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Research Article

A Note on the Lebesgue-Radon-Nikodym Theorem with respect to Weighted p-adic Invariant Integral on $\mathbb{Z}_p$

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We will give the Lebesgue-Radon-Nikodym theorem with respect to weighted p-adic q-measure on $\mathbb{Z}_p$. In special case, $q = 1$, we can derive the same result as Kim, 2012; Kim et al., 2011.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is defined by $|x|_p = p^{-v_p(x)} = p^{-r}$ for $x = p^r(s/t)$ where s and t are integers with $(p,s) = (p,t) = 1$ and $r \in \mathbb{Q}$ (see [1–12]).

When one speaks of q-extension, q can be regarded as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-1/(p-1)}$ and we use the notations of q-numbers as follows:

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. The fermionic invariant measure on $\mathbb{Z}_p$ is defined by Kim as follows:

$$\mu_{-1}\left(a + p^n\mathbb{Z}_p\right) = (-1)^a, \tag{1.2}$$

where

$$a+p^n\mathbb{Z}_p=\{x\in\mathbb{Z}_p \mid x\equiv a \pmod{p^n}\},\tag{1.3}$$

and $a\in\mathbb{Z}$ with $0\le a<p^n$ (see [2–4, 7]).

From (1.2), the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x,\tag{1.4}$$

where $f\in C(\mathbb{Z}_p)$ (see [2–4, 6–9]).

The idea for generalizing the fermionic integral is replacing the fermionic Haar measure with weakly (strongly) fermionic measure $\mathbb{Z}_p$ satisfying

$$\left|\mu_{-1}(a+p^n\mathbb{Z}_p)-\mu_{-1}\left(a+p^{n+1}\mathbb{Z}_p\right)\right|_p\le\delta_n,\tag{1.5}$$

(see [4, 5, 10]), where $\delta_n\to 0$, a is a element of $\mathbb{Z}_p$, and δ_n is independent of a (for strongly fermionic measure, δ_n is replaced by Cp^{-n}, where C is a positive constant).

Let $f(x)$ be a function defined on $\mathbb{Z}_p$. The fermionic integral of f with respect to a weakly fermionic measure μ_{-1} is

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x)=\lim_{n\to\infty}\sum_{x=0}^{p^n-1} f(x)\mu_{-1}(x+p^n\mathbb{Z}_p),\tag{1.6}$$

if the limit exists.

If μ_{-1} is a weakly fermionic measure on $\mathbb{Z}_p$, then we can define the Radon-Nikodym derivative of μ_{-1} with respect to the Haar measure on $\mathbb{Z}_p$ as follows:

$$f_{\mu_{-1}}(x)=\lim_{n\to\infty}\mu_{-1}(x+p^n\mathbb{Z}_p)\tag{1.7}$$

(see [4, 11]). Note that $f_{\mu_{-1}}$ is only a continuous function on $\mathbb{Z}_p$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in\mathrm{UD}(\mathbb{Z}_p)$, let us define $\mu_{-1,f}$ as follows:

$$\mu_{-1,f}(x+p^n\mathbb{Z}_p)=\int_{x+p^n\mathbb{Z}_p} f(x)d\mu_{-1}(x)\tag{1.8}$$

(see [4, 11]), where the integral is the fermionic p-adic invariant integral. From (1.8), we can easily note that $\mu_{-1,f}$ is a strongly fermionic measure on $\mathbb{Z}_p$. Since then

$$\begin{aligned}\left|\mu_{-1,f}(x+p^n\mathbb{Z}_p)-\mu_{-1,f}\left(x+p^{n+1}\mathbb{Z}_p\right)\right|_p &= \left|\sum_{x=0}^{p^n-1} f(x)(-1)^x-\sum_{x=0}^{p^n} f(x)(-1)^x\right|_p \\ &= \left|\frac{f(p^n)}{p^n}\right|\left|p^n\right| \leq Cp^{-n},\end{aligned} \tag{1.9}$$

where C is positive constant.

The purpose of this paper is to derive a Lebesgue-Radon-Nikodym type theorem with respect to the fermionic weighted p-adic q-measure on $\mathbb{Z}_p$.

