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Research Article

Neimark-Sacker Bifurcation Analysis for a Discrete-Time System of Two Neurons

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A class of discrete-time system modelling a network with two neurons is considered. First, we investigate the global stability of the given system. Next, we study the local stability by techniques developed by Kuznetsov to discrete-time systems. It is found that Neimark-Sacker bifurcation (or Hopf bifurcation for map) will occur when the bifurcation parameter exceeds a critical value. A formula determining the direction and stability of Neimark-Sacker bifurcation by applying normal form theory and center manifold theorem is given. Finally, some numerical simulations for justifying the theoretical results are also provided.

1. Introduction

Since one of the models with electric circuit implementation was proposed by Hopfield [1], the dynamical behaviors (including stability, instability, periodic oscillatory, bifurcation, and chaos) of the continuous-time neural networks have received increasing interest due to their promising potential applications in many fields, such as signal processing, pattern recognition, optimization, and associative memories (see [2–5]).

For computer simulation, experimental or computational purposes, it is common to discrete the continuous-time neural networks. Certainly, the discrete-time analog inherits the dynamical characteristics of the continuous-time neural networks under mild or no restriction on the discretional step size and also remains functionally similar to the continuous-time system and any physical or biological reality that the continuous-time system has. We refer to [6, 7] for related discussions on the importance and the need for discrete-time analog to reflect the dynamics of their continuous-time counterparts. Recently, Zhao et al. [8] discussed the stability and Hopf bifurcation on discrete-time Hopfield neural networks with delay. Yu and Cao [9] studied the stability and Hopf bifurcation on a four-neuron BAM neural network with time delays. Xiao and Cao [10] considered the stability and pitchfork bifurcation, flip bifurcation, and Neimark-Sacker bifurcation. Yuan et al. [11] investigated the stability and Neimark-Sacker bifurcation of a discrete-time neural network. Yuan et al. [12] made a discussion on the stability and Neimark-Sacker bifurcation on a discrete-time neural network. For more knowledge about neural networks, one can see [13–18].

It will be pointed that two neurons have the same transfer function f in [11] and two neurons have different transfer functions f in [12] (i.e., the transfer function of the first neuron is f_1 and the transfer function of the second neuron is f_2). In this paper, we assume that there are same transfer function f_1 in the first equation and there are same transfer function f_2 in the second equation, then we obtain the following discrete-time neural network model with self-connection in the form:

$$\begin{aligned} x_1(n+1) &= \beta x_1(n) + a_{11} f_1(x_1(n)) + a_{12} f_1(x_2(n)), \\ x_2(n+1) &= \beta x_2(n) + a_{21} f_2(x_1(n)) + a_{22} f_2(x_2(n)), \end{aligned} \tag{1.1}$$

where x_i $(i = 1, 2)$ denotes the activity of the ith neuron, $\beta \in (0, 1)$ is internal delay of neurons, the constants a_{ij} $(i, j = 1, 2)$ denote the connection weights, $f_i : R \to R$ is a continuous transfer function, and $f_i(0) = 0$ $(i = 1, 2)$.

The discrete-time system (1.1) can be regarded as a discrete analogy of the differential system

$$\begin{aligned} \dot{x}_1(t) &= -\mu x_1(t) + w_{11} f_1(x_1(t)) + w_{12} f_1(x_2(t)), \\ \dot{x}_2(t) &= -\mu x_2(t) + w_{21} f_2(x_1(t)) + w_{22} f_2(x_2(t)), \end{aligned} \tag{1.2}$$

or the system with a piecewise constant arguments:

$$\begin{aligned} \dot{x}_1(t) &= -\mu x_1(t) + w_{11} f_1(x_1([t])) + w_{12} f_1(x_2([t])), \\ \dot{x}_2(t) &= -\mu x_2(t) + w_{21} f_2(x_1([t])) + w_{22} f_2(x_2([t])), \end{aligned} \tag{1.3}$$

where $\mu > 0$ and $[\cdot]$ denotes the greatest integer function. For the method of discrete analogy, we refer to [19–21]. The motivation of this research is system (1.1) which includes the discrete version of system (1.2) and (1.3). On the other hand, the wide application of differential equations with piecewise constant argument in certain biomedical models [22] and much progress have been made in the study of system with the piecewise arguments since the pioneering work of Cooke and Wiener [23] and Shah and Wiener [24].

In this paper, we investigate the nonlinear dynamical behavior of a discrete-time system of two neurons, namely, (1.1), and prove that Neimark-Sacker bifurcation will occur in the discrete-time system. Using techniques developed by Kuznetsov to discrete-time systems [25], we obtain the stability of the bifurcating periodic solution and the direction of Neimark-Sacker bifurcation.

The organization of this paper is as follows. In Section 2, we will discuss the stability of the trivial solutions and the existence of Neimark-Sacker bifurcation. In Section 3, a formula for determining the direction of Neimark-Sacker bifurcation and the stability of bifurcating periodic solution will be given by using the normal form method and the center manifold theory for discrete-time system developed by Kuznetsov [25]. In Section 4, numerical simulations aimed at justifying the theoretical analysis will be reported.

