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Research Article

Backward Bifurcation of an Epidemic Model with Infectious Force in Infected and Immune Period and Treatment

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An epidemic model with infectious force in infected and immune period and treatment rate of infectious individuals is proposed to understand the effect of the capacity for treatment of infective on the disease spread. It is assumed that treatment rate is proportional to the number of infective below the capacity and is constant when the number of infective is greater than the capacity. It is proved that the existence and stability of equilibria for the model is not only related to the basic reproduction number but also the capacity for treatment of infective. It is found that a backward bifurcation occurs if the capacity is small. It is also found that there exist bistable endemic equilibria if the capacity is low.

1. Introduction

Recently, mathematical models describing the dynamics of human infectious diseases have played an important role in the disease control in epidemiology. Researchers have proposed many epidemic models to understand the mechanism of disease transmission. We assume that a susceptible individual first goes through a latent period after infection before becoming infectious. The resulting models are of SEI, SEIR, or SEIRS type, respectively. Zhang and Ma [1] studied the global stability of an SEI model with general contact rate. Yuan et al. [2] considered the local stability of the model having infectious force in both latent period and infected period. Li and Jin [3–5] studied the global stability of the epidemic model having infectious force in both latent period and infected period. Usually, these classical epidemic models have only one endemic equilibrium when the basic reproduction number $R_0 > 1$, and the disease-free equilibrium is always stable when $R_0 < 1$ and unstable when $R_0 > 1$. So the bifurcation leading from a disease-free equilibrium to an endemic equilibrium is forward.

But in recent years, the phenomenon of the backward bifurcations has arisen the interests in disease control (see [6–15]). In this case, the basic reproduction number cannot describe the necessary disease elimination effort any more. Thus, it is important to identify backward bifurcations and establish thresholds for the control of diseases.

In classical epidemic models, the treatment rate of the infective is assumed to be proportional to the number of the infective. Because the resources of treatment should be limited, every community should have a suitable capacity for treatment. This hypothesis is satisfactory when the number of the infective is small and the resources of treatment are enough and unsatisfactory when the number of the infective is large and the resources of treatment are limited. Thus, it is important to determine a suitable capacity for the treatment of a disease. A constant treatment rate of disease is adopted in [16]. Note that a constant treatment rate is suitable when the number of infective is large. In [17], the treatment rate of the disease is modified into

$$T(I) = \begin{cases} rI & \text{if } 0 \leqslant I \leqslant I_0, \\ k & \text{if } I > I_0, \end{cases} \tag{1.1}$$

where $k = rI_0$, r and I_0 are positive constant. This means that the treatment rate of disease is proportional to the number of the infective when the capacity of treatment is not reached and, otherwise, takes the maximal capacity. This improves the classical proportional treatment and the constant treatment in [16].

In this paper, we study the backward bifurcation and global dynamics of an epidemic model with infectious force in infected and immune period and treatment function. To formulate our model, we will consider a population that is divided into three types: susceptible, infective, and recovered. Let $S(t)$, $I(t)$, and $R(t)$ denote the numbers of susceptible, infective, recovered individuals at time t, respectively. The total population size at time t is denoted by $N(t)$.

The basic assumptions in the paper are as the follows.

(i) There is a positive constant recruitment rate of the population A.

(ii) Positive constant d is the nature death rate of population.

(iii) β_1, β_2 are the rate of the efficient contact in the infected and recovered period, respectively.

(iv) Positive constant γ is the natural recovery rate of infective individuals.

(v) Positive constant ϵ is the disease-related death rate.

(vi) The treatment of a disease is $T(I)$ in (1.1).

Under the assumptions above, an epidemic model to be studied takes the following form:

$$\begin{gathered} \frac{dS}{dt} = A - dS - \beta_1 SI - \beta_2 SR, \\ \frac{dI}{dt} = \beta_1 SI + \beta_2 SR - (d + \gamma + \epsilon)I - T(I), \\ \frac{dR}{dt} = \gamma I + T(I) - dR, \end{gathered} \tag{1.2}$$

where $S(t) + I(t) + R(t) = N(t)$. It is easy to verify that R_+^3 is positive invariant for system (1.2).

According to $S(t) + I(t) + R(t) = N(t)$ and (1.1), $N(t)$ satisfies the following equation:

$$\frac{dN}{dt} = A - dN - \epsilon I. \tag{1.3}$$

Then system (1.2) is equivalent to

$$\begin{aligned} \frac{dN}{dt} &= A - dN - \epsilon I, \\ \frac{dI}{dt} &= (\beta_1 I + \beta_2 R)(N - I - R) - (d + \gamma + \epsilon)I - T(I), \\ \frac{dR}{dt} &= \gamma I + T(I) - dR. \end{aligned} \tag{1.4}$$

It is easy to verify that all solutions of system (1.4) initiating in set $\{(N, I, R) \mid N > 0, I \geqslant 0, R \geqslant 0, I + R \leqslant N\}$ eventually enter the set $\Omega = \{(N, I, R) \mid 0 < N \leqslant A/d, I \geqslant 0, R \geqslant 0, I + R \leqslant N\}$. Therefore, Ω is positively invariant for system (1.4). We consider the solutions of system (1.4) in Ω below.

