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Research Article

α-Well-Posedness for Quasivariational Inequality Problems

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We introduce and study the concepts of α-well-posedness and L-α-well-posedness for quasivariational inequality problems having a unique solution and the concepts of α-well-posedness in the generalized sense and L-α-well-posedness in the generalized sense for quasivariational inequality problems having more than one solution. We present some necessary and/or sufficient conditions for the various kinds of well-posedness to occur. Our results generalize and strengthen previously known results for quasivariational inequality problems.

1. Introduction

Let E be a reflexive real Banach space and let K be a nonempty closed convex subset of E. Let S be a set-valued mapping from K to K and let A be an operator from E to the dual space E^*. Bensoussan and Lions [1], Baiocchi and Capelo [2], and Mosco [3] considered the following quasivariational inequality (in short, (QVIP)), which is to find a point $u_0 \in K$ such that

$$u_0 \in S(u_0), \quad \langle Au_0, u_0 - v\rangle \leq 0, \quad \forall v \in S(u_0). \tag{1.1}$$

The interest in quasivariational inequality problems lies in the fact that many economic or engineering problems are modeled through them, as explained in [4, 5] where a wide bibliography on variational inequalities, quasivariational inequality problems, and related problems is contained. Moreover, under suitable assumptions, a quasivariational inequality is equivalent to a generalized Nash equilibrium problem [3].

On the other hand, well-posedness plays a crucial role in the stability theory for optimization problems, which guarantees that, for an approximating solution sequence, there exists a subsequence which converges to a solution [6]. The study of well-posedness for scalar minimization problems started from Tikhonov [7] and Levitin and Polyak [8]. Since the convergence of numerical methods for quasivariational inequality Problems can be obtained also with the aid of well-posedness theory, Lignola [9] introduced and investigated the concepts of well-posedness and L-well-posedness for quasivariational inequalities having a unique solution and the concepts of well-posedness and L-well-posedness in the generalized sense for quasivariational inequality problems having more than one solution.

In this paper, inspired by the above concepts of well-posedness for (QVIP), we introduce and study the concepts of α-well-posedness and L-α-well-posedness for quasivariational inequality Problems having a unique solution and the concepts of α-well-posedness in the generalized sense and L-α-well-posedness in the generalized sense for quasivariational inequality Problems having more than one solution. The results in this paper generalize and improve the results in [9, 10].

2. Preliminaries

Denote by Γ the solution set of (QVIP). Let $\alpha > 0$. In order to investigate the α-well-posed for (QVIP), we need the following definitions.

First we recall the notion of Mosco convergence [11]. A sequence $(H_n)_n$ of subsets of E Mosco converges to a set H if

$$H = \liminf_n H_n = w - \limsup_n H_n, \tag{2.1}$$

where $\lim_n \inf H_n$ and $w{-}\lim_n \sup H_n$ are, respectively, the Painlevé-Kuratowski strong limit inferior and weak limit superior of a sequence $(H_n)_n$, that is,

$$\begin{gathered}\liminf_n H_n = \left\{y \in E : \ \exists y_n \in H_n,\ n \in N,\ \text{with } y_n \longrightarrow y\right\}, \\ w - \limsup_n H_n = \left\{y \in E : \ \exists n_k \uparrow +\infty,\ n_k \in N,\ \exists y_{n_k} \in H_{n_k},\ k \in N,\ \text{with } y_{n_k} \rightharpoonup y\right\},\end{gathered} \tag{2.2}$$

where "$\rightharpoonup$" means weak convergence, "$\to$" means strong convergence.

If $H = \lim_n \inf H_n$, we call the sequence $(H_n)_n$ of subsets of E Lower Semi-Mosco which converges to a set H.

It is easy to see that a sequence $(H_n)_n$ of subsets of E Mosco converges to a set H which implies that the sequence $(H_n)_n$, also Lower Semi-Mosco, converges to the set H, but the converse is not true in general.

We will use the usual abbreviations usc and lsc for "upper semicontinuous" and "lower semicontinuous," respectively. Let E be a reflexive real Banach space with dual E^*. An operator $A : E \to E^*$ will be called hemicontinuous if it is continuous from every segment of E to E^* endowed with the weak topology. $A : E \to E^*$ will be called monotone if $\langle Au - Av,\ u - v\rangle \geq 0$ for every u, and $v \in E$. $A : E \to E^*$ will be called pseudomonotone if $\langle Au,\ u - v\rangle \leq 0 \Rightarrow \langle Av,\ u - v\rangle \leq 0$ for every u and $v \in E$.

