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Research Article

Boundary Stabilization of Memory Type for the Porous-Thermo-Elasticity System

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We consider the one-dimensional viscoelastic Porous-Thermo-Elastic system. We establish a general decay results. The usual exponential and polynomial decay rates are only special cases.

1. Introduction

An increasing interest has been developed in recent years to determine the decay behavior of the solutions of several elasticity problems. It is known that combining the elasticity equations with thermal effects provokes stability of solutions in the one-dimensional case [1]. Several results concerning the exponential or the polynomial decay of solutions for the thermoelastic systems were obtained by [2–6].

A sample model describing the one-dimensional porous-thermo-elasticity, which was developed in [7, 8], is given by the following system:

$$\begin{aligned}
\rho u_{tt} &= \mu u_{xx} + b v_x - \beta \theta_x, \quad \text{in } (0,L)\times\mathbb{R}^+,\\
J v_{tt} &= \alpha v_{xx} - b u_x - \xi v + m\theta - \tau v_t, \quad \text{in } (0,L)\times\mathbb{R}^+,\\
c\theta_t &= \kappa\theta_{xx} - \beta u_{xt} - m v_t, \quad \text{in } (0,L)\times\mathbb{R}^+,
\end{aligned} \tag{1.1}$$

where t denotes the time variable, x is the space variable, the functions $u$is the displacement, v is the volume fraction of the solid elastic material, and the function θ is the temperature difference. The coefficients $\rho,\ \mu,\ J,\ \alpha,\ \xi,\ \tau,\ c$, and κ are positive constants. b is a constant such that $b^2 < \mu\xi$.

Casas and Quintanilla [7] considered the above system and used the semigroup theory and the method developed by Liu and Zheng [4] to establish the exponential decay of the solution under the boundary conditions of the form

$$u(x,t) = v_x(x,t) = \theta_x(x,t) = 0, \quad x = 0,\ L, t \in (0,\infty). \tag{1.2}$$

Soufyane [9] considered the following system:

$$\begin{gathered}
u_{tt} = u_{xx} + v_x - \theta_x, \quad \text{in } (0,L) \times \mathbb{R}^+, \\
v_{tt} = v_{xx} - u_x - v + \theta - \int_0^t g(t-s) v_{xx}(s) ds, \quad \text{in } (0,L) \times \mathbb{R}^+, \\
\theta_t = \theta_{xx} - u_{xt} - v_t, \quad \text{in } (0,\ L) \times \mathbb{R}^+, \\
u(x,0) = u^0(x), \qquad u_t(x,0) = u^1(x), \qquad v(x,0) = v^0(x), \\
v_t(x,0) = v^1(x), \qquad \theta(x,0) = \theta^0(x), \\
u(0,t) = u(L,t) = v_x(0,t) = v_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \quad t \geq 0.
\end{gathered} \tag{1.3}$$

He proved that the solution of (1.3) decays exponentially if the function g decays exponentially, and the solutions (1.3) decay polynomially if the function g decays polynomially.

Recently Pamplona et al. [10] considered the follwing system:

$$\begin{gathered}
\rho u_{tt} = \mu u_{xx} + b v_x - \beta \theta_x + \gamma u_{xxt}, \quad \text{in } (0,\pi) \times \mathbb{R}^+, \\
J v_{tt} = \alpha v_{xx} - b u_x - \xi v + m\theta, \quad \text{in } (0,\pi) \times \mathbb{R}^+, \\
c\theta_t = \kappa \theta_{xx} - \beta u_{xt} - m v_t, \quad \text{in } (0,\pi) \times \mathbb{R}^+, \\
u(x,0) = u^0(x), \qquad u_t(x,0) = u^1(x), \qquad v(x,0) = v^0(x), \\
v_t(x,0) = v^1(x), \qquad \theta(x,0) = \theta^0(x), \\
u(0,t) = u(\pi,t) = v(0,t) = v(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0, \quad t \geq 0.
\end{gathered} \tag{1.4}$$

They proved that the system is not exponential stable, and they showed that the solution decays polynomially.