2. The Lebesgue-Radon-Nikodym Theorem with Respect to the Weighted p-adic q-Measure

For any positive integer a and n with $a < p^n$ and $f \in \mathrm{UD}(\mathbb{Z}_p)$, we define $\tilde{\mu}_{f,-q}$, weighted fermionic measure on $\mathbb{Z}_p$ as follows:

$$\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) = \int_{a+p^n\mathbb{Z}_p} q^x f(x)d\mu_{-1}(x), \tag{2.1}$$

where the integral is the fermionic p-adic invariant integral on $\mathbb{Z}_p$.

From (2.1), we note that

$$\begin{aligned}\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) &= \lim_{m\to\infty}\sum_{x=0}^{p^m-1} f(a+p^n x)(-1)^{a+p^n x}q^{a+p^n x} \\ &= (-1)^a q^a \lim_{m\to\infty}\sum_{x=0}^{p^{m-n}-1} f(a+p^n x)(-1)^x q^{p^n x} \\ &= (-1)^a\int_{\mathbb{Z}_p} f(a+p^n x)q^{a+p^n x}d\mu_{-1}(x).\end{aligned} \tag{2.2}$$

By (2.2), we get

$$\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) = (-1)^a\int_{\mathbb{Z}_p} f(a+p^n x)q^{a+p^n x}d\mu_{-1}(x). \tag{2.3}$$

Thus, by (2.3), we have

$$\tilde{\mu}_{\alpha f+\beta g,-q} = \alpha\tilde{\mu}_{f,-q}+\beta\tilde{\mu}_{g,-q}, \tag{2.4}$$

where $f, g \in \mathrm{UD}(\mathbb{Z}_p)$ and α, β are positive constants.

By (2.1), (2.2), (2.3), and (2.4), we get

$$\left|\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p)\right|_p \le \|f_q\|_\infty, \tag{2.5}$$

where $\|f_q\|_\infty = \sup_{x\in\mathbb{Z}_p}|f(x)q^x|_p$.

Let $P(x) \in \mathbb{C}_p[[x]_q]$ be an arbitrary q-polynomial. Now we show that $\tilde{\mu}_{P,-q}$ is a strongly weighted fermionic p-adic invariant measure on $\mathbb{Z}_p$. Without a loss of generality, it is enough to prove the statement for $P(x) = [x]_q^k$.

For $a \in \mathbb{Z}$ with $0 \le a < p^n$, we have

$$\tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p) = \lim_{m\to\infty}(-q)^a \sum_{i=0}^{p^{m-n}-1} q^{p^n i}[a+ip^n]_q^k(-1)^i. \tag{2.6}$$

Note that

$$q^{p^n i} = \sum_{l=0}^{i}\binom{i}{l}[p^n]_q^l(q-1)^l, \tag{2.7}$$

and

$$[a+ip^n]_q^k = \left([a]_q + q^a[p^n]_q[i]_{q^{p^n}}\right)^k. \tag{2.8}$$

By (2.6) and (2.8), we easily get

$$\begin{aligned}\tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p) &\equiv (-1)^a q^a [a]_q^k \left(\operatorname{mod}[p^n]_q\right)\\ &\equiv (-1)^a P(a) q^a \left(\operatorname{mod}[p^n]_q\right).\end{aligned} \tag{2.9}$$

For $x \in \mathbb{Z}_p$, let $x \equiv x_n \pmod{p^n}$ and $x \equiv x_{n+1} \pmod{p^{n+1}}$, where $x_n, x_{n+1} \in \mathbb{Z}$ with $0 \le x_n < p^n$ and $0 \le x_{n+1} < p^{n+1}$. Then we have

$$\left|\tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}_{P,-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\right|_p \le Cp^{-v_p(1-q^{p^n})}, \tag{2.10}$$

where C is positive constant and $n \gg 0$.