Definition 2.1. A sequence $(u_n)_n$ is an α-approximating sequence for (QVIP) if

(i) $(u_n) \in K$, for all $n \in N$;

(ii) there exists a sequence $(\varepsilon_n)_n,\ \varepsilon_n > 0$, decreasing to 0 such that

$$\begin{gathered} d(u_n, S(u_n)) \leq \varepsilon_n, \text{ that is, } u_n \in B(S(u_n), \varepsilon_n), \quad \forall n \in N, \\ \langle Au_n, u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2 \leq \varepsilon_n, \quad \forall v \in S(u_n),\ \forall n \in N. \end{gathered} \tag{2.3}$$

Definition 2.2. A quasivariational inequality (QVIP) is said to be α-well-posed (resp., α-well-posed in the generalized sense) if it has a unique solution u_0 and every α-approximating sequence $(u_n)_n$ strongly converges to u_0 (resp., if the solution set Γ of (QVIP) is nonempty and for every α-approximating sequence $(u_n)_n$ has a subsequence which strongly converges to a point of Γ).

Definition 2.3. A sequence $(u_n)_n$ is an L-α-approximating sequence for (QVIP) if:

(i) $(u_n) \in K$, for all $n \in N$;

(ii) there exists a sequence $(\varepsilon_n)_n,\ \varepsilon_n > 0$, decreasing to 0 such that $d(u_n, S(u_n)) \leq \varepsilon_n$, and

$$\langle Av, u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2 \leq \varepsilon_n, \quad \forall v \in S(u_n),\ \forall n \in N. \tag{2.4}$$

Definition 2.4. A quasivariational inequality (QVIP) is said to be L-α-well-posed (resp., L-α-well-posed in the generalized sense) if it has a unique solution u_0 and every L-α-approximating sequence $(u_n)_n$ strongly converges to u_0 (resp., if the solution set Γ of (QVIP) is nonempty and for every L-α-approximating sequence $(u_n)_n$ has a subsequence which strongly converges to a point of Γ).

It is worth noting that if $\alpha = 0$, then the definitions of α-well-posedness, α-well-posedness in the generalized sense, L-α-well-posedness, and L-α-well-posedness in the generalized sense for (QVIP), respectively, reduce to those of the well-posedness, well-posedness in the generalized sense, L-well-posedness, and L-well-posedness in the generalized sense for (QVIP) in [9]. We also note that Definition 2.2 generalizes and extends α-well-posedness and α-well-posedness in the generalized sense of variational inequalities in [10] which are related to the continuously differentiable gap function of variational inequality Problems introduced by Fukushima [12].

We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.5 (see [13]). *Let $(H_n)_n$ be a sequence of nonempty subsets of the space E such that*

(i) *H_n is convex for every $n \in N$;*

(ii) *$H_0 \subseteq \lim_n \inf H_n$;*

(iii) *there exists $m \in N$ such that $\operatorname{int} \bigcap_{n \geq m} H_n \neq \emptyset$.*

Then, for every $u_0 \in \operatorname{int} H_0$, there exists a positive real number δ such that $B(u_0, \delta) \subseteq H_n$, for all $n \geq m$.

If E is a finite dimensional space, then assumption (iii) can be replaced by

(iii)′ $\operatorname{int} H_0 \neq \emptyset$.

The following Lemmas 2.6 and 2.7 play important roles in this paper. Now we present a Minty type lemma for quasivariational inequalities as follows.

Lemma 2.6. *Suppose that set-valued mapping S is nonempty convex-valued, the operator A is hemicontinuous and monotone, $u_0 \in S(u_0)$. Then the following conditions are equivalent:*

(i) $\langle Au_0, u_0 - v\rangle - (\alpha/2)\|u_0 - v\|^2 \le 0$, *for all* $v \in S(u_0)$,

(ii) $\langle Av, u_0 - v\rangle - (\alpha/2)\|u_0 - v\|^2 \le 0$, *for all* $v \in S(u_0)$.

Proof. We first prove that (ii) implies (i). Let v be a arbitrary point of $S(u_0)$. For every number $t \in [0,1]$, since the set-valued mapping S is convex-valued and $u_0 \in S(u_0)$, the point $v_t = tv + (1-t)u_0$ belongs to $S(u_0)$. It follows from (ii) that

$$\langle Av_t, u_0 - v_t\rangle - \frac{\alpha}{2}\|u_0 - v_t\|^2 \le 0. \tag{2.5}$$

From the definition of v_t, one has

$$\lim_{t\to 0}\left(\langle Av_t, u_0 - v\rangle - \frac{\alpha}{2}t\|u_0 - v\|^2\right) \le 0, \tag{2.6}$$

and it follows from the hemicontinuity of A that

$$\langle Au_0, u_0 - v\rangle \le 0, \tag{2.7}$$

then

$$\langle Au_0, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \le 0, \quad \forall v \in S(u_0). \tag{2.8}$$

The converse is an easy consequence of monotonicity of A. □

Lemma 2.7. *Assume that set-valued mapping S is nonempty convex-valued, then $u_0 \in \Gamma$ if and only if the following conditions hold:*

$$u_0 \in S(u_0), \quad \langle Au_0, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \le 0, \quad \forall v \in S(u_0). \tag{2.9}$$

Proof. The necessity is clearly held. Now we prove the sufficiency. Let for all $v \in S(u_0)$, for all $t \in [0,1]$, $v_t = tv + (1-t)u_0$. Since S is convex-valued, $v_t \in S(u_0)$, one has

$$\langle Au_0, u_0 - v_t\rangle - \frac{\alpha}{2}\|u_0 - v_t\|^2 \le 0, \quad \forall t \in (0,1], \tag{2.10}$$

which implies that

$$\langle Au_0, u_0 - v\rangle - t\frac{\alpha}{2}\|u_0 - v\|^2 \le 0, \quad \forall t \in (0,1],\ \forall v \in S(u_0). \tag{2.11}$$

The above inequality implies, for t converging to zero, that u_0 is a solution of (QVIP). This completes the proof. □

3. Case of a Unique Solution

In this section, we investigate some metric characterizations of α-well-posedness and L-α-well-posedness for (QVIP).