When $0 \leqslant I \leqslant I_0$, system (1.4) becomes

$$\begin{aligned} \frac{dN}{dt} &= A - dN - \epsilon I, \\ \frac{dI}{dt} &= (\beta_1 I + \beta_2 R)(N - I - R) - (d + \gamma + \epsilon + r)I, \\ \frac{dR}{dt} &= (\gamma + r)I - dR. \end{aligned} \tag{1.5}$$

When $I > I_0$, system (1.4) becomes

$$\begin{aligned} \frac{dN}{dt} &= A - dN - \epsilon I, \\ \frac{dI}{dt} &= (\beta_1 I + \beta_2 R)(N - I - R) - (d + \gamma + \epsilon)I - k, \\ \frac{dR}{dt} &= \gamma I + k - dR. \end{aligned} \tag{1.6}$$

The purpose of this paper is to show that system (1.4) has a backward bifurcation if the capacity for treatment is small. We obtain the sufficient conditions that the disease-free equilibrium and endemic equilibria of system (1.4) are stable. It is shown that (1.4) has bistable endemic equilibria if the capacity is small. The organization of this paper is as follows. In next section, we study the existence and bifurcations of equilibria for (1.4). We analyze the stability of equilibria for (1.4) and present the numerical simulations in Section 3.

2. The Existence of Equilibria

In this section, we consider the equilibria of system (1.4). Obviously, $E_0(A/d, 0, 0)$ is the disease-free equilibrium of (1.4). For the endemic equilibrium $E(N, I, R)$ of (1.4), N, I and R satisfy

$$\begin{gathered} A - dN - \epsilon I = 0, \\ (\beta_1 I + \beta_2 R)(N - I - R) - (d + \gamma + \epsilon)I - T(I) = 0, \\ \gamma I + T(I) - dR = 0. \end{gathered} \tag{2.1}$$

When $0 \leqslant I \leqslant I_0$, system (2.1) becomes

$$\begin{gathered} A - dN - \epsilon I = 0, \\ (\beta_1 I + \beta_2 R)(N - I - R) - (d + \gamma + \epsilon + r)I = 0, \\ (\gamma + r)I - dR = 0. \end{gathered} \tag{2.2}$$

When $I > I_0$, system (2.1) becomes

$$\begin{gathered} A - dN - \epsilon I = 0, \\ (\beta_1 I + \beta_2 R)(N - I - R) - (d + \gamma + \epsilon)I - k = 0, \\ \gamma I + k - dR = 0. \end{gathered} \tag{2.3}$$

Form (2.2), I satisfies the following equation:

$$\left(\beta_1 + \beta_2 \frac{\gamma + r}{d}\right) \frac{A - (d + \epsilon + \gamma + r)I}{d} = d + \epsilon + \gamma + r. \tag{2.4}$$

Therefore, we obtain

$$I = \frac{A - d(d + \epsilon + \gamma + r)/(\beta_1 + \beta_2((\gamma + r)/d))}{d + \epsilon + \gamma + r}. \tag{2.5}$$

Let

$$R_0 = \frac{A(\beta_1 + \beta_2((\gamma + r)/d))}{d(d + \epsilon + \gamma + r)}. \tag{2.6}$$

Then R_0 is a basic reproduction number of (1.4). If $R_0 > 1$, then $I > 0$; (2.2) admits a unique positive solution $E_* = (N_*, I_*, R_*)$, where

$$N_* = \frac{A - \epsilon I_*}{d},$$
$$I_* = \frac{A - d(d+\epsilon+\gamma+r)/(\beta_1 + \beta_2((\gamma+r)/d))}{d+\epsilon+\gamma+r}, \tag{2.7}$$
$$R_* = \frac{(\gamma+r)I_*}{d}.$$

Clearly, E_* is an endemic equilibrium of (1.4) if and only if

$$1 < R_0 \leqslant 1 + \frac{\beta_1 + \beta_2((\gamma+r)/d)}{d} I_0. \tag{2.8}$$

According to (2.3), I satisfies the following equation:

$$a_0 I^2 + a_1 I + a_2 = 0, \tag{2.9}$$

where $a_0 = (\beta_1 + \beta_2(\gamma/d))(d+\epsilon+\gamma)$,

$$a_1 = d(d+\epsilon+\gamma) + \beta_2 \frac{k}{d}(d+\epsilon+\gamma) - \left(\beta_1 + \beta_2 \frac{\gamma}{d}\right)(A-k),$$
$$a_2 = dk - \beta_2 \frac{k}{d}(A-k). \tag{2.10}$$