2. Stability and Existence of Neimark-Sacker Bifurcation

In this section, we discuss the global and local stability of the equilibrium $(0,0)$ of system (1.1). In order to prove our results, we need the following hypothesis:

(H1) $f_i : R \to R$ is globally Lipschitz with Lipschitz constant $L_i > 0$, $(i = 1,2)$, that is,

$$\left| f_i(u) - f_i(v) \right| \le L_i |u - v| \quad \text{for } u, v \in R. \tag{2.1}$$

Theorem 2.1. *Let* $\Delta = (|a_{11}|L_1 + |a_{22}|L_2)^2 + 4|a_{12}||a_{21}|L_1 L_2$. *Suppose that hypothesis (H1) and the inequality*

$$\left| 2\beta + |a_{11}|L_1 + |a_{22}|L_2 \pm \sqrt{\Delta} \right| < 2 \tag{2.2}$$

are satisfied, then $(x_1(n), x_2(n)) \to (0,0)$ *as* $n \to \infty$.

Proof. It follows from system (1.1) that

$$\begin{pmatrix} |x_1(n+1)| \\ |x_2(n+1)| \end{pmatrix} \le \begin{pmatrix} \beta + |a_{11}|L_1 & |a_{12}|L_1 \\ |a_{21}|L_2 & \beta + |a_{22}|L_2 \end{pmatrix} \begin{pmatrix} |x_1(n)| \\ |x_2(n)| \end{pmatrix}. \tag{2.3}$$

Set

$$M = \begin{pmatrix} \beta + |a_{11}|L_1 & |a_{12}|L_1 \\ |a_{21}|L_2 & \beta + |a_{22}|L_2 \end{pmatrix}. \tag{2.4}$$

Clearly, the eigenvalues of M are given by

$$\lambda_{1,2} = \frac{2\beta + |a_{11}|L_1 + |a_{22}|L_2 \pm \sqrt{\Delta}}{2}, \tag{2.5}$$

which implies that $|\lambda_{1,2}| < 1$. Thus the eigenvalues of M are inside the unit circle and $(x_1(n), x_2(n)) \to (0,0)$ as $n \to \infty$.

Next, we will analyze the local stability of the equilibrium $(0,0)$. For most of models in the literature, including the ones [20, 26, 27], the transfer function f is $f(u) = \tanh(cu)$. However, we only make the following assumption on functions f_i:

(H2) $f_i \in C^1(R)$ and $f_i(0) = 0$ $(i = 1,2)$.

For the sake of simplicity and the need of discussion, we define the following parameters:

$$\begin{aligned} P_1 &= \beta + a_{11} f_1'(0), \\ P_2 &= \beta + a_{22} f_2'(0), \\ D &= -4 a_{12} a_{21} f_1'(0) f_2'(0). \end{aligned} \tag{2.6}$$

□

Theorem 2.2. *The zero solution of* (1.1) *is asymptotically stable if (H2) is satisfied and* $(P_1, P_2, D) \in X_0$*, where*

$$\begin{aligned} X_0 &= X_1 \cap X_2 \cup X_3, \\ X_1 &= \left\{ (P_1, P_2, D) \in R^3,\ D > 4(P_1 + P_2 - P_1P_2 - 1),\ P_1 + P_2 < 2,\ (P_1 - P_2)^2 \geq D \right\}, \\ X_2 &= \left\{ (P_1, P_2, D) \in R^3,\ D > -4(P_1 + P_2 + P_1P_2 + 1),\ P_1 + P_2 > -2,\ (P_1 - P_2)^2 \geq D \right\}, \\ X_3 &= \left\{ (P_1, P_2, D) \in R^3,\ D < 4(1 - P_1P_2),\ (P_1 - P_2)^2 < D \right\}. \end{aligned} \tag{2.7}$$

Proof. Under (H2), using Taylor expansion, we can expand the right-hand side of system (1, 1) into first-, second-, third-, and other higher-order terms about the equilibrium (0, 0), and we have

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} \beta + a_{11} f_1'(0) & a_{12} f_1'(0) \\ a_{21} f_2'(0) & \beta + a_{22} f_2'(0) \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} + \begin{pmatrix} F_1(x, D) \\ F_2(x, D) \end{pmatrix}, \tag{2.8}$$

where $x = (x_1, x_2)^T \in R^2$, and

$$\begin{aligned} F_1(x, D) &= \frac{a_{11} f_1''(0)}{2} x_1^2(n) + \frac{a_{12} f_1''(0)}{2} x_2^2(n) + \frac{a_{11} f_1'''(0)}{6} x_1^3(n) + \frac{a_{12} f_1'''(0)}{6} x_2^3(n) + \cdots, \\ F_2(x, D) &= \frac{a_{21} f_2''(0)}{2} x_1^2(n) + \frac{a_{22} f_2''(0)}{2} x_2^2(n) + \frac{a_{21} f_2'''(0)}{6} x_1^3(n) + \frac{a_{22} f_2'''(0)}{6} x_2^3(n) + \cdots. \end{aligned} \tag{2.9}$$

The associated characteristic equation of its linearized system is

$$\lambda^2 - (P_1 + P_2)\lambda + P_1P_2 + \frac{1}{4}D = 0. \tag{2.10}$$

In order to make the equilibrium (0, 0) be locally asymptotically stable, it is necessary and sufficient that all the roots of (2.10) are inside the unit circle. Hence, we will discuss the following two cases.