For any $\varepsilon > 0$, we consider the set

$$Q_\varepsilon = \left\{u \in K : u \in B(S(u), \varepsilon),\ \langle Au, u - v\rangle - \frac{\alpha}{2}\|u - v\|^2 \le \varepsilon,\ \forall v \in S(u)\right\}$$
$$L_\varepsilon = \left\{u \in K : u \in B(S(u), \varepsilon),\ \langle Av, u - v\rangle - \frac{\alpha}{2}\|u - v\|^2 \le \varepsilon,\ \forall v \in S(u)\right\}. \tag{3.1}$$

Theorem 3.1. *Let the same assumptions be as in Lemma 2.7. Then, one has*

(a) *(QVIP) is α-well-posed if and only if the solution set Γ of (QVIP) is nonempty and* $\lim_{\varepsilon\to 0} \operatorname{diam} Q_\varepsilon = 0$;

(b) *moreover, if $A : E \to E^*$ is pseudomonotone, then (QVIP) is L-α-well-posed if and only if the solution set Γ of (QVIP) is nonempty and* $\lim_{\varepsilon\to 0} \operatorname{diam} L_\varepsilon = 0$.

Proof. We only prove (a). The proof of (b) is similar and is omitted here. Suppose that (QVIP) is α-well-posed, then $\Gamma \neq \emptyset$. It follows from Lemma 2.7 that $Q_\varepsilon \neq \emptyset$. Suppose by contradiction that there exists a real number β, such that $\lim_{\varepsilon\to 0} \operatorname{diam} Q_\varepsilon > \beta > 0$, then there exists $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, and $(w_n)_n, (z_n)_n \in Q_{\varepsilon_n}$, such that $\|w_n - z_n\| > \beta$, for all $n \in N$. Since the sequences $(w_n)_n, (z_n)_n$ are both α-approximating sequences for (QVIP), $(w_n)_n$ and $(z_n)_n$ strongly converge to the unique solution u_0, and this gives a contradiction. Therefore, $\lim_{\varepsilon\to 0} \operatorname{diam} Q_\varepsilon = 0$.

Conversely, let $(u_n)_n, u_n \in K$, be an α-approximating sequence for (QVIP). Then there exists a sequence $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, such that

$$d(u_n, S(u_n)) \le \varepsilon_n, \quad \forall n \in N,$$
$$\langle Au_n, u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2 \le \varepsilon_n, \quad \forall v \in S(u_n),\ \forall n \in N, \tag{3.2}$$

that is, $u_n \subset Q_{\varepsilon_n}$, for all $n \in N$. It is easy to see $\lim_{\varepsilon\to 0} \operatorname{diam} Q_\varepsilon = 0$ and $\Gamma \neq \emptyset$ implying that Γ is a singleton point set. Indeed, if there exist two different solutions z_1, z_2, then from Lemma 2.7, we know that $z_1, z_2 \in Q_\varepsilon$, for all $\varepsilon > 0$. Thus, $\lim_{\varepsilon\to 0} \operatorname{diam} Q_\varepsilon \ge \|z_1 - z_2\| \neq 0$, a contraction. Let u_0 be the unique solution of (QVIP). It follows from Lemma 2.7 that $u_0 \in Q_{\varepsilon_n}$. Thus, $\lim_{n\to 0}\|u_n - u_0\| \le \lim_{n\to 0} \operatorname{diam} Q_{\varepsilon_n} = 0$. So $(u_n)_n$ strongly converge to u_0. Therefore, (QVIP) is α-well-posed. □

Theorem 3.2. *Let $\alpha > 0$ and the following assumptions hold:*

(i) *the set-valued mapping S is nonempty convex-valued, and, for each sequence $(u_n)_n$ in K converges to u_0, the sequence $(S(u_n))_n$ Lower Semi-Mosco converging to $S(u_0)$;*

(ii) *for every converging sequence* $(h_n)_n$, *there exists* $m \in N$, *such that*

$$\operatorname{int} \bigcap_{n \geq m} S(h_n) \neq \emptyset; \tag{3.3}$$

(iii) *the operator* A *is hemicontinuous and monotone on* K.

Then, (QVIP) is α*-well-posed if and only if*

$$Q_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \operatorname{diam} Q_\varepsilon = 0. \tag{3.4}$$

Proof. The necessity has been proved in Theorem 3.1(a).