We only consider the case of $a_2 > 0$. If $a_1 \geqslant 0$, it is clear that (2.9) does not have positive real root. Let us suppose $a_1 < 0$ below. Note that $a_1 < 0$ is equivalent to

$$R_0 \geqslant 1 + \frac{\beta_2(k/d)(d+\epsilon+\gamma) + \beta_2(r/d)A + (\beta_1 + \beta_2(\gamma/d))k - dr}{d(d+\epsilon+\gamma+r)} =: p_*. \tag{2.11}$$

It is easy that

$$\begin{aligned}
\Delta = a_1^2 - 4a_0a_2 &= R_0^2 d^2 (d+\epsilon+\gamma+r)^2 \\
&\quad - 2R_0 d(d+\epsilon+\gamma+r)\left[\left(d + \beta_2\frac{k}{d}\right)(d+\epsilon+\gamma) + \beta_2 \frac{r}{d}A + \left(\beta_1 + \beta_2\frac{\gamma}{d}\right)k\right] \\
&\quad + \left[\left(d + \beta_2\frac{k}{d}\right)(d+\epsilon+\gamma) + \beta_2 \frac{r}{d}A + \left(\beta_1 + \beta_2\frac{\gamma}{d}\right)k\right]^2 \\
&\quad - 4\left(\beta_1 + \beta_2\frac{\gamma}{d}\right)(d+\epsilon+\gamma)\left(dk - \beta_2\frac{k}{d}(A-k)\right).
\end{aligned} \tag{2.12}$$

It follows that $\Delta \geqslant 0$ is equivalent to

$$R_0 \geqslant p_* + \frac{2\sqrt{(\beta_1 + \beta_2(\gamma/d))(d+\epsilon+\gamma)[dk - \beta_2(k/d)(A-k)]}}{d(d+\epsilon+\gamma+r)} =: p_0, \tag{2.13}$$

or

$$R_0 \leqslant p_* - \frac{2\sqrt{(\beta_1 + \beta_2(\gamma/d))(d+\epsilon+\gamma)[dk - \beta_2(k/d)(A-k)]}}{d(d+\epsilon+\gamma+r)}. \tag{2.14}$$

Thus $a_1 < 0$ and $\Delta \geqslant 0$ if and only if (2.13) holds. Let us suppose that (2.13) holds. Then (2.9) has two positive solutions I_1 and I_2 where

$$I_1 = \frac{-a_1 - \sqrt{\Delta}}{2a_0}, \qquad I_2 = \frac{-a_1 + \sqrt{\Delta}}{2a_0}. \tag{2.15}$$

Set $N_i = (A - \epsilon I_i)/d$, $R_i = (\gamma I_i + k)/d$ and $E_i(N_i, I_i, R_i)$ $(i = 1,2)$. If $I_i > I_0$ $(i = 1,2)$, then E_i is an endemic equilibrium of (1.6).

By the definition of I_1, we notice that $I_1 > I_0$ is equivalent to

$$-\sqrt{\Delta} > 2a_0 I_0 + a_1. \tag{2.16}$$

This implies that $2a_0 I_0 + a_1 < 0$. By immediate calculation, $2a_0 I_0 + a_1 < 0$ is equivalent to

$$R_0 > p_* + \frac{2(\beta_1 + \beta_2(\gamma/d))(d+\epsilon+\gamma)I_0}{d(d+\epsilon+\gamma+r)} =: p_1. \tag{2.17}$$

Further, $I_1 > I_0$ demands that

$$(2a_0 I_0 + a_1)^2 > \Delta. \tag{2.18}$$

By immediate calculation,

$$R_0 < 1 + \frac{\beta_1 + \beta_2((\gamma+r)/d)}{d} I_0 =: p_2. \tag{2.19}$$

Therefore, $I_1 > I_0$ holds if and only if $R_0 > p_1$ and $R_0 < p_2$.

By similar discussions as previously mentioned, we have that $I_2 > I_0$ holds if and only if either $R_0 > p_1$, or $R_0 > p_2$, $R_0 < p_1$.

Summarizing the discussions above, we have the following conclusion.

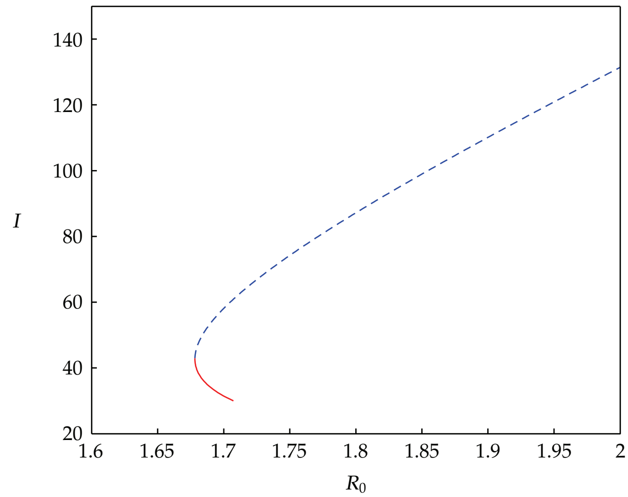

Figure 1: The figure of infective sizes at equilibria versus R_0 when $I_0 = 30$, $A = 80$, $\beta_1 = 0.01$, $\beta_2 = 0.01$, $\gamma = 0.01$, $d = 0.9$, $\epsilon = 0.01$, $r = 1$, $p_0 = 1.6782$, $p_1 = 1.541$, $p_2 = 1.7074$, where (i) of Theorem 2.2 holds. The bifurcation from the disease-free equilibrium at $R_0 = 1$ is forward, and there is a backward bifurcation from an endemic equilibrium at $R_0 = 1$, which leads to the existence of multiple endemic equilibria.