Let

$$f_{\tilde{\mu}_{P,-q}}(a) = \lim_{n\to\infty}\tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p). \tag{2.11}$$

Then, by (2.9) and (2.10), we see that

$$f_{\tilde{\mu}_{P,-q}}(a) = (-1)^a q^a [a]_q^k = (-1)^a q^a P(a). \tag{2.12}$$

Since $f_{\tilde{\mu}_{P,-q}}(x)$ is continuous function on $\mathbb{Z}_p$. For $x \in \mathbb{Z}_p$, we have

$$f_{\tilde{\mu}_{P,-q}}(x) = (-1)^x q^x [x]_q^k, \qquad (k \in \mathbb{Z}_+). \tag{2.13}$$

Let $g \in \mathrm{UD}(\mathbb{Z}_p)$. Then, by (2.10), (2.12), and (2.13), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{P,-q}(x) &= \lim_{n\to\infty} \sum_{x=0}^{p^n-1} g(x) \tilde{\mu}_{P,-q}(x + p^n\mathbb{Z}_p) \\ &= \lim_{n\to\infty} \sum_{x=0}^{p^n-1} g(x) q^x [x]_q^k (-1)^x \\ &= \int_{\mathbb{Z}_p} g(x) q^x [x]_q^k d\mu_{-1}(x).\end{aligned} \tag{2.14}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 2.1. *Let $P(x) \in \mathbb{C}_p[[x]_q]$ be an arbitrary q-polynomial. Then $\tilde{\mu}_{P,-q}$ is a strongly weighted fermionic p-adic invariant measure on $\mathbb{Z}_p$; that is,*

$$f_{\tilde{\mu}_{P,-q}}(x) = (-1)^x q^x P(x), \qquad \forall x \in \mathbb{Z}_p. \tag{2.15}$$

Furthermore, for any $g \in UD(\mathbb{Z}_p)$,

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{P,-q}(x) = \int_{\mathbb{Z}_p} g(x) P(x) q^x d\mu_{-1}(x), \tag{2.16}$$

where the second integral is weighted fermionic p-adic invariant integral on $\mathbb{Z}_p$.

Let $f(x) = \sum_{n=0}^{\infty} a_{n,q} \binom{x}{n}_q$ be the Mahler q-expansion of continuous function on $\mathbb{Z}_p$, where

$$\binom{x}{n}_q = \frac{[x]_q [x-1]_q \cdots [x-n+1]_q}{[n]_q!}. \tag{2.17}$$

Then we note that $\lim_{n\to\infty} |a_{n,q}| = 0$.

Let

$$f_m(x) = \sum_{i=0}^{m} a_{i,q} \binom{x}{i}_q \in \mathbb{C}_p\left[[x]_q\right]. \tag{2.18}$$

Then

$$\| f - f_m \|_\infty \le \sup_{n \le m} |a_{n,q}|. \tag{2.19}$$

The function $f(x)$ can be rewritten as $f = f_m + f - f_m$. Thus, by (2.4) and (2.19), we get

$$\begin{aligned}
&\left|\widetilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) - \widetilde{\mu}_{f,-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\right| \\
&\quad \le \max\left\{\left|\widetilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p) - \widetilde{\mu}_{f_m,-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\right|,\right. \\
&\qquad \left.\left|\widetilde{\mu}_{f-f_m,-q}(a+p^n\mathbb{Z}_p) - \widetilde{\mu}_{f-f_m,-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\right|\right\}.
\end{aligned} \tag{2.20}$$

From Theorem 2.1, we note that

$$\left|\widetilde{\mu}_{f-f_m,-q}(a+p^n\mathbb{Z}_p)\right|_p \le \|f - f_m\|_\infty \le C_1 p^{-2v_p(1-q^{p^n})}, \tag{2.21}$$

where C_1 are positive constants. For $m \gg 0$, we have $\|f\|_\infty = \|f_m\|_\infty$. So, we see that

$$\begin{aligned}
&\left|\widetilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p) - \widetilde{\mu}_{f_m,-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\right|_p \\
&\quad = \left|\frac{f_m\left([p^n]_q\right)q^{p^n}}{[p^n]_q^2}\right|_p \left|[p^n]_q^2\right|_p \le \|f_m q^x\|_\infty \left|[p^n]_q^2\right|_p \le C_2 p^{-2v_p(1-q^{p^n})},
\end{aligned} \tag{2.22}$$

where C_2 is a positive constant.