Conversely, assume that (3.4) holds. It is easy to see that (3.4) implies that the solution set Γ of (QVIP) is a singleton point set. Let $(u_n)_n$ be an α-approximating sequence for (QVIP), that is, there exists a sequence $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, such that

$$\begin{gathered} d(u_n, S(u_n)) \leq \varepsilon_n, \quad \forall n \in N, \\ \langle Au_n, u_n - v \rangle - \frac{\alpha}{2}\|u_n - v\|^2 \leq \varepsilon_n, \quad \forall v \in S(u_n), \ \forall n \in N. \end{gathered} \tag{3.5}$$

Therefore, $u_n \subset Q_{\varepsilon_n}$, for all $n \in N$. In light of (3.4), $(u_n)_n$ is a Cauchy sequence and strongly converges to a point $u_0 \in K$. In order to obtain that u_0 solves (QVIP), we start to prove that $u_0 \in S(u_0)$. For each $n \in N$, choose $u_n' \in S(u_n)$, such that $\|u_n - u_n'\| < d(u_n, S(u_n)) + \varepsilon_n \leq 2\varepsilon_n$. It follows from $u_n \to u_0$ and $\varepsilon_n \to 0$ that $u_n' \to u_0$. It follows from the assumption (i) that $\lim_n \inf S(u_n) = S(u_0)$. Thus, $u_0 \in S(u_0)$.

To complete the proof, consider an arbitrary point $v \in S(u_0)$. By Lower Semi-Mosco convergence again, we have $S(u_0) \subseteq \lim_n \inf S(u_n)$. Also observe that assumption (ii) applied to the constant sequence $h_n = u_0$, for all $n \in N$, implies that $\operatorname{int} S(u_0) \neq \emptyset$. From Lemma 2.5, it follows that if $v \in \operatorname{int} S(u_0)$, then there exist $m \in N$ and $\delta > 0$ such that $\operatorname{int} B(v, \delta) \subseteq S(u_n)$, for all $n > m$. Thus, $v \in S(u_n)$ for n sufficiently large. Notice the A is monotone and the sequence $(u_n)_n$ is an α-approximating sequence for (QVIP), then we have

$$\langle Av, u_0 - v \rangle = \lim_n \langle Av, u_n - v \rangle \leq \lim_n \inf \langle Au_n, u_n - v \rangle \leq \lim_n \left(\varepsilon_n + \frac{\alpha}{2}\|u_n - v\|^2 \right) = \frac{\alpha}{2}\|u_0 - v\|^2. \tag{3.6}$$

If $v \in S(u_0) \setminus \operatorname{int} S(u_0)$, let $(v_n)_n$ be a sequence converging to v, whose point belongs to a segment contained in $\operatorname{int} S(u_0)$. Since $v_n \in \operatorname{int} S(u_0)$, for all $n \in N$, one has

$$\langle Av_n, u_0 - v_n \rangle \leq \frac{\alpha}{2}\|u_0 - v_n\|^2. \tag{3.7}$$

Since the hemicontinuity of A,

$$\langle Av, u_0 - v \rangle \leq \frac{\alpha}{2}\|u_0 - v\|^2, \quad \forall v \in S(u_0). \tag{3.8}$$

It follows from Lemma 2.6 that

$$\langle Au_0, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \le 0, \quad \forall v \in S(u_0), \tag{3.9}$$

then, by Lemma 2.7, we obtain that u_0 solves (QVIP). This completes the proof. □

Now, we present a result in which assumption (ii) of above theorem is dropped, while the continuity assumption on the operator A is strengthened.

Theorem 3.3. *Let the following assumptions hold:*

(i) *the set-valued mapping S is nonempty convex-valued, and, for each sequence $(u_n)_n$ in K converging to u_0, the sequence $(S(u_n))_n$ Lower Semi-Mosco converges to $S(u_0)$;*

(ii) *the operator A is (s, w)-continuous on K.*

Then, (QVIP) is α-well-posed if and only if (3.4) *holds.*

Proof. The necessity follows from Theorem 3.1 and Lemma 2.7.

Conversely, let $(u_n)_n$ be an α-approximating sequence for (QVIP) and (3.4) holds. From (3.4) and the proof of Theorem 3.2, we can obtain that $(u_n)_n$ strongly converges to u_0, with $u_0 \in S(u_0)$. Since Lower Semi-Mosco convergence implies for every $v \in S(u_0)$, there exists sequence $(v_n)_n$ strongly converging to v such that $v_n \in S(u_n)$. Since the operator A is (s, w)-continuous and $(u_n)_n$ is an α-approximating sequence for (QVIP), we have

$$\langle Au_0, u_0 - v\rangle = \lim_n \langle Au_n, u_n - v_n\rangle \le \lim_n\left(\varepsilon_n + \frac{\alpha}{2}\|u_n - v_n\|^2\right) = \frac{\alpha}{2}\|u_0 - v\|^2. \tag{3.10}$$