Case 1 ($(P_1 - P_2)^2 \geq D$). In this case, the roots of (2.10) are given by

$$\lambda_{1,2} = \frac{1}{2}\left[P_1 + P_2 \pm \sqrt{(P_1 - P_2)^2 - D}\right]. \tag{2.11}$$

Obviously, we obtained that the modulus of eigenvalues $\lambda_{1,2}$ are less than 1 if and only if $(P_1, P_2, D) \in X_1 \bigcup X_2$, where

$$\begin{aligned} X_1 &= \left\{(P_1, P_2, D) \in R^3, D > 4(P_1 + P_2 - P_1P_2 - 1), P_1 + P_2 < 2, (P_1 - P_2)^2 \geq D\right\}, \\ X_2 &= \left\{(P_1, P_2, D) \in R^3, D > -4(P_1 + P_2 + P_1P_2 + 1), P_1 + P_2 > -2, (P_1 - P_2)^2 \geq D\right\}. \end{aligned} \tag{2.12}$$

Thus, we obtain that the eigenvalues $\lambda_{1,2}$ are inside the unit circle when $(P_1, P_2, D) \in X_1 \bigcup X_2$ is satisfied.

Case 2 ($(P_1 - P_2)^2 < D$). In this case, the characteristic equation of (2.10) has a pair of conjugate complex roots:

$$\lambda_{1,2} = \frac{1}{2}\left[P_1 + P_2 \pm i\sqrt{D - (P_1 - P_2)^2}\right]. \tag{2.13}$$

It is easy to verify that $|\lambda_{1,2}| < 1$ if and only if $(P_1, P_2, D) \in X_3$.

Combining case 1 with case 2 yields that the the eigenvalues $\lambda_{1,2}$ are inside the unit circle for $(P_1, P_2, D) \in X_0 = X_1 \bigcup X_2 \bigcup X_3$ and the zero solution of (1.1) is asymptotically stable.

In what follows, we will choose D as the bifurcation parameter to study the Neimark-Sacker bifurcation at $(0, 0)$. For $(P_1 - P_2)^2 < D$, we denote

$$\lambda(D) = \frac{1}{2}\left[P_1 + P_2 + i\sqrt{D - (P_1 - P_2)^2}\right]. \tag{2.14}$$

Then the eigenvalues of (2.10) are conjugate complex $\lambda(D)$ and $\overline{\lambda(D)}$. The modulus of eigenvalue is $|\lambda(D)| = (1/2)\sqrt{D + 4P_1P_2}$. Clearly, $|\lambda(D)| = 1$ if and only if

$$D = D^* = 4(1 - P_1P_2). \tag{2.15}$$

When the parameter D passes through such critical value of $D^* = 4(1 - P_1P_2)$, a Neimark-Sacker bifurcation may be expected. Obviously, we have

$$|\lambda(D)| < 1 \quad \text{for } (P_1 - P_2)^2 < D < D^*. \tag{2.16}$$

Since the modulus of eigenvalue $|\lambda(D^*)| = 1$, we know that D^* is a critical value which destroies the stability of $(0, 0)$. The following lemma is helpful to study bifurcation of $(0, 0)$. □

Lemma 2.3. *If (H2) and* $0 < P_1 + P_2 < 2$ *are satisfied, then*

(i) $((d/dD)|\lambda(D)|)_{D=D^*} > 0$,

(ii) $\lambda^k(D^*) \neq 1$, *for* $k = 1, 2, 3, 4$,

where $\lambda(D)$ *and* $D = D^*$ *are given by* (2.14) *and* (2.15), *respectively.*

Proof. Under the assumption $0 < P_1 + P_2 < 2$, we have

$$\left(\frac{d}{dD} |\lambda(D)| \right)_{D=D^*} = \frac{1}{8} > 0, \tag{2.17}$$

which implies (i) holds. On the other hand, $\lambda^k(D^*) = 1$ for some $k \in \{1,2,3,4\}$ if and only if the argument $\lambda(D^*) \in \{0, \pm\pi/2, \pm 2\pi/3, \pi\}$. Since $|\lambda(D^*)| = 1$, $\operatorname{Re}\lambda(D^*) > 0)$, $\operatorname{Im}\lambda(D^*) > 0$, it follows that $\arg\lambda(D^*) \notin \{0, \pm\pi/2, \pm 2\pi/3, \pi\}$. Hence the condition (ii) of Lemma 2.3 is also satisfied. The proof is complete. □

By Lemma 2.2 in [28], we obtain the following results.

Theorem 2.4. *Suppose that (H2) and* $0 < P_1 + P_2 < 2$ *are satisfied, then one has the following.*

(i) *If* $(P_1 - P_2)^2 < D < D^*$, *then the equilibrium* $(0, 0)$ *is asymptotically stable.*

(ii) *If* $D > D^*$, *then the equilibrium* $(0, 0)$ *is unstable.*

(iii) *The Neimark-Sacker bifurcation occurs at* $D = D^*$. *That is, system* (1.1) *has a unique close invariant curve bifurcating from the equilibrium* $(0, 0)$.