Theorem 2.1. *$E_0(A/d, 0, 0)$ is always the disease-free equilibrium of (1.5). $E_*(N_*, I_*, R_*)$ is an endemic equilibrium of system (1.4) if and only if $1 < R_0 \leqslant p_2$. Furthermore, E_* is the unique equilibrium of system (1.4) if $1 < R_0 \leqslant p_2$, and one of the following conditions is satisfied:*

(i) $R_0 < p_0$,

(ii) $p_0 \leqslant R_0 < p_1$.

By calculation, we have $p_2 - p_1 = [d - \beta_2(A-k)/d]r - (\beta_1 + \beta_2((\gamma + r)/d))I_0(d + \epsilon + \gamma)$. Note that $[d - \beta_2(A-k)/d]r > (\beta_1 + \beta_2(\gamma/d))(d + \epsilon + \gamma)I_0$ is equivalent to that $p_1 < p_2$.

Theorem 2.2. *Endemic equilibria E_1 and E_2 do not exist if $R_0 < p_0$. Further, if $R_0 \geqslant p_0$, we have the following:*

(i) *if $[d - \beta_2(A-k)/d]r > (\beta_1 + \beta_2(\gamma/d))(d + \epsilon + \gamma)I_0$, then both E_1 and E_2 exist when $p_1 < R_0 < p_2$,*

(ii) *if $[d - \beta_2(A-k)/d]r > (\beta_1 + \beta_2(\gamma/d))(d + \epsilon + \gamma)I_0$, then E_1 does not exist but E_2 exists if $R_0 \geqslant p_2$,*

(iii) *letting $[d - \beta_2(A-k)/d]r \leqslant (\beta_1 + \beta_2(\gamma/d))(d + \epsilon + \gamma)I_0$, then E_1 does not exist. Further, E_2 exists when $R_0 > p_2$, and E_2 does not exist when $R_0 \leqslant p_2$.*

We consider $p_0 > 1$. If $[d - \beta_2(A-k)/d]r > (\beta_1 + \beta_2(\gamma/d))(d + \epsilon + \gamma)I_0$, a typical bifurcation diagram is illustrated in Figure 1, where the bifurcation from the disease-free equilibrium at $R_0 = 1$ is forward and there is a backward bifurcation from an endemic equilibrium at $R_0 = 1.71$, which gives rise to the existence of multiple endemic equilibria. Further, if $[d - \beta_2(A - k)/d]r \leqslant (\beta_1 + \beta_2(\gamma/d))(d + \epsilon + \gamma)I_0$, a typical bifurcation diagram is illustrated in Figure 2, where the bifurcation at $R_0 = 1$ is forward, and (1.4) has one unique endemic equilibrium for all $R_0 > 1$.

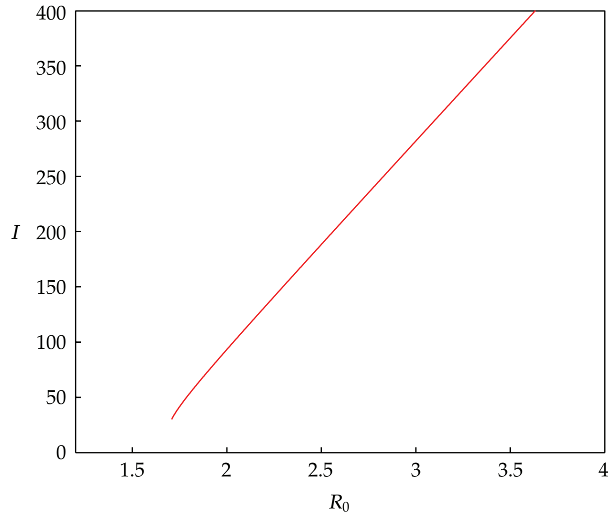

Figure 2: The diagram of I_*, I_2 versus R_0 when $I_0 = 30$, $A = 100$, $\beta_1 = 0.01$, $\beta_2 = 0.01$, $\gamma = 0.01$, $d = 0.9$, $\epsilon = 0.01$, $r = 1$, $p_0 = 1.6889$, $p_1 = 1.7882$, $p_2 = 1.7074$, where (iii) of Theorem 2.2 holds. The bifurcation at $R_0 = 1$ is forward, and (1.4) has a unique endemic equilibrium for $R_0 > 1$.