By Lemma 2.7, we obtain that u_0 solves (QVIP). This completes the proof. □

Theorem 3.4. *Let the following assumptions hold:*

(i) *the set-valued mapping S is nonempty convex-valued, and, for each sequence $(u_n)_n$ in K converges to u_0, the sequence $(S(u_n))_n$ Lower Semi-Mosco converging to $S(u_0)$;*

(ii) *for every converging sequence $(h_n)_n$, there exists $m \in N$, such that*

$$\operatorname{int} \bigcap_{n \ge m} S(h_n) \ne \emptyset; \tag{3.11}$$

(iii) *the operator A is hemicontinuous and monotone on K.*

Then, (QVIP) is L-α-well-posed if and only if

$$L_\varepsilon \ne \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \operatorname{diam} L_\varepsilon = 0. \tag{3.12}$$

Proof. Assume that (QVIP) is L-α-well-posed, then it follows from the monotonicity of A that $\emptyset \ne \Gamma \ne L_\varepsilon$, for all $\varepsilon > 0$. It follows from Theorem 3.1(b) that the necessity can be completed.

Assume that (3.12) holds. Let $(u_n)_n$ be an L-α-approximating sequence for (QVIP), then there exists a sequence $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, such that $u_n \in L_{\varepsilon_n}$, for all $n \in N$. Following the same argument as the proof of Theorem 3.1, it is easy to see $\lim_{\varepsilon\to 0}\operatorname{diam} L_\varepsilon = 0$ and $\Gamma \neq \emptyset$ imply that Γ is a singleton point set. In light of the assumption, $(u_n)_n$ is a Cauchy sequence and strongly converges to a point $u_0 \in K$ and $u_0 \in S(u_0)$. Let $v \in \operatorname{int} S(u_0)$ and using Lemma 2.5, one has $v \in S(u_n)$, for n sufficiently large. Then, we get

$$\langle Av, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 = \lim_n \left[\langle Av, u_n - v\rangle - \frac{\alpha}{2}\|u_n - v\|^2\right] \leq \lim_n \varepsilon_n = 0. \tag{3.13}$$

If $v \in S(u_0) \setminus \operatorname{int} S(u_0)$, let a sequence v_n converges to v, whose points belong to a segment contained in $\operatorname{int} S(u_0)$. Since

$$\langle Av_n, u_0 - v_n\rangle - \frac{\alpha}{2}\|u_0 - v_n\|^2 \leq 0 \tag{3.14}$$

and the operator A is hemicontinuous, one gets

$$\langle Av, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \leq 0. \tag{3.15}$$

According to Lemmas 2.6 and 2.7, u_0 is the solution of (QVIP). □

Theorem 3.5. *Let the following assumptions hold:*

(i) *the set-valued mapping S is nonempty convex-valued, and, for each sequence $(u_n)_n$ in K converging to u_0, the sequence $(S(u_n))_n$ Lower Semi-Mosco converges to $S(u_0)$;*

(ii) *the operator A is (s,w)-continuous and monotone on K.*

Then, (QVIP) is L-α-well-posed if and only if (3.12) *holds.*

Proof. Assume (3.12) holds. Let $(u_n)_n$ be an L-α-approximating sequence for (QVIP), then there exists a sequence $\varepsilon_n > 0$, with $\varepsilon_n \to 0$, such that $(u_n)_n \subset L_{\varepsilon_n}$, for all $n \in N$. Since $\lim_{\varepsilon\to 0}\operatorname{diam} L_\varepsilon = 0$, $(u_n)_n$ is a Cauchy sequence and converges to u_0. As the similar proof of Theorem 3.2, $u_0 \in S(u_0)$. Let $v \in S(u_0)$. Since Lower Semi-Mosco convergence implies for every $v \in S(u_0)$, there exists a sequence $(v_n)_n$ converging to v, such that $v_n \in S(u_n)$. Since A is (s,w)-continuous and $(u_n)_n$ is an L-α-approximating sequence for (QVIP), one has

$$\langle Av, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 = \lim_n \left[\langle Av_n, u_n - v_n\rangle - \frac{\alpha}{2}\|u_n - v_n\|^2\right] \leq \lim_n \varepsilon_n = 0. \tag{3.16}$$

Applying Lemmas 2.6 and 2.7, we have that (QVIP) is L-α-well-posed.

The necessity can be completed as Theorem 3.3. □

4. α-Well-Posedness in the Generalized Sense

In this section, we introduce and investigate some metric characterizations of α-well-posedness in the generalized sense and L-α-well-posedness in the generalized sense for (QVI).

Definition 4.1 (see [11]). Let (X, d) be a metric space and let A, B be nonempty subsets of X. The Hausdorff distance $H(\cdot, \cdot)$ between A and B is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\}, \tag{4.1}$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$.

Definition 4.2 (see [11]). Let A be a nonempty subset of X. The measure of non compactness μ of the set A is defined by

$$\mu(A) = \inf\left\{ \varepsilon > 0 : \ A \subseteq \bigcup_{i=1}^{n} A_i, \ \operatorname{diam} A_i < \varepsilon, \ i = 1, 2, \ldots, n \right\}, \tag{4.2}$$

where diam means the diameter of a set.