Proof. Obviously, we have $|\lambda| < 1$ for $(P_1 - P_2)^2 < D < D^*$ and $|\lambda| > 1$ for $D > D^*$, which means (i) and (ii) are true. The conclusions in Lemma 2.3 indicate the transversality condition for the Neimark-Sacker bifurcation is satisfied, so the Neimark-Sacker bifurcation occurs at $D = D^*$. Conclusion (iii) follows. □

3. Direction and Stability of Neimark-Sacker Bifurcation

In the above section, we have shown that Neimark-Sacker bifurcation occurs at some value $D = D^*$ for system (1.1) under condition (H2) and $(P_1 - P_2)^2 < D$, $0 < P_1 + P_2 < 2$. In this section, by employing the normal form method and the center manifold theory for discrete-time system developed by Kuznetsov [25], we will study the direction and stability of Neimark-Sacker bifurcation. In what follows, we make the following further assumption:

(H3) $f \in C^3(R)$.

Now system (1.1) can be rewritten as

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} \longmapsto A(D) \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} + \begin{pmatrix} F_1(x, D) \\ F_2(x, D) \end{pmatrix}, \tag{3.1}$$

where $x = (x_1, x_2)^T \in R^2$ and

$$A(D) = \begin{pmatrix} \beta + a_{11} f_1'(0) & a_{12} f_1'(0) \\ a_{21} f_2'(0) & \beta + a_{22} f_2'(0) \end{pmatrix},$$

$$F_1(x, D) = \frac{a_{11} f_1''(0)}{2} x_1^2(n) + \frac{a_{12} f_1''(0)}{2} x_2^2(n) + \frac{a_{11} f_1'''(0)}{6} x_1^3(n) + \frac{a_{12} f_1'''(0)}{6} x_2^3(n) + \cdots, \tag{3.2}$$

$$F_2(x, D) = \frac{a_{21} f_2''(0)}{2} x_1^2(n) + \frac{a_{22} f_2''(0)}{2} x_2^2(n) + \frac{a_{21} f_2'''(0)}{6} x_1^3(n) + \frac{a_{22} f_2'''(0)}{6} x_2^3(n) + \cdots.$$

Denote

$$A = A(D) = \begin{pmatrix} \beta + a_{11} f_1'(0) & a_{12} f_1'(0) \\ a_{21} f_2'(0) & \beta + a_{22} f_2'(0) \end{pmatrix}. \tag{3.3}$$

Suppose that $q \in C^2$ is an eigenvector of $A(D)$ corresponding to eigenvalue $\lambda(D)$ given by (2.14) and $p \in C^2$ is an an eigenvector of $A^T(D)$ corresponding to eigenvalue $\overline{\lambda(D)}$. Then

$$A(D)q(D) = \lambda(D)q(D), \qquad A^T(D)q(D) = \overline{\lambda(D)}q(D). \tag{3.4}$$

By direct calculation, we obtain that

$$q \sim \left(1, -\frac{2\beta f'(0)}{r}\right)^T, \qquad p \sim \left(1, -\frac{2}{\overline{\overline{r}}}\right)^T, \tag{3.5}$$

where

$$r = -\frac{1}{2}\left[P_1 + P_2 \pm \sqrt{D - (P_1 - P_2)^2} i\right] - \left(\beta + a_{22} f_2'(0)\right) \overset{\text{def}}{=} A_1 + iA_2. \tag{3.6}$$

For the eigenvector q, to normalize p, let

$$p = \frac{|\overline{r}|^2}{|\overline{r}|^2 + a_{12} a_{21} f_1'(0) f_2'(0)} \left(1, -\frac{2}{\overline{\overline{r}}}\right)^T. \tag{3.7}$$

We have $\langle q, p \rangle = 1$, where $\langle \cdot \rangle$ means the standard scalar product in $C^2 : \langle q, p \rangle = \overline{p}_1 q_1 + \overline{p}_2 q_2$. Any vector $x \in R^2$ can be represented for D near D^* as

$$x = zq(D) + \overline{zq(D)}. \tag{3.8}$$

For some complex z, obviously,

$$z = \langle p(D), x \rangle. \tag{3.9}$$

Thus, system (3.1) can be transformed for D near D^* into the following form:

$$\dot{z} = \lambda(D)z + g(z, \overline{z}, D), \tag{3.10}$$

where $\lambda(D)$ that can be written as $\lambda(D) = (1 + \varphi(D))e^{i\theta(D)}(\varphi(D)$ is a smooth function with $\varphi(D^*) = 0)$ and

$$g(z, \overline{z}, D) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(D) z^k z^l. \tag{3.11}$$

We know that F_i $(i = 1, 2)$ in (3.1) can be expanded as

$$\begin{aligned} F_1(\xi, D) &= \frac{a_{11} f_1''(0)}{2} \xi_1^2 + \frac{a_{12} f_1''(0)}{2} \xi_2^2 + \frac{a_{11} f_1'''(0)}{6} \xi_1^3 + \frac{a_{12} f_1'''(0)}{6} \xi_2^3 + \cdots, \\ F_2(\xi, D) &= \frac{a_{21} f_2''(0)}{2} \xi_1^2 + \frac{a_{22} f_2''(0)}{2} \xi_2^2 + \frac{a_{21} f_2'''(0)}{6} \xi_1^3 + \frac{a_{22} f_2'''(0)}{6} \xi_2^3 + \cdots. \end{aligned} \tag{3.12}$$