Note that a backward bifurcation with endemic equilibria when $R_0 < 1$ is very interesting in applications. We present the following corollary to give conditions for such a backward bifurcation to occur.

Corollary 2.3. *Equation* (1.4) *has a backward bifurcation with endemic equilibria when* $R_0 < 1$ *if* $[d - \beta_2(A-k)/d]r > (\beta_1 + \beta_2(\gamma/d))(d + \epsilon + \gamma)I_0$ *and* $p_0 < 1$.

Example 2.4. Fix $I_0 = 10$, $A = 60$, $\beta_1 = 0.01$, $\beta_2 = 0.005$, $\gamma = 0.1$, $d = 1$, $\epsilon = 0.1$, and $r = 3$. Then $p_1 \approx 0.6779$, $p_0 \approx 0.8878$, $p_2 \approx 1.225$ and $[d - \beta_2(A-k)/d]r - (\beta_1 + \beta_2\gamma/d)I_0(d + \epsilon + \gamma) = 2.424$. Thus, (1.4) has a backward bifurcation with endemic equilibria when $R_0 < 1$ in this case (see Figure 3).

As I_0 (the capacity of treatment resources) increases, by the definition we see that p_0 increases. When I_0 is so large that $p_0 > 1$, it follows from Theorem 2.2 that there is no backward bifurcation with endemic equilibria when $R_0 < 1$. If we increase I_0 to $R_0 < p_0$, (1.4) does not have a backward bifurcation because endemic equilibria E_1 and E_2 do not exist. This means that an insufficient capacity for treatment is a source of the backward bifurcation.

3. The Stability of Equilibria

We first determine the stability of the disease-free equilibrium $E_0(A/d, 0, 0)$. The Jacobian matrix of (1.4) at $E_0(A/d, 0, 0)$ is

$$\begin{pmatrix} -d & -\epsilon & 0 \\ 0 & \beta_1 \dfrac{A}{d} - (d + \epsilon + \gamma + r) & \beta_2 \dfrac{A}{d} \\ 0 & \gamma + r & -d \end{pmatrix}. \tag{3.1}$$

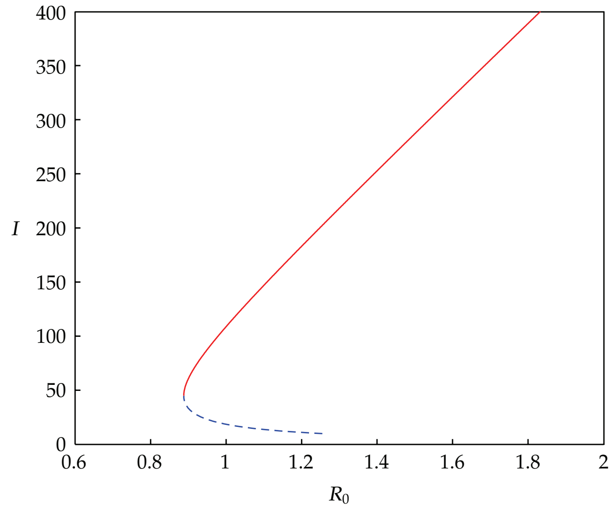

Figure 3: The figure of I_*, I_1 and I_2 versus R_0 that shows a backward bifurcation with endemic equilibrium when $R_0 < 1$, where Corollary 2.3 holds.

Its characteristic equation is

$$(\lambda + d)\left[\lambda^2 + \left(d - \beta_1 \frac{A}{d} + (d + \epsilon + \gamma + r)\right)\lambda + d(d + \epsilon + \gamma + r)\beta_1 \frac{A}{d} - \beta_2 \frac{A}{d}(\gamma + r)\right] = 0. \quad (3.2)$$

We obtain

$$\begin{gathered} \lambda_1 = -d < 0, \\ d\left[(d + \epsilon + \gamma + r) - \beta_1 \frac{A}{d}\right] - \beta_2 \frac{A}{d}(\gamma + r) = d(d + \epsilon + \gamma + r)(1 - R_0). \end{gathered} \quad (3.3)$$

Therefore, we get the following theorem.

Theorem 3.1. *The disease-free equilibrium $E_0(A/d, 0, 0)$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

Next, the stability of endemic equilibrium $E_*(N_*, I_*, R_*)$ is analyzed. The Jacobian matrix of (1.4) at $E_*(N_*, I_*, R_*)$ is

$$J_* = \begin{pmatrix} -d & -\epsilon & 0 \\ b_1 & \beta_1 b_2 - a_1 - (d + \epsilon + \gamma + r) & \beta_2 b_2 - b_1 \\ 0 & \gamma + r & -d \end{pmatrix}, \quad (3.4)$$

where $c_0 = \beta_1 I_* + \beta_2 R_*$, $b_0 = N_* - I_* - R_*$.