Theorem 4.3. *Let the same assumptions be as in Lemma 2.7. Then, one has the following.*

(a) *(QVIP) is α-well-posed in the generalized sense if and only if the solution set Γ of (QVIP) is nonempty compact and $e(Q_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$.*

(b) *Moreover, if A is pseudomonotone, then (QVIP) is L-α-well-posed in the generalized sense if and only if the solution set Γ of (QVIP) is nonempty compact and $e(L_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$.*

Proof. We only prove (a), the proof of (b) is similar and is omitted here. Assume that (QVIP) is α-well-posed in the generalized sense, then the Γ is nonempty and compact. It follows from Lemma 2.7 that $Q_\varepsilon \neq \emptyset$. Now we prove $e(Q_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$. Suppose by contradiction that there exists $\beta > 0$, $\varepsilon_n \to 0$, and $w_n \in Q_{\varepsilon_n}$, such that $d(w_n, \Gamma) \geq \beta$. It follows from $w_n \in Q_{\varepsilon_n}$ that $(w_n)_n$ is an α-approximating sequence for (QVIP). (QVIP) is α-well-posedness in the generalized sense, then there exists a subsequence $(w_{n_k})_k$ of $(w_n)_n$ strongly converging to a point of Γ. This contradicts $d(w_n, \Gamma) \geq \beta$. Thus, $e(Q_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$.

For the converse, let $(u_n)_n$ be an α-approximating sequence for (QVIP), then $u_n \in Q_{\varepsilon_n}$. It follows from $e(Q_{\varepsilon_n}, \Gamma) \to 0$ that there exists a sequence $z_n \subset \Gamma$, such that $d(u_n, z_n) \to 0$. Since Γ is compact, there exists a subsequence $(z_{n_k})_k$ of $(z_n)_n$ strongly converging to $u_0 \in \Gamma$. Thus there exists the corresponding subsequence $(u_{n_k})_k$ of $(u_n)_n$ strongly converging to u_0. Therefore, (QVIP) is α-well-posed in the generalized sense. □

Theorem 4.4. *(a) If (QVIP) is α-well-posed in the generalized sense, then*

$$Q_\varepsilon \neq \emptyset, \qquad \forall \varepsilon > 0, \qquad \lim_{\varepsilon \to 0} \mu(Q_\varepsilon) = 0. \tag{4.3}$$

(b) If (4.3) *and the following assumptions hold:*

(i) *the set-valued mapping S is nonempty convex-valued, and, for each sequence $(u_n)_n$ in K converges to u_0, the sequence $(S(u_n))_n$ Lower Semi-Mosco converging to $S(u_0)$;*

(ii) *the operator A is (s, w)-continuous on K,*

then, (QVIP) is α-well-posed in the generalized sense.

Proof. (a) Suppose that (QVIP) is α-well-posed in the generalized sense. So $Q_\varepsilon \neq \emptyset$, for all $\varepsilon > 0$. By Theorem 4.3(a), Γ is nonempty compact and $e(Q_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$. For any $\varepsilon > 0$, we have

$$H(Q_\varepsilon, \Gamma) = \max\{e(Q_\varepsilon, \Gamma), e(\Gamma, Q_\varepsilon)\} = e(Q_\varepsilon, \Gamma), \tag{4.4}$$

and since Γ is compact, $\mu(\Gamma) = 0$. For every $n \in N$, the following relation holds [14]:

$$\mu(Q_\varepsilon) \leq 2H(Q_\varepsilon, \Gamma) + \mu(\Gamma) = 2H(Q_\varepsilon, \Gamma) = 2e(Q_\varepsilon, \Gamma). \tag{4.5}$$

It follows from $e(Q_\varepsilon, \Gamma) \to 0$, as $\varepsilon \to 0$, that $\lim_{\varepsilon \to 0} \mu(Q_\varepsilon) = 0$.

(b) Assume that (4.3) holds. Then, for any $\varepsilon > 0$, $\mathrm{cl}(Q_\varepsilon)$ is nonempty closed and increasing with $\varepsilon > 0$. By (4.3), $\lim_{\varepsilon \to 0} \mu(\mathrm{cl}(Q_\varepsilon)) = \lim_{\varepsilon \to 0} \mu(Q_\varepsilon) = 0$, where $\mathrm{cl}(Q_\varepsilon)$ is the closure of Q_ε. By the generalized Cantor theorem [11, page 412], we know that

$$\lim_{\varepsilon \to 0} H(\mathrm{cl}(Q_\varepsilon), \Delta) = 0, \quad \text{as } \varepsilon \longrightarrow 0, \tag{4.6}$$

where $\Delta = \bigcap_{\varepsilon > 0} \mathrm{cl}(Q_\varepsilon)$ is nonempty compact.

