$ and $\sigma_a(T + N) = \sigma_a(T)$. Hence, by [19, Theorem 2.2] and [20, Theorem 3.2], we have that $\Pi(T + N) = \sigma(T + N) \setminus \sigma_D(T + N) = \sigma(T) \setminus \sigma_D(T) = \Pi(T) = \Pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T) = \sigma_a(T + N) \setminus \sigma_{LD}(T + N) = \Pi_a(T + N)$. By Lemma 2.1 again, $T + N$ satisfies property (gb). □

The following example, which is a revised version of [15, Example 3.11], shows that the hypothesis of commutativity in Theorem 2.6 is crucial.

Example 2.7. Let $T : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be the unilateral right shift operator defined by

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots) \quad \forall (x_n) \in l_2(\mathbb{N}). \tag{2.3}$$

Let $N : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be a nilpotent operator with rank one defined by

$$N(x_1, x_2, \ldots) = (0, -x_1, 0, \ldots) \quad \forall (x_n) \in l_2(\mathbb{N}). \tag{2.4}$$

Then $TN \neq NT$. Moreover,

$$\begin{gathered}\sigma(T) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\}, \\ \sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \\ \sigma(T + N) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\}, \\ \sigma_a(T + N) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}.\end{gathered} \tag{2.5}$$

It follows that $\Pi_a(T) = \Pi(T) = \emptyset$ and $\{0\} = \Pi_a(T + N) \neq \Pi(T + N) = \emptyset$. Hence, by Lemma 2.1, $T + N$ does not satisfy property (gb). But since T has SVEP, T satisfies a-Browder's theorem or equivalently, by [8, Theorem 2.2], T satisfies generalized a-Browder's theorem. Therefore, by Lemma 2.1 again, T satisfies property (gb).

To continue the discussion of this paper, we recall some classical definitions. Using the isomorphism $X/\mathcal{N}(T^d) \approx \mathcal{R}(T^d)$ and following [21], a topology on $\mathcal{R}(T^d)$ is defined as follows.

Definition 2.8. Let $T \in \mathcal{B}(X)$. For every $d \in \mathbb{N}$, the operator range topological on $\mathcal{R}(T^d)$ is defined by the norm $\|\cdot\|_{\mathcal{R}(T^d)}$ such that for all $y \in \mathcal{R}(T^d)$,

$$\|y\|_{\mathcal{R}(T^d)} := \inf\left\{\|x\| : x \in X, y = T^d x\right\}. \tag{2.6}$$

For a detailed discussion of operator ranges and their topologies, we refer the reader to [22, 23].

Definition 2.9. Let $T \in \mathcal{B}(X)$ and let $d \in \mathbb{N}$. Then T has *uniform descent* for $n \geq d$ if $k_n(T) = 0$ for all $n \geq d$. If in addition $\mathcal{R}(T^n)$ is closed in the operator range topology of $\mathcal{R}(T^d)$ for all $n \geq d$, then we say that T has *eventual topological uniform descent*, and, more precisely, that T has *topological uniform descent for* $n \geq d$.

Operators with eventual topological uniform descent are introduced by Grabiner in [21]. It includes many classes of operators introduced in the introduction of this paper, such as upper semi-B-Fredholm operators, left Drazin invertible operators, and Drazin invertible operators. It also includes many other classes of operators such as operators of Kato type, quasi-Fredholm operators, operators with finite descent, and operators with finite essential descent. A very detailed and far-reaching account of these notations can be seen in [11, 18, 24]. Especially, operators which have topological uniform descent for $n \geq 0$ are precisely the *semi-regular* operators studied by Mbekhta in [25]. Discussions of operators with eventual topological uniform descent may be found in [21, 26–29].

Lemma 2.10. *If $T \in \mathcal{B}(X)$ and F is a finite rank operator commuting with T, then*

(1) $\sigma_{SBF_+^-}(T+F) = \sigma_{SBF_+^-}(T)$,

(2) $\sigma_D(T+F) = \sigma_D(T)$.

Proof. (1) Without loss of generality, we need only to show that $0 \notin \sigma_{SBF_+^-}(T+F)$ if and only if $0 \notin \sigma_{SBF_+^-}(T)$. By symmetry, it suffices to prove that $0 \notin \sigma_{SBF_+^-}(T+F)$ if $0 \notin \sigma_{SBF_+^-}(T)$.