Making use of (2.2), the characteristic equation of J_* is simplified into

$$\begin{aligned}(\lambda + d)\Big[\lambda^2 &+ (d + c_0 + (d + \epsilon + \gamma + r) - \beta_1 b_0)\lambda \\ &+ d(c_0 + d + \epsilon + \gamma + r - \beta_1 b_0) + (c_0 - \beta_2 b_0)(\gamma + r) + c_0\epsilon\Big] = 0,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}&d + c_0 + (d + \epsilon + \gamma + r) - \beta_1 b_0 \\ &\quad = d + \left(\beta_1 + \beta_2\frac{\gamma + r}{d}\right)I_* + (d + \epsilon + \gamma + r)\left(1 - \frac{\beta_1}{\beta_1 + \beta_2((\gamma + r)/d)}\right) > 0, \\ &d(c_0 + d + \epsilon + \gamma + r - \beta_1 b_0) + (c_0 - \beta_2 b_0)(\gamma + r) + c_0\epsilon \\ &\quad = \left(\beta_1 + \beta_2\frac{\gamma + r}{d}\right)I_*(d + \epsilon + \gamma + r) > 0.\end{aligned} \tag{3.6}$$

Therefore, the real part of the all eigenvalues of J_* is negative when $1 < R_0 \leqslant p_2$.

Theorem 3.2. *If $1 < R_0 \leqslant p_2$, then the endemic equilibrium E_* of* (1.4) *is locally asymptotically stable.*

Afterwards, we study the stability of endemic equilibrium $E_1(N_1, I_1, R_1)$. The characteristic equation of Jacobian matrix of (1.4) at $E_1(N_1, I_1, R_1)$ is

$$(\lambda + d)\Big[\lambda^2 + (d + c_1 + (d + \epsilon + \gamma) - \beta_1 b_1)\lambda + d(c_1 + d + \epsilon + \gamma - \beta_1 b_1) + \gamma(c_1 - \beta_2 b_1) + c_1\epsilon\Big] = 0, \tag{3.7}$$

where $c_1 = \beta_1 I_1 + \beta_2 R_1$, $b_1 = N_1 - I_1 - R_1$. After some calculations, we obtain

$$d(c_1 + d + \epsilon + \gamma - \beta_1 b_1) + \gamma(c_1 - \beta_2 b_1) + c_1\epsilon = 2a_0 I_1 + a_1 = -\sqrt{\Delta} < 0. \tag{3.8}$$

Therefore, (3.7) has positive real part eigenvalues. Thus $E_1(N_1, I_1, R_1)$ is unstable.

Theorem 3.3. *If the endemic equilibrium $E_1(N_1, I_1, R_1)$ of system* (1.4) *exists, then it is unstable.*

Finally, we analyze the stability of endemic equilibrium $E_2(N_2, I_2, R_2)$. Its characteristic equation is

$$(\lambda + d)\Big[\lambda^2 + (d + c_2 + (d + \epsilon + \gamma) - \beta_1 b_2)\lambda + d(c_2 + d + \epsilon + \gamma - \beta_1 b_2) + \gamma(c_2 - \beta_2 b_2) + c_2\epsilon\Big] = 0, \tag{3.9}$$

where $c_2 = \beta_1 I_2 + \beta_2 R_2$, $b_2 = N_2 - I_2 - R_2$. By some calculations, we obtain

$$\begin{aligned}
&d(c_2 + d + \epsilon + \gamma - \beta_1 b_2) + \gamma(c_2 - \beta_2 b_2) + c_2\epsilon \\
&\qquad = 2a_0 I_2 + a_1 = \sqrt{\Delta} > 0, \\
&d + c_2 + (d + \epsilon + \gamma) - \beta_1 b_2 \\
&\qquad = d + \left(\beta_1 + \beta_2 \frac{\gamma}{d}\right) I_2 + (d + \epsilon + \gamma) + \frac{\beta_2 k}{d} + \frac{\beta_1 (d + \epsilon + \gamma) I_2}{d} + \frac{\beta_1 (k - A)}{d}.
\end{aligned} \tag{3.10}$$

It follows that $d + c_2 + (d + \epsilon + \gamma) - \beta_1 b_2 > 0$ is equivalent to

$$\sqrt{\Delta} > a_1 + \frac{-2a_0 \left[d + \beta_2 k/d + (d + \gamma + \epsilon) - \beta_1((A - k)/d)\right]}{\beta_1 + \beta_2(\gamma/d) + \beta_1((d + \gamma + \epsilon)/d)}. \tag{3.11}$$

If

$$a_1 \left[\beta_1 + \beta_2 \frac{\gamma}{d} + \beta_1 \frac{d + \gamma + \epsilon}{d}\right] - 2a_0 \left[d + \frac{\beta_2 k}{d} + (d + \gamma + \epsilon) - \beta_1 \frac{A - k}{d}\right] < 0, \tag{3.12}$$

then $d + c_2 + (d + \epsilon + \gamma) - \beta_1 b_2 > 0$. Thus $E_2(N_2, I_2, R_2)$ is locally asymptotically stable.