Now we show that

$$\Gamma = \Delta. \tag{4.7}$$

It follows from Lemma 2.7 that $\Gamma \subseteq \Delta$. So we need to prove that $\Delta \subseteq \Gamma$. Indeed, let $u_0 \in \Delta$. Then, $d(u_0, Q_\varepsilon) = 0$ for every $\varepsilon > 0$. Given $\varepsilon_n > 0$, $\varepsilon_n \to 0$, for every n, there exists $u_n \in Q_{\varepsilon_n}$ such that $d(u_0, u_n) < \varepsilon_n$. Hence, $u_n \to u_0$ and

$$d(u_n, S(u_n)) \leq \varepsilon_n, \tag{4.8}$$

$$\langle Au_n, u_n - v \rangle \leq \varepsilon_n + \frac{\alpha}{2}\|u_n - v\|^2, \quad \forall v \in S(u_n). \tag{4.9}$$

It follows from (4.8), $u_n \to u_0$, and the proof of Theorem 3.2 that $u_0 \in S(u_0)$.

Since Lower Semi-Mosco convergence implies that, for every $v \in S(u_0)$, there exists a sequence $v_n \in S(u_n)$, for all $n \in N$, such that $\lim_n v_n = v$ in the strongly topology.

Since the operator A is (s, w)-continuous on K, hence

$$\langle Au_0, u_0 - v \rangle - \frac{\alpha}{2}\|u_0 - v\|^2 = \lim_n \left[\langle Au_n, u_n - v_n \rangle - \frac{\alpha}{2}\|u_n - v_n\|^2 \right] \leq \lim_n \varepsilon_n = 0. \tag{4.10}$$

By Lemma 2.7, we know $u_0 \in \Gamma$. Thus, $\Delta \subseteq \Gamma$. It follows from (4.6) and (4.7) that $\lim_{\varepsilon \to 0} e(Q_\varepsilon, \Gamma) = 0$. It follows from the compactness of Γ and Theorem 4.3(a) that (QVIP) is α-well-posed in the generalized sense. The proof is completed. □

Theorem 4.5. *Let K be a nonempty, compact, and convex subset of E, let the set-valued mapping S be nonempty convex-valued, and, for each sequence $(u_n)_n$ in K converging to u_0, the sequence $(S(u_n))_n$ Lower Semi-Mosco converges to $S(u_0)$, and the operator A is (s, w)-continuous on K. Then, (QVIP) is α-well-posed in the generalized sense.*

Proof. Let $(u_n)_n$ be an α-approximating sequence for (QVIP). Since the set K is compact, there exists subsequence $(u_{n_k})_k$ of $(u_n)_n$ strongly converging to a point $u_0 \in K$. Reasoning as in Theorem 3.3, we get $u_0 \in S(u_0)$ and u_0 solves (QVIP). Therefore, (QVIP) is α-well-posed in the generalized sense. □

Theorem 4.6. *Let the following assumptions hold:*

(i) *the set-valued mapping S is nonempty convex-valued, and, for each sequence $(u_n)_n$ in K converging to u_0, the sequence $(S(u_n))_n$ Lower Semi-Mosco converges to $S(u_0)$;*

(ii) *the operator A is (s, w)-continuous and monotone on K.*

Then, (QVIP) is L-α-well-posed in the generalized sense if and only if

$$L_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \quad \lim_{\varepsilon \to 0} \mu(L_\varepsilon) = 0. \tag{4.11}$$

Proof. Assume that (QVIP) is L-α-well-posed in the generalized sense. It follows from Lemma 2.7 and the monotonicity of A that $\Gamma \subset L_\varepsilon$, for all $\varepsilon > 0$. And so $L_\varepsilon \neq \emptyset$, for each $\varepsilon > 0$. By Theorem 4.3(b), we can get $e(L_\varepsilon, \Gamma) \to 0$ as $\varepsilon \to 0$. From the proof of Theorem 4.4, we also obtain

$$\mu(L_\varepsilon) \leq 2H(L_\varepsilon, \Gamma) + \mu(\Gamma) = 2H(L_\varepsilon, \Gamma) = 2e(L_\varepsilon, \Gamma). \tag{4.12}$$

Thus, $\lim_{\varepsilon \to 0} \mu(L_\varepsilon) = 0$.

Conversely, assume (4.11) holds. Then, for any $\varepsilon > 0$, $\mathrm{cl}(L_\varepsilon)$ is nonempty closed and increasing with $\varepsilon > 0$. By (4.11), $\lim_{\varepsilon \to 0} \mu(\mathrm{cl}(L_\varepsilon)) = \lim_{\varepsilon \to 0} \mu(L_\varepsilon) = 0$, where $\mathrm{cl}(L_\varepsilon)$ is the closure of L_ε. By the generalized Cantor theorem [11, page 412], we know that

$$\lim_{\varepsilon \to 0} H(\mathrm{cl}(L_\varepsilon), \Delta) = 0, \quad \text{as } \varepsilon \longrightarrow 0, \tag{4.13}$$

where $\Delta = \bigcap_{\varepsilon > 0} \mathrm{cl}(L_\varepsilon)$ is nonempty compact.