It follows that

$$B_1(x, y) = \sum_{j,k}^{2} \left. \frac{\partial^2 F_1(\xi, D^*)}{\partial \xi_j \partial \xi_k} \right|_{\xi=0} x_j x_k = a_{11} f_1''(0) x_1 y_1 + a_{12} f_1''(0) x_2 y_2, \tag{3.13}$$

$$B_2(x, y) = \sum_{j,k}^{2} \left. \frac{\partial^2 F_2(\xi, D^*)}{\partial \xi_j \partial \xi_k} \right|_{\xi=0} x_j x_k = a_{21} f_2''(0) x_1 y + a_{22} f_2''(0) x_2 y_2, \tag{3.14}$$

$$\begin{aligned} C_1(x, y, u) &= \sum_{j,k,l}^{2} \left. \frac{\partial^3 F_1(\xi, D^*)}{\partial \xi_j \partial \xi_k \partial \xi_l} \right|_{\xi=0} x_j x_k u_l \\ &= \left(a_{11} f_1'''(0) x_1 y_1 u_1 + a_{12} f_1'''(0) \right) x_2 y_2 u_2, \end{aligned} \tag{3.15}$$

$$\begin{aligned} C_2(x, y, u) &= \sum_{j,k,l}^{2} \left. \frac{\partial^3 F_2(\xi, D^*)}{\partial \xi_j \partial \xi_k \partial \xi_l} \right|_{\xi=0} x_j x_k u_l \\ &= \left(a_{21} f_1'''(0) x_1 y_1 u_1 + a_{22} f_1'''(0) \right) x_2 y_2 u_2. \end{aligned} \tag{3.16}$$

By (3.11)–(3.16) and the following formulas:

$$\begin{aligned} g_{20}(D^*) &= \langle p, B(q, q) \rangle, \qquad g_{11}(D^*) = \langle p, B(q, \overline{q}) \rangle, \\ g_{02}(D^*) &= \langle p, B(\overline{q}, \overline{q}) \rangle, \qquad g_{21}(D^*) = \langle p, B(q, q, \overline{q}) \rangle, \end{aligned} \tag{3.17}$$

we obtain

$$\begin{aligned}
g_{11}(D^*) &= g_{02}(D^*) = g_{20}(D^*) \\
&= \frac{r^2}{r^2 + a_{12}a_{21}f_1'(0)f_2'(0)}\left(a_{11}f_1''(0) + a_{12}f_1''(0)\right) \\
&\quad + \frac{r^2}{r^2 + a_{12}a_{21}f_1'(0)f_2'(0)} \times \frac{a_{12}f_1'}{r} \times \left(a_{21}f_2''(0) + a_{22}f_2''(0)\right) \\
&= \frac{r^2\left(a_{11}f_1''(0) + a_{12}f_1''(0)\right) + ra_{21}f_1'(0)\left(a_{21}f_2''(0) + a_{22}f_2''(0)\right)}{r^2 + a_{11}a_{12}f_1'(0)f_2'(0)} \\
&= \frac{Q_1 + iQ_2}{T_1 + iT_2} = \frac{(Q_1T_1 + Q_2T_2) + i(Q_2T_1 - Q_1T_2)}{T_1^2 + T_2^2},
\end{aligned} \tag{3.18}$$

where

$$\begin{gathered}
T_1 = A_1^2 - A_2^2 + a_{12}a_{21}f_1'(0)f_2'(0), \qquad T_2 = 2A_1A_2, \\
Q_1 = \left(A_1^2 - A_2^2\right)\left(a_{11}f_1''(0) + a_{12}f_1''(0)\right) + A_1a_{12}f_1'(0)\left(a_{21}f_2''(0) + a_{22}f_2''(0)\right), \\
Q_2 = 2A_1A_2\left(a_{11}f_1''(0) + a_{12}f_1''(0)\right) + A_2a_{12}f_1'(0)\left(a_{21}f_2''(0) + a_{22}f_2''(0)\right), \\
A_1 = \frac{1}{2}(P_1 + P_2) - \left(\beta + a_{22}f_2'(0)\right), \qquad A_2 = \frac{1}{2}\sqrt{D - (P_1 - P_2)^2}i.
\end{gathered} \tag{3.19}$$

Noting that $e^{-i\theta(D^*)} = \overline{\lambda(D^*)}$, we can compute the coefficient $a(D^*)$ which determines the direction of the appearance of the invariant curve in system (1.1) exhibiting the Neimark-Sacker bifurcation:

$$\begin{aligned}
a(D^*) = {}& \operatorname{Re}\left[\frac{e^{-i\theta(D^*)}g_{21}}{2}\right] - \operatorname{Re}\left[\frac{\left(1 - 2e^{i\theta(D^*)}\right)e^{-i\theta(D^*)}}{2\left(1 - e^{i\theta(D^*)}\right)}g_{20}g_{11}\right] \\
& - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2.
\end{aligned} \tag{3.20}$$