Suppose that $0 \notin \sigma_{SBF_+^-}(T)$. Then T is an upper semi-B-Fredholm operator and $\operatorname{ind}(T) \leq 0$. Hence, it follows from [24, Theorem 3.6] and [20, Theorem 3.2] that $T+F$ is also an upper semi-B-Fredholm operator. Thus, by [21, Theorem 5.8], $\operatorname{ind}(T+F) = \operatorname{ind}(T) \leq 0$. Consequently, $T+F$ is an upper semi-B-Weyl operator, that is, $0 \notin \sigma_{SBF_+^-}(T)$, and this completes the proof of (1).

(2) Noting that an operator is Drazin invertible if and only if it is of finite ascent and finite descent, the conclusion follows from [19, Theorem 2.2]. $\square$

Theorem 2.11. *If $T \in \mathcal{B}(X)$ satisfies $S(T^*) \subseteq \sigma_{SBF_+^-}(T)$ and F is a finite rank operator commuting with T, then $T+F$ satisfies property (gb).*

Proof. Since F is a finite rank operator commuting with T, by Lemma 2.10, $\sigma_{SBF_+^-}(T+F) = \sigma_{SBF_+^-}(T)$ and $\sigma_D(T+F) = \sigma_D(T)$. Since $S(T^*) \subseteq \sigma_{SBF_+^-}(T)$, by Lemma 2.3, $\sigma_{SBF_+^-}(T) = \sigma_D(T)$. Thus, $\sigma_{SBF_+^-}(T+F) = \sigma_D(T+F)$. By Lemma 2.2, $T+F$ satisfies property (gb). □

The following example illustrates that property (gb) in general is not preserved under commuting finite rank perturbations.

Example 2.12. Let $U : l_2(\mathbb{N}) \longrightarrow l_2(\mathbb{N})$ be the unilateral right shift operator defined by

$$U(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots) \quad \forall (x_n) \in l_2(\mathbb{N}). \tag{2.7}$$

For fixed $0 < \varepsilon < 1$, let $F_\varepsilon : l_2(\mathbb{N}) \longrightarrow l_2(\mathbb{N})$ be a finite rank operator defined by

$$F_\varepsilon(x_1, x_2, \ldots) = (-\varepsilon x_1, 0, 0, \ldots) \quad \forall (x_n) \in l_2(\mathbb{N}). \tag{2.8}$$

We consider the operators T and F defined by $T = U \oplus I$ and $F = 0 \oplus F_\varepsilon$, respectively. Then F is a finite rank operator and $TF = FT$. Moreover,

$$\begin{gathered}
\sigma(T) = \sigma(U) \cup \sigma(I) = \{\lambda \in \mathbb{C} : 0 \le |\lambda| \le 1\}, \\
\sigma_a(T) = \sigma_a(U) \cup \sigma_a(I) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \\
\sigma(T+F) = \sigma(U) \cup \sigma(I + F_\varepsilon) = \{\lambda \in \mathbb{C} : 0 \le |\lambda| \le 1\}, \\
\sigma_a(T+F) = \sigma_a(U) \cup \sigma_a(I + F_\varepsilon) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{1 - \varepsilon\}.
\end{gathered} \tag{2.9}$$

It follows that $\Pi_a(T) = \Pi(T) = \emptyset$ and $\{1-\varepsilon\} = \Pi_a(T+F) \neq \Pi(T+F) = \emptyset$. Hence, by Lemma 2.1, $T+F$ does not satisfy property (gb). But since T has SVEP, T satisfies a-Browder's theorem or equivalently, by [8, Theorem 2.2], T satisfies generalized a-Browder's theorem. Therefore by Lemma 2.1 again, T satisfies property (gb).

Rashid gives in [15, Theorem 3.15] that if $T \in \mathcal{B}(X)$ and Q is a quasinilpotent operator that commute with T, then

$$\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T). \tag{2.10}$$

The next example shows that this equality does not hold in general.