In this paper we are concerned with the following model:

$$\begin{gathered}
\rho u_{tt} = \mu u_{xx} + b v_x - \beta \theta_x, \quad \text{in } (0,L) \times \mathbb{R}^+, \\
J v_{tt} = \alpha v_{xx} - b u_x - \xi v + m\theta, \quad \text{in } (0,L) \times \mathbb{R}^+, \\
c\theta_t = \kappa \theta_{xx} - \beta u_{xt} - m v_t, \quad \text{in } (0,L) \times \mathbb{R}^+,
\end{gathered} \tag{1.5}$$

with the initial conditions

$$\begin{gathered}
u(x,0) = u^0(x), \qquad u_t(x,0) = u^1(x), \qquad v(x,0) = v^0(x), \\
v_t(x,0) = v^1(x), \qquad \theta(x,0) = \theta^0(x),
\end{gathered} \tag{1.6}$$

and the boundary conditions

$$u(0,t) = v(0,t) = \theta(0,t) = \theta(L,t) = 0, \quad t \geq 0, \tag{1.7}$$

$$u(L,t) = -\int_0^t g_1(t-s)[\mu u_x(L,s) + bv(L,s)]ds, \tag{1.8}$$

$$v(L,t) = -\int_0^t g_2(t-s)\alpha v_x(L,s)ds. \tag{1.9}$$

Our main interest concerns the asymptotic behavior of the solution of the system above. That is, whether the dissipation given by the boundary memory effect is strong enough to stabilize the whole system. And what type of rate of decay may we expect (exponential decay or polynomial decay?). We obtain an exponential decay or polynomial decay result under some conditions on g_i $(i = 1,2)$. Our proof is based on the multiplier techniques.

This work is divided into four sections. In Section 2 we introduce some notations and some material needed for our work. In Section 3 we state and prove the exponential decay of the solutions of our studied system. Section 4 is devoted to the polynomial decay.

2. Preliminaries

In this section we introduce some notations and we study the existence of regular and weak solutions to system (1.5)–(1.9). First, we will use (1.8) and (1.9) to estimate the boundary terms $u_x(L,t)$ and $v_x(L,t)$.

Defining the convolution product operator by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)ds, \tag{2.1}$$

and differentiating equation (1.8) we obtain

$$\mu u_x(L,t) + bv(L,t) + \frac{1}{g_1(0)}(g_1' * u)(L,t) = -\frac{1}{g_1(0)}u_t(L,t) \quad \forall t \geq 0. \tag{2.2}$$

Applying Volterra's inverse operator, we get

$$\mu u_x(L,t) + bv(L,t) = -\frac{1}{g_1(0)}[u_t(L,t) + (k_1 * u)(L,t)], \quad \forall t \geq 0, \tag{2.3}$$

where the resolvent kernel k_1 satisfies

$$k_1(t) + \frac{1}{g_1(0)}(g_1' * k_1)(t) = -\frac{1}{g_1(0)}g_1'(t). \tag{2.4}$$

Denoting by $\eta_1 = 1/g_1(0)$, we arrive at

$$\mu u_x(L,t) + bv(L,t) = -\eta_1[u_t(L,t) + k_1(0)u(L,t) - k_1(t)u(L,0) + (k_1' * u)(L,t)], \quad \forall t \geq 0. \tag{2.5}$$

A similar procedure leads to

$$\alpha v_x(L,t) = -\eta_2[v_t(L,t) + k_2(0)v(L,t) - k_2(t)v(L,0) + (k_2' * v)(L,t)], \quad \forall t \geq 0, \tag{2.6}$$

where $\eta_2 = 1/g_2(0)$.

Reciprocally, taking, in a natural way, the initial data $u^0(L) = v^0(L) = 0$, the identities (2.5) and (2.6) imply (1.8) and (1.9).

Since we are interested in relaxation functions of exponential or polynomial type and identities (2.5)-(2.6) involve the resolvent kernels k_i $(i = 1, 2)$, we want to know if k_i has the same properties. The following lemma answers this question. Let h be a relaxation function and k its resolvent kernel, that is,

$$k(t) - (k * h)(t) = h(t). \tag{2.7}$$

Lemma 2.1 (see [11]). *If h is a positive continuous function, then k is also positive and continuous. Moreover,*

(1) *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that*

$$h(t) \leq c_0 e^{-\gamma t}, \tag{2.8}$$

then the function k satisfies

$$k(t) \leq \frac{c_0(\gamma - \epsilon)}{(\gamma - \epsilon - c_0)} e^{-\epsilon t}, \tag{2.9}$$

for all $0 < \epsilon < \gamma - c_0$.