By complicated calculation, if $a_1[\beta_1 + \beta_2(\gamma/d) + \beta_1((d + \gamma + \epsilon)/d)] - 2a_0[d + \beta_2 k/d + (d + \gamma + \epsilon) - \beta_1((A - k)/d)] > 0$, then (3.11) is equivalent to

$$\begin{aligned}
a_2 \left[\beta_1 + \beta_2 \frac{\gamma}{d} + \beta_1 \frac{d + \gamma + \epsilon}{d}\right]^2 &< a_1 \left[\beta_1 + \beta_2 \frac{\gamma}{d} + \beta_1 \frac{d + \gamma + \epsilon}{d}\right] \left[d + \frac{\beta_2 k}{d} + (d + \gamma + \epsilon) - \beta_1 \frac{A - k}{d}\right] \\
&\quad - a_0 \left[d + \frac{\beta_2 k}{d} + (d + \gamma + \epsilon) - \beta_1 \frac{A - k}{d}\right]^2.
\end{aligned} \tag{3.13}$$

Theorem 3.4. *Suppose the endemic equilibrium $E_2(N_2, I_2, R_2)$ of system* (1.4) *exists; if either*

$$a_1 \left[\beta_1 + \beta_2 \frac{\gamma}{d} + \beta_1 \frac{d + \gamma + \epsilon}{d}\right] - 2a_0 \left[d + \frac{\beta_2 k}{d} + (d + \gamma + \epsilon) - \beta_1 \frac{A - k}{d}\right] < 0, \tag{3.14}$$

or

$$\begin{aligned}
&a_1 \left[\beta_1 + \beta_2 \frac{\gamma}{d} + \beta_1 \frac{d + \gamma + \epsilon}{d}\right] - 2a_0 \left[d + \frac{\beta_2 k}{d} + (d + \gamma + \epsilon) - \beta_1 \frac{A - k}{d}\right] > 0, \\
&a_2 \left[\beta_1 + \beta_2 \frac{\gamma}{d} + \beta_1 \frac{d + \gamma + \epsilon}{d}\right]^2 < a_1 \left[\beta_1 + \beta_2 \frac{\gamma}{d} + \beta_1 \frac{d + \gamma + \epsilon}{d}\right] \left[d + \frac{\beta_2 k}{d} + (d + \gamma + \epsilon) - \beta_1 \frac{A - k}{d}\right] \\
&\qquad - a_0 \left[d + \frac{\beta_2 k}{d} + (d + \gamma + \epsilon) - \beta_1 \frac{A - k}{d}\right]^2,
\end{aligned} \tag{3.15}$$

then it is locally asymptotically stable.

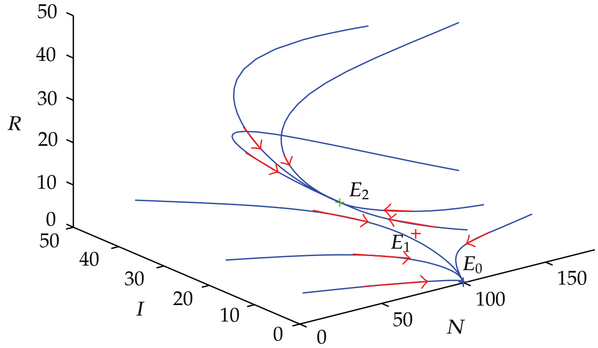

Figure 4: The phase diagram of system (1.4) when $I_0 = 5$, $A = 80$, $\beta_1 = 0.015$, $\beta_2 = 0.001$, $\gamma = 0.01$, $d = 0.8$, $\epsilon = 0.01$, $r = 1$, $p_0 = 0.8607$, $p_1 = 0.6589$, $p_2 = 1.1016$, $R_0 = 0.8935$.

Theorem 3.5. *The disease-free equilibrium E_0 of system* (1.4) *is globally asymptotically stable, if one of the following conditions is satisfied:*

(i) $R_0 < 1$ *and* $p_0 > 1$,

(ii) $R_0 < 1$, $p_0 < 1$ *and* $p_1 \geqslant 1$.

Proof. $R_0 < 1$ implies that E_* does not exist. Suppose $p_0 \geqslant 1$. It follows from the discussions for Theorem 2.2 that E_1 or E_2 exists only if $R_0 > p_0$, which is impossible since we have $R_0 < 1$. Let us now suppose $p_0 < 1$ and $p_1 \geqslant 1$. If $[d-\beta_2(A-k)/d]r > (\beta_1+\beta_2(\gamma/d)I_0(d+\epsilon+\gamma)$, since $p_1 < p_2$, it follows from the discussions for (i), (ii) of Theorem 2.2 that E_1 or E_2 exists only if $R_0 > p_1$, which is impossible since we have $R_0 < 1$. If $[d-\beta_2(A-k)/d]r > (\beta_1+\beta_2\gamma/d)I_0(d+\epsilon+\gamma)$, since $1 < p_2$, it follows from (iii) of Theorem 2.2 that E_1 and E_2 do not exist. In summary, endemic equilibria do not exist under the assumptions. □

4. The Simulation of Model

In this section, we give the numerical simulations of system (1.4) for the conclusions gained previously.