We calculate every term, respectively,

(i)

$$\begin{aligned}
\operatorname{Re}\left[\frac{e^{-i\theta(D^*)}g_{21}}{2}\right] = {}& \frac{1}{4\left(M_1^2 + M_2^2\right)} \\
& \times \Bigg[(P_1 + P_2)(B_1M_1 + B_2M_2) \\
& \quad \times (B_2M_1 + B_1M_2)\sqrt{D^* - (P_1 - P_2)^2}\Bigg],
\end{aligned} \tag{3.21}$$

where

$$
\begin{aligned}
B_1 &= a_{11} f_1'''(0)\left(A_1^4 - A_2^4\right) + a_{12} f_1'''(0)\left(a_{21} f_2'(0)\right)^3 A_1 \\
&\quad + a_{12} f_1'(0)\left[a_{21} f_1'''(0)\left(A_1^2 + A_2^2\right) A_1 + a_{22} f_2'''(0)\left(a_{21} f_2'(0)\right)^3\right], \\
B_2 &= 2a_{11} f_1'''(0) A_1 A_2\left(A_1^2 + A_2^2\right) + a_{12} f_1'''(0)\left(a_{21} f_2'(0)\right)^3 A_2 \\
&\quad + a_{12} a_{21} f_1'''(0) A_1\left(A_1^2 + A_2^2\right), \\
M_1 &= \left(A_1^2 + A_2^2\right)\left[A_1^2 - A_2^2 + a_{12} a_{21} f_1'(0) f_2'(0)\right], \\
M_2 &= 2A_1 A_2\left(A_1^2 + A_2^2\right),
\end{aligned}
\tag{3.22}
$$

(ii)

$$
\operatorname{Re}\left[\frac{\left(1 - 2e^{i\theta(D^*)}\right)e^{-2i\theta(D^*)}}{2\left(1 - e^{i\theta(D^*)}\right)} g_{20} g_{11}\right] = \frac{K_1}{K_2},
\tag{3.23}
$$

where

$$
\begin{aligned}
K_1 &= \left[(Q_1T_1 + Q_2T_2)^2 - (Q_2T_1 - Q_1T_2)^2\right] \\
&\quad \times \left[(1 - P_1 - P_2)\left(2P_1^2 + 2P_2^2 - D^*\right) - 2(P_1 + P_2)\left(D^* - (P_1 - P_2)^2\right)\right] \\
&\quad + 2(Q_1T_1 + Q_2T_2)(Q_2T_1 - Q_1T_2) \\
&\quad \times \left[\left(2P_1^2 + 2P_2^2 - D^*\right)\sqrt{D^* - (P_1 - P_2)^2} + 2(1 - P_1 - P_2)\right. \\
&\qquad \left. \times (P_1 + P_2)\sqrt{D^* - (P_1 - P_2)^2}\right], \\
K_2 &= 4\left(T_1^2 + T_2^2\right)^2\left[(2 - P_1 - P_2)^2 + D^* - \left(P_1 - p_2\right)^2\right],
\end{aligned}
\tag{3.24}
$$

(iii)

$$
\begin{aligned}
\frac{1}{2}\left|g_{11}\right|^2 - \frac{1}{4}\left|g_{02}\right|^2 = \frac{1}{4}\left|g_{11}\right|^2 &= \frac{1}{4}\left|\frac{(Q_1T_1 + Q_2T_2) + i(Q_2T_1 - Q_1T_2)}{\left(T_1^2 + T_1^2\right)}\right|^2 \\
&= \frac{(Q_1T_1 + Q_2T_2)^2 + (Q_2T_1 - Q_1T_2)^2}{4\left(T_1^2 + T_1^2\right)^2}.
\end{aligned}
\tag{3.25}
$$

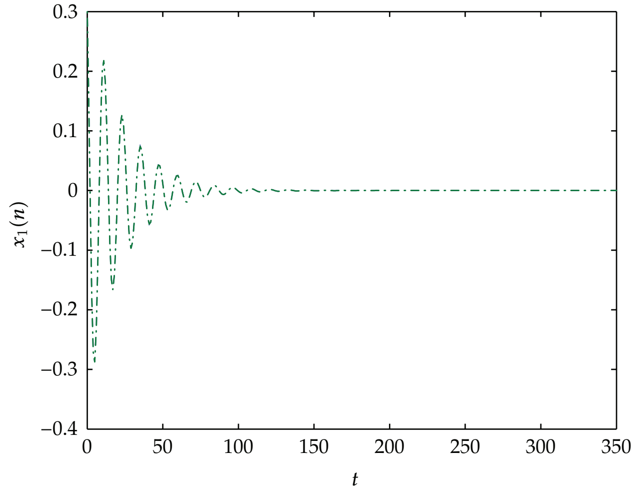

Figure 1: The equilibrium $(0,0)$ is asymptotically stable. The initial value and parameter D are $(2,3)$ and $38/15$, respectively.

Thus

$$a(D^*) = \frac{1}{4\left(M_1^2+M_2^2\right)}(P_1+P_2)(B_1M_1+B_2M_2)(B_2M_1+B_1M_2) \times \sqrt{D^* - (P_1-P_2)^2} + \frac{K_1}{K_2} + \frac{(Q_1T_1+Q_2T_2)^2+(Q_2T_1-Q_1T_2)^2}{4\left(T_1^2+T_1^2\right)^2}. \tag{3.26}$$

Theorem 3.1. *Suppose that condition (H3) holds and* $f_i(0)=0$, $(P_1-P_2)^2<D$, $0<P_1+P_2<2$, *then the direction of the Neimark-Sacker bifurcation and stability of bifurcating periodic solution can be determined by the sign of* $a(D^*)$. *In fact, if* $a(D^*)<0(>0)$, *then the Neimark-Sacker bifurcation is supercritical (subcritical) and the bifurcating periodic solution is asymptotically stable (unstable), where* D^* *is given by* (2.15).

Remark 3.2. This method is introduced by Kuznetsov in [25].