Now we show that

$$\Gamma = \Delta. \tag{4.14}$$

It follow from Lemma 2.7 and the monotonicity of A that $\Gamma \subseteq \Delta$. So we need to prove that $\Delta \subseteq \Gamma$. Indeed, let $u_0 \in \Delta$. Then $d(u_0, L_\varepsilon) = 0$ for every $\varepsilon > 0$. Given $\varepsilon_n > 0$, $\varepsilon_n \to 0$, for every n, there exists $u_n \in L_{\varepsilon_n}$ such that $d(u_0, u_n) < \varepsilon_n$. Hence, $u_n \to u_0$ and

$$d(u_n, S(u_n)) \leq \varepsilon_n, \tag{4.15}$$

$$\langle Av, u_n - v \rangle \leq \varepsilon_n + \frac{\alpha}{2} \|u_n - v\|^2, \quad \forall v \in S(u_n). \tag{4.16}$$

It follows from (4.15), $x_n \to x_0$, and the proof of Theorem 3.2 that $u_0 \in S(u_0)$.

Since $S(u_n)$ Lower Semi-Mosco converges to $S(u_0)$, for every $v \in S(u_0)$, there exists a sequence $v_n \in S(u_n)$, for all $n \in N$, such that $\lim_n v_n = v$ in the strong topology.

Since the operator A is (s,w)-continuous on K, hence

$$\langle Av, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 = \lim_n\left[\langle Av_n, u_n - v_n\rangle - \frac{\alpha}{2}\|u_n - v_n\|^2\right] \le \lim_n \varepsilon_n = 0. \tag{4.17}$$

By Lemma 2.6 we know that $u_0 \in S(u_0)$, such that

$$\langle Au_0, u_0 - v\rangle - \frac{\alpha}{2}\|u_0 - v\|^2 \le 0, \quad \forall v \in S(u_0). \tag{4.18}$$

It follow from Lemma 2.7 that $u_0 \in \Gamma$. Thus, $\Delta \subseteq \Gamma$. It follows from (4.13) and (4.14) that $\lim_{\varepsilon\to 0} e(L_\varepsilon, \Gamma) = 0$. It follows from the compactness of Γ and Theorem 4.3(b) that (QVIP) is L-α-well-posed in the generalized sense. The problem is completed. □

Remark 4.7. It is easy to see that if $\alpha = 0$, then by the main results in our paper, we can recover the corresponding results in [9] with the weaker condition that $S(x_n)$ Lower Semi-Mosco converges to $S(x_0)$ instead of the condition that S is (s,w)-closed and (s,w)-subcontinuous, and (s,s)-lower semicontinuous.

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References

[1] A. Bensoussan and J.-L. Lions, “Contrôle impulsionnel et inéquations quasi-variationnelles d’evolution,” *Comptes Rendus de l’Académe des Sciences de Paris*, vol. 276, pp. A1333–A1338, 1973.

[2] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free Boundary Problems*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1984.

[3] U. Mosco, “Implicit variational problems and quasi variational inequalities,” in *Nonlinear Operators and the Calculus of Variations (Brussels, 1975)*, Lecture Notes in Mathematics, Vol. 543, pp. 83–156, Springer, Berlin, Germany, 1976.

[4] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. 1 of *Springer Series in Operations Research*, Springer, New York, NY, USA, 2003.

[5] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, vol. 2 of *Springer Series in Operations Research*, Springer, New York, NY, USA, 2003.

[6] A. J. Zaslavski, *Optimization on Metric and Normed Spaces*, vol. 44 of *Springer Optimization and Its Applications*, Springer, New York, NY, USA, 2010.

[7] A. N. Tikhonov, “On the stability of the functional optimization problem,” *USSR Computational Mathematics and Mathematical Physics*, vol. 6, no. 4, pp. 28–33, 1966.

[8] E. S. Levitin and B. T. Polyak, “Convergence of minimizing sequences in conditional extremum problem,” *Soviet Mathematics Doklady*, vol. 7, pp. 764–767, 1966.

[9] M. B. Lignola, “Well-posedness and L-well-posedness for quasivariational inequalities,” *Journal of Optimization Theory and Applications*, vol. 128, no. 1, pp. 119–138, 2006.

[10] M. B. Lignola and J. Morgan, “Approximating solutions and α-well-posedness for variational inequalities and Nash equilibria,” in *Decision and Control in Management Science*, pp. 367–378, Kluwer Academic Publishers, Dodrecht, The Netherlands, 2002.

[11] K. Kuratowski, *Topology*, vol. 1, 2, Academic Press, New York, NY, USA, 1966.
[12] M. Fukushima, "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems," *Mathematical Programming*, vol. 53, no. 1, pp. 99–110, 1992.
[13] M. B. Lignola and J. Morgan, "Semicontinuity and episemicontinuity: equivalence and applications," *Bollettino dell'Unione Matematica Italiana B*, vol. 8, no. 1, pp. 1–16, 1994.
[14] M. Furi and A. Vignoli, "About well-posed optimization problems for functionals in metric spaces," *Journal of Optimization Theory and Applications*, vol. 5, no. 3, pp. 225–229, 1970.