Example 4.1. For system (1.4), if $R_0 < 1$ and $R_0 > p_0$ and $p_1 < R_0 < p_2$, then the equilibrium E_* does not exist, and there are three equilibria E_0, E_1, and E_2. Its phase diagram is illustrated in Figure 4. Numerical calculations show that E_0 and E_2 are stable, but E_1 is unstable.

Example 4.2. For system (1.4), if $R_0 > 1$ and $R_0 < p_0$, there is the unique equilibrium E_* which is stable. Its phase diagram is illustrated in Figure 5. Numerical calculations show that the unique equilibrium E_* is globally stable.

Example 4.3. For system (1.4), if $R_0 > 1$ and $R_0 > p_0$ and $p_1 < R_0 < p_2$, the equilibria E_2 and E_* are stable, and E_0 and E_1 are unstable; its phase diagram is illustrated in Figure 6. Numerical calculations show that the equilibria E_2 and E_* are stable, and E_0 and E_1 unstable. Thus, we have bistable endemic equilibria.

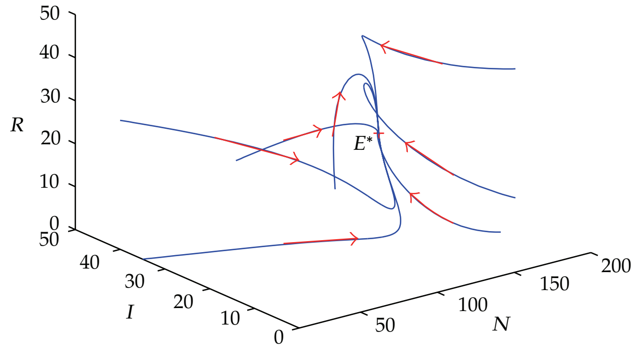

Figure 5: The phase diagram of system (1.4) when $I_0 = 40$, $A = 100$, $\beta_1 = 0.02$, $\beta_2 = 0.01$, $\gamma = 0.01$, $d = 0.9$, $\epsilon = 0.01$, $r = 1.5$, $p_0 = 2.6340$, $p_1 = 2.6606$, $p_2 = 2.6346$, $R_0 = 1.6886$.

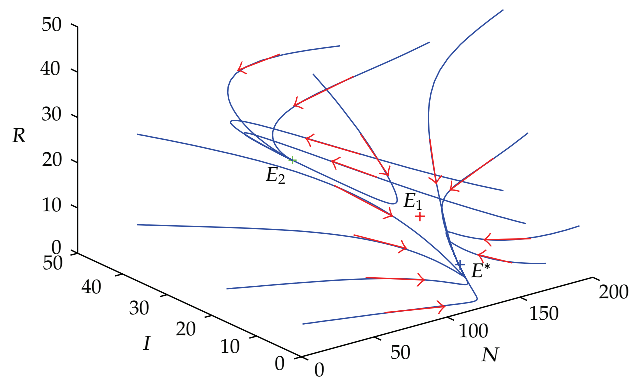

Figure 6: The phase diagram of system (1.4) when $I_0 = 10$, $A = 100$, $\beta_1 = 0.015$, $\beta_2 = 0.001$, $\gamma = 0.01$, $d = 0.8$, $\epsilon = 0.01$, $r = 1$, $p_0 = 1.0462$, $p_1 = 0.8156$, $p_2 = 1.2033$, $R_0 = 1.1169$.

5. Discussion

In this paper, we have proposed an epidemic model with infectious force in infected and immune period and treatment rate of infectious individuals to understand the effect of the capacity for treatment of infective on the disease transmission, which can occur when patients have to be hospitalized but there are limited beds or medical establishments in hospitals, or there is not enough medicine for treatment. We have shown in Theorem 2.2 and Corollary 2.3 that backward bifurcations occur because of the insufficient capacity for treatment. We have also shown that system (1.4) has bistable endemic equilibria because of the limited resources. This means that the basic reproduction number $R_0 < 1$ and small treatment rate are not enough to eradicate the disease, but the basic reproduction number $R_0 < 1$ and large treatment rate may eradicate the disease. The disease cannot be eradicated for any treatment rate if the basic reproduction number $R_0 > 1$. Therefore, the level of initial infectious invasion must be lowered to a threshold so that the disease dies out or approaches a lower endemic steady state for a range of parameters.

In Sections 2 and 3, when $I > I_0$, with respect to the existence and the local stability of the endemic equilibrium we only proved for the model (1.6) under the restriction $a_2 > 0$. But the case of $a_2 < 0$ is an unsolved question.

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