(2) *If*

$$c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p} ds < +\infty, \tag{2.10}$$

for a given $p > 1$ and if there exists a positive constant c_0 with $c_0 c_p < 1$, for which

$$h(t) \leq c_0 (1+t)^{-p}, \tag{2.11}$$

then the function k satisfies

$$k(t) \leq \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}. \tag{2.12}$$

Based on Lemma 2.1, we will use (2.5)-(2.6) instead of (1.8)-(1.9). We then define

$$(g\, o\, \varphi)(t) := \int_0^t g(t-s)|\varphi(t)-\varphi(s)|^2 ds,$$
$$(g \odot \varphi)(t) := \int_0^t g(t-s)(\varphi(t)-\varphi(s)) ds. \tag{2.13}$$

By using Hölder's inequality for $0 \le \mu \le 1$, we have

$$|(g \odot \varphi)(t)|^2 \le \left(\int_0^t |g(s)|^{2(1-\mu)} ds\right)(|g|^{2\mu} o\varphi)(t). \tag{2.14}$$

Lemma 2.2 (see [12]). *If $g, \varphi \in C^1(\mathbb{R}^+)$, then*

$$(g * \varphi)\varphi_t = -\frac{1}{2} g(t)|\varphi(t)|^2 + \frac{1}{2} g' o\, \varphi - \frac{1}{2}\frac{d}{dt}\left(g\, o\, \varphi - \left(\int_0^t g(s)ds\right)|\varphi(t)|^2\right). \tag{2.15}$$

3. Exponential Decay

In this section we study the asymptotic behavior of the solutions of system (1.5)–(1.9), when the resolvent kernels k_i $(i = 1, 2)$ satisfy, for $\gamma_i > 0$, the following conditions:

$$k_i(0) > 0, \qquad k_i(t) \ge 0, \qquad k_i'(t) \le 0, \qquad k_i''(t) \ge -\gamma_i k_i'(t). \tag{3.1}$$

These assumptions imply that k_i' converges exponentially to 0, that is,

$$0 \le -k_i'(t) \le Ce^{-\gamma_i t}. \tag{3.2}$$

We define the first-order energy of system (1.5)–(1.9) by

$$\begin{aligned} E(t) := {} & \frac{1}{2}\int_0^L \left[\rho|u_t(x,t)|^2 + J|v_t(x,t)|^2 + c|\theta(x,t)|^2 + \mu|u_x(x,t)|^2\right] dx \\ & + \frac{1}{2}\int_0^L \left[\alpha|v_x(x,t)|^2 + \xi|v(x,t)|^2 + 2bu_x(x,t)v(x,t)\right] dx \\ & + \frac{\eta_1}{2}(k_1(t)u^2(L,t) - (k_1'\, ou)(L,t) + \frac{\eta_2}{2}(k_2(t)v^2(L,t) - (k_2' ov)(L,t). \end{aligned} \tag{3.3}$$

In the sequel we define by $V_1 := \{u \in H^1(0,L) : u(0) = 0\}$. We are now ready to state our first result.

Theorem 3.1. *Given* $((u^0,u^1),(v^0,v^1),\theta^0) \in ((V_1 \times L^2(0,L))^2 \times L^2(0,L))$, *assume that* (3.1) *holds with*

$$\sup_{t\in[t_0,\infty[} (k_i(t)) \text{ small enough.} \tag{3.4}$$

Assume further that b is a small number, then the energy E satisfies the following decay estimates:

$$E(t) \leq c_1 E(0) e^{-\omega t}, \quad \text{if } u^0(L) = v^0(L) = 0. \tag{3.5}$$

Otherwise,

$$E(t) \leq c_1 E(0) e^{-\omega t}\left(1 + \int_0^t k_1^2(s) e^{\omega s} ds + \int_0^t k_2^2(s) e^{\omega s} ds\right), \tag{3.6}$$

where c_1 and ω are positive constants independent of the initial data.