4. Numerical Examples

In this section, we give numerical simulations to support our theoretical analysis. Let $\beta = 1/2$, $a_{11}=1$, $a_{12}=-1$, $a_{22}=-1$, $f_1(u)=\sin u$, and $f_2(u)=\arctan(u/3)$ in system (1.1); namely, system (1.1) has the following form:

$$\begin{aligned} x_1(n+1) &= \frac{1}{2}x_1(n)+\sin(x_1(n))-\sin(x_2(n)), \\ x_2(n+1) &= \frac{1}{2}x_2(n)+a_{21}\arctan\frac{x_1(n)}{3}-\arctan\frac{x_2(n)}{3}. \end{aligned} \tag{4.1}$$

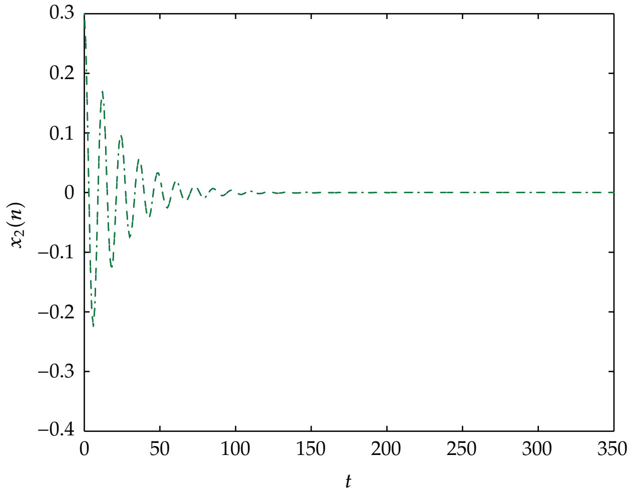

Figure 2: The equilibrium $(0, 0)$ is asymptotically stable. The initial value and parameter D are $(2, 3)$ and $38/15$, respectively.

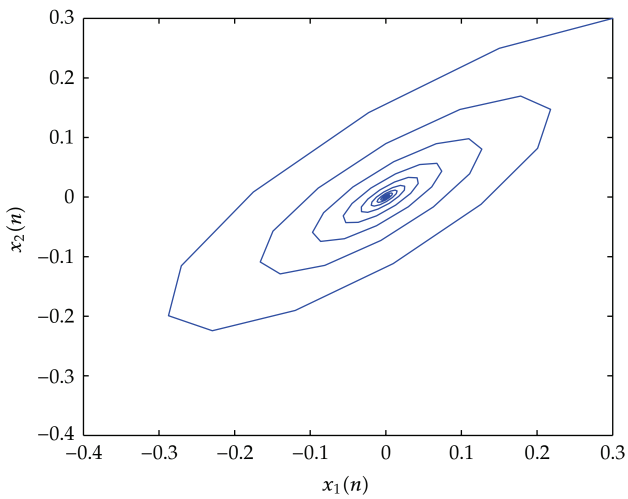

Figure 3: The equilibrium $(0, 0)$ is asymptotically stable. The initial value and parameter D are $(2, 3)$ and $38/15$, respectively.

By the simple calculation, we obtain

$$f_1'(0) = 1, \qquad f_1''(0) = 0, \qquad f_1'''(0) = -1, \qquad f_2'(0) = \frac{1}{3}, \qquad f_2''(0) = -\frac{2}{27}, \qquad f_2'''(0) = 0,$$
$$P_1 = \beta + a_{11} f_1'(0) = -\frac{3}{2}, \qquad P_2 = \beta + a_{22} f_2'(0) = \frac{1}{6}, \qquad D^* = 4(1 - P_1 P_2) = 5. \tag{4.2}$$

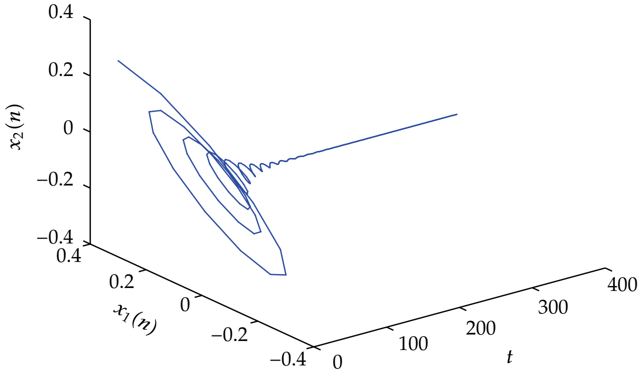

Figure 4: The equilibrium $(0,0)$ is asymptotically stable. The initial value and parameter D are $(2,3)$ and $38/15$, respectively.

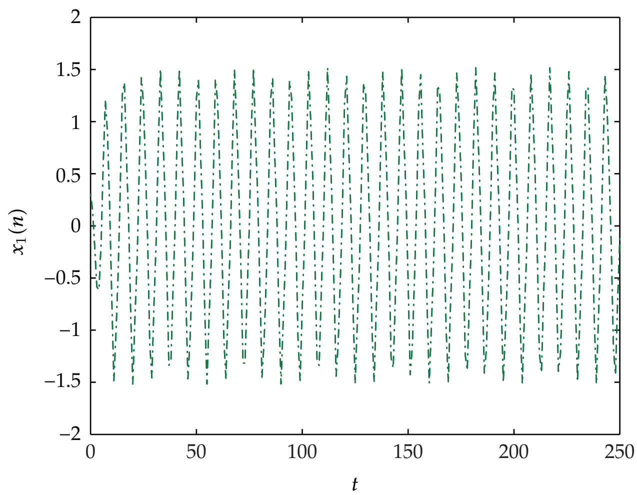

Figure 5: An invariant closed circle bifurcates from equilibrium $(0,0)$. The initial value and parameter D are $(2,3)$ and $14/3$, respectively.