Example 2.13. Let Q denote the Volterra operator on the Banach space $C[0,1]$ defined by

$$(Qf)(t) = \int_0^t f(s)\mathrm{d}s \quad \forall f \in C[0,1] \quad t \in [0,1]. \tag{2.11}$$

Q is injective and quasinilpotent. Hence, it is easy to see that $\mathcal{R}(Q^n)$ is not closed for every $n \in \mathbb{N}$. Let $T = 0 \in \mathcal{B}(C[0,1])$. It is easy to see that $TQ = 0 = QT$ and $0 \notin \sigma_{SBF_+^-}(0) = \sigma_{SBF_+^-}(T)$, but $0 \in \sigma_{SBF_+^-}(Q) = \sigma_{SBF_+^-}(0+Q) = \sigma_{SBF_+^-}(T+Q)$. Hence, $\sigma_{SBF_+^-}(T+Q) \neq \sigma_{SBF_+^-}(T)$.

Rashid claims in [15, Theorem 3.16] that property (gb) is stable under commuting quasinilpotent perturbations, but its proof relies on [15, Theorem 3.15] which, by Example 2.13, is not always true. The following example shows that property (gb) in general is not preserved under commuting quasinilpotent perturbations.

Example 2.14. Let $U : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be the unilateral right shift operator defined by

$$U(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots) \quad \forall (x_n) \in l_2(\mathbb{N}). \tag{2.12}$$

Let $V : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be a quasinilpotent operator defined by

$$V(x_1, x_2, \ldots) = \left(0, x_1, 0, \frac{x_3}{3}, \frac{x_4}{4}, \ldots\right) \quad \forall (x_n) \in l_2(\mathbb{N}). \tag{2.13}$$

Let $N : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ be a quasinilpotent operator defined by

$$N(x_1, x_2, \cdots) = \left(0, 0, 0, -\frac{x_3}{3}, -\frac{x_4}{4}, \ldots\right) \quad \forall (x_n) \in l_2(\mathbb{N}). \tag{2.14}$$

It is easy to verify that $VN = NV$. We consider the operators T and Q defined by $T = U \oplus V$ and $Q = 0 \oplus N$, respectively. Then Q is quasinilpotent and $TQ = QT$. Moreover,

$$\begin{aligned} \sigma(T) &= \sigma(U) \cup \sigma(V) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\}, \\ \sigma_a(T) &= \sigma_a(U) \cup \sigma_a(V) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}, \\ \sigma(T+Q) &= \sigma(U) \cup \sigma(V+N) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\}, \\ \sigma_a(T+Q) &= \sigma_a(U) \cup \sigma_a(V+N) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}. \end{aligned} \tag{2.15}$$

It follows that $\Pi_a(T) = \Pi(T) = \emptyset$ and $\{0\} = \Pi_a(T+Q) \neq \Pi(T+Q) = \emptyset$. Hence, by Lemma 2.1, $T+Q$ does not satisfy property (gb). But since T has SVEP, T satisfies a-Browder's theorem or equivalently, by [8, Theorem 2.2], T satisfies generalized a-Browder's theorem. Therefore, by Lemma 2.1 again, T satisfies property (gb).

Our last result, which also improves [14, Theorem 2.5] from a different standpoint, gives the correct version of [15, Theorem 3.16].

Theorem 2.15. *Suppose that $T \in \mathcal{B}(X)$ obeys property (gb) and that $Q \in \mathcal{B}(X)$ is a quasinilpotent operator commuting with T. If T is a-polaroid, then $T+Q$ obeys (gb).*

Proof. Since T satisfies property (gb), by Lemma 2.1, T satisfies generalized a-Browder's theorem and $\Pi(T) = \Pi_a(T)$. Hence, $T+Q$ satisfies generalized a-Browder's theorem. In order to show that $T+Q$ satisfies property (gb), by Lemma 2.1 again, it suffices to show that $\Pi(T+Q) = \Pi_a(T+Q)$. Since $\Pi(T+Q) \subseteq \Pi_a(T+Q)$ is always true, one needs only to show that $\Pi_a(T+Q) \subseteq \Pi(T+Q)$.

Let $\lambda \in \Pi_a(T+Q) = \sigma_a(T+Q) \setminus \sigma_{LD}(T+Q) = \mathrm{iso}\,\sigma_a(T+Q) \setminus \sigma_{LD}(T+Q)$. Then by [18], $\lambda \in \mathrm{iso}\,\sigma_a(T)$. Since T is a-polaroid, $\lambda \in \Pi_a(T) = \Pi(T)$. Thus by [29, Theorem 3.12], $\lambda \in \Pi(T+Q)$. Therefore, $\Pi_a(T+Q) \subseteq \Pi(T+Q)$, and this completes the proof. □

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