By (2.21), we get

$$\begin{aligned}
&\left|(-1)^a f(a) q^a - \widetilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p)\right|_p \\
&\quad \le \max\left\{\left|q^a f(a) - f_m(a)q^a\right|_p, \left|q^a f_m(a) - \widetilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p)\right|_p, \left|\widetilde{\mu}_{f-f_m,-q}(a+p^n\mathbb{Z}_p)\right|_p\right\} \\
&\quad \le \max\left\{\left|f(a) - f_m(a)\right|_p, \left|f_m(a) - \widetilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p)\right|_p, \|f - f_m\|_\infty\right\}.
\end{aligned} \tag{2.23}$$

Let us assume that we fix $\epsilon > 0$ and fix m such that $\|f - f_m\| < \epsilon$. Then we have

$$\left|(-q)^a f(a) - \widetilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p)\right|_p \le \epsilon, \qquad \text{for } n \gg 0. \tag{2.24}$$

Thus, by (2.24), we have

$$f_{\widetilde{\mu}_{f,-q}}(a) = \lim_{n\to\infty} \widetilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) = (-1)^a q^a f(a). \tag{2.25}$$

Let m be the sufficiently large number such that $\|f - f_m\|_\infty \le p^{-n}$. Then we get

$$\begin{aligned}
\widetilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) &= \widetilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p) + \widetilde{\mu}_{f-f_m,-q}(a+p^n\mathbb{Z}_p) \\
&= (-1)^a q^a f(a) \left(\bmod [p^n]_q^2\right).
\end{aligned} \tag{2.26}$$

For $g \in \mathrm{UD}(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x) d\tilde{\mu}_{f,-q}(x) = \int_{\mathbb{Z}_p} f(x) g(x) q^x d\mu_{-1}(x). \tag{2.27}$$

Let f be the function from $\mathrm{UD}(\mathbb{Z}_p)$ to $\mathrm{Lip}(\mathbb{Z}_p)$. We easily see that $q^x \mu_{-1}(x + p^n\mathbb{Z}_p)$ is a strongly weighted p-adic invariant measure on $\mathbb{Z}_p$ and

$$\left| (f_q)_{\mu_{-1}}(a) - q^a \mu_{-1}(a + p^n\mathbb{Z}_p) \right|_p \le C_3 p^{-2v_p(1-q^{p^n})}, \tag{2.28}$$

where $f_q(x) = f(x)q^x$ and C_3 is positive constant and $n \in \mathbb{Z}_+$.

If $\tilde{\mu}_{1,-q}$ is associated with strongly weighted fermionic invariant measure on $\mathbb{Z}_p$, then we have

$$\left| \tilde{\mu}_{1,-q}(a + p^n\mathbb{Z}_p) - (f_q)_{\mu_{-1}}(a) \right|_p \le C_4 p^{-2v_p(1-q^{p^n})}, \tag{2.29}$$

where $n > 0$ and C_4 is positive constant.

For $n \gg 0$, we have

$$\begin{aligned} &\left| q^a \mu_{-1}(a + p^n\mathbb{Z}_p) - \tilde{\mu}_{1,-q}(a + p^n\mathbb{Z}_p) \right|_p \\ &\quad \le \left| q^a \mu_{-1}(a + p^n\mathbb{Z}_p) - (f_q)_{\tilde{\mu}_{-1}}(a) \right|_p + \left| (f_q)_{\tilde{\mu}_{-1}}(a) - \tilde{\mu}_{1,-q}(a + p^n\mathbb{Z}_p) \right|_p \le K, \end{aligned} \tag{2.30}$$

where K is positive constant.

Hence, $q\mu_{-1} - \tilde{\mu}_{1,-q}$ is a weighted measure on $\mathbb{Z}_p$. Therefore, we obtain the following theorem.

Theorem 2.2. *Let $q\mu_{-1}$ be a strongly weighted p-adic invariant measure on $\mathbb{Z}_p$, and assume that the fermionic weighted Radon-Nikodym derivative $(f_q)_{\mu_{-1}}$ on $\mathbb{Z}_p$ is uniformly differentiable function. Suppose that $\tilde{\mu}_{1,-q}$ is the strongly weighted fermionic p-adic invariant measure associated with $(f_q)_{\mu_{-1}}$. Then there exists a weighted measure $\tilde{\mu}_{2,-q}$ on $\mathbb{Z}_p$ such that*

$$q^x \mu_{-1}(x + p^n\mathbb{Z}_p) = \tilde{\mu}_{1,-q}(x + p^n\mathbb{Z}_p) + \tilde{\mu}_{2,-q}(x + p^n\mathbb{Z}_p). \tag{2.31}$$

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