Choose $a_{21} = 0.4$ so that $D = 38/15 < D^* = 3$, $(P_1, P_2, D) \in X_0$. By Theorem 2.4, we know that the origin is asymptotically stable. The corresponding waveform and phase plots are shown in Figures 1, 2, 3, and 4. Choose $a_{21} = 4$, then $D = 14/3 > D^* = 3$. By Theorem 2.4, we know that a Neimark-Sacker bifurcation occurs when $D = D^* = 3$. By a series of complicated computation, we obtain $g_{20} = g_{11} = g_{02} = 0$, $a(D^*) \approx -1.214 < 0$. By Theorem 3.1, we know that the periodic solution is stable. The corresponding phase plot is shown in Figures 5, 6, 7, and 8.

5. Conclusions

The discrete-time delay system of neural networks provides some dynamical behaviors which enrich the theory of continuous system and have potential applications in neural networks. Although the system discussed in this paper is quite simple, it is potentially useful

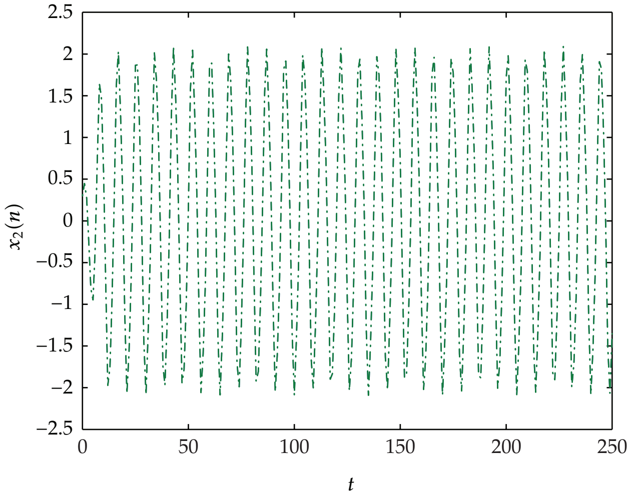

Figure 6: An invariant closed circle bifurcates from equilibrium $(0,0)$. The initial value and parameter D are $(2,3)$ and $14/3$, respectively.

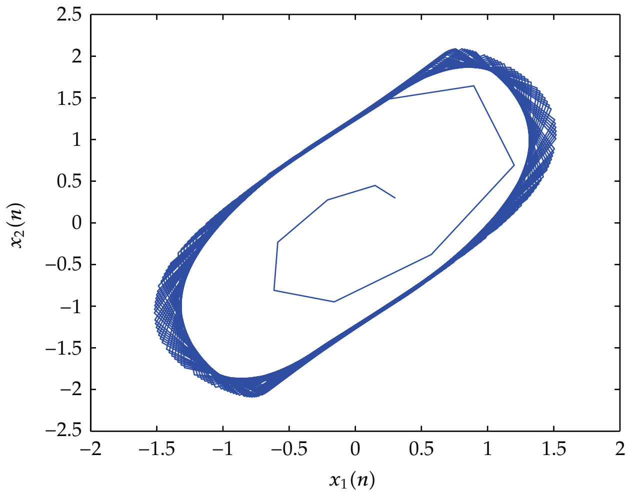

Figure 7: An invariant closed circle bifurcates from equilibrium $(0,0)$. The initial value and parameter D are $(2,3)$ and $14/3$, respectively.

applications as the complexity which has been carried over to the other models with delay. By choosing a proper bifurcation parameter, we have shown that a Neimark-Sacker bifurcation occurs when this parameter passes through a critical value. We have also determined the direction of the Neimark-Sacker bifurcation and the stability of periodic solutions by applying the normal form theory and the center manifold reduction. Our simulation results have verified and demonstrated the correctness of the theoretical results. Our work is a excellent complementary to the known results [11, 12] in the literatures.

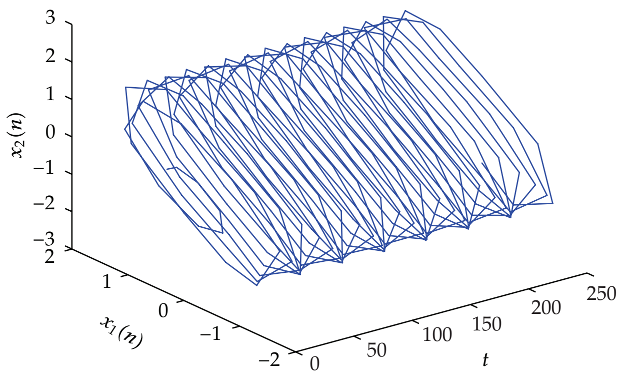

Figure 8: An invariant closed circle bifurcates from equilibrium $(0, 0)$. The initial value and parameter D are $(2, 3)$ and $14/3$, respectively.

Acknowledgments

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