Proof. The main idea is to construct a Lyapunov functional $\mathcal{L}(t)$ equivalent to $E(t)$. To do this we use the multiplier techniques. The proof of Theorem 3.1 will be achieved with the help of a sequence of lemmas. □

Lemma 3.2. *Under the assumptions of Theorem 3.1, the energy of the solution of* (1.5)–(1.9) *satisfies*

$$\begin{aligned}\frac{dE}{dt} \leq{}& -\kappa \int_0^L |\theta_x|^2 dx - \frac{\eta_1}{2} u_t^2(L,t) + \frac{\eta_1}{2} k_1^2(t) \left|u^0(L)\right|^2 \\ &+ \frac{\eta_1}{2} k_1'(t) |u(L,t)|^2 - \frac{\eta_1}{2} (k_1'' o u)(L,t) \\ &- \frac{\eta_2}{2} v_t^2(L,t) + \frac{\eta_2}{2} k_2^2(t) \left|v^0(L)\right|^2 \\ &+ \frac{\eta_2}{2} k_2'(t) |v(L,t)|^2 - \frac{\eta_2}{2} (k_2'' o v)(L,t).\end{aligned} \tag{3.7}$$

Proof. Multiplying first equation of (1.5) by u_t, multiplying second equation of (1.5) by v_t and third equation of (1.5) by θ, and integrating by parts over $[0,L]$, we obtain

$$\begin{aligned}\frac{d}{2dt}\int_0^L \left(\rho|u_t|^2 + \mu|u_x|^2\right) dx &= -b\int_0^L u_{tx} v\, dx + \beta \int_0^L u_{xt}\theta\, dx + [\mu u_x(L,t) + b v(L,t)] u_t(L,t), \\ \frac{d}{2dt}\int_0^L \left(J|v_t|^2 + \alpha|v_x|^2 + \xi|v|^2\right) dx &= -b\int_0^L v_t u_x\, dx + m\int_0^L \theta v_t\, dx + \alpha v_x(L) v_t(L), \\ \frac{d}{2dt}\int_0^L c|\theta|^2 dx &= -\kappa\int_0^L |\theta_x|^2 dx - m\int_0^L \theta v_t\, dx - \beta\int_0^L \theta u_{xt}\, dx.\end{aligned} \tag{3.8}$$

By a summation of these three identities, we get

$$\begin{aligned}&\frac{d}{2dt}\int_{\Omega}\left(\rho|u_t|^2+\mu|u_x|^2+J|v_t|^2+\alpha|v_x|^2+\xi|v|^2+c|\theta|^2+2bu_xv\right)dx\\&\quad\le -\kappa\int_0^L|\theta_x|^2dx+\left[\mu u_x(L,t)+bv(L,t)\right]u_t(L,t)+\alpha v_x(L,t)v_t(L,t),\end{aligned}\tag{3.9}$$

using (2.5), (2.6), and Lemma 2.2, we obtain

$$\begin{aligned}\frac{dE}{dt}\le&-\kappa\int_0^L|\theta_x|^2dx-\frac{\eta_1}{2}u_t^2(L,t)+\frac{\eta_1}{2}k_1^2(t)\left|u^0(L)\right|^2\\&+\frac{\eta_1}{2}k_1'(t)|u(L,t)|^2-\frac{\eta_1}{2}(k_1''ou)(L,t)\\&-\frac{\eta_2}{2}v_t^2(L,t)+\frac{\eta_2}{2}k_2^2(t)\left|v^0(L)\right|^2\\&+\frac{\eta_2}{2}k_2'(t)|v(L,t)|^2-\frac{\eta_2}{2}(k_2''ov)(L,t),\end{aligned}\tag{3.10}$$

which ends the proof of Lemma 3.2. □

Lemma 3.3. *Under the assumptions of Theorem 3.1, the energy of the solution of* (1.5)–(1.9) *satisfies*

$$\begin{aligned}&\frac{d}{dt}\int_0^L(2xu_x+(1-\varepsilon_0)u)\rho u_tdx+\frac{d}{dt}\int_0^L(2xv_x+(1-\varepsilon_0)v)Jv_tdx\\&\quad\le-\varepsilon_0\int_0^L\rho u_t^2dx-\frac{(2-\varepsilon_0)}{2}\mu\int_0^Lu_x^2dx-2b(1-\varepsilon_0)\int_0^Lu_xvdx\\&\qquad-\varepsilon_0\int_0^LJv_t^2dx-\frac{(2-\varepsilon_0)}{2}\alpha\int_0^Lv_x^2dx+C\int_0^L\theta_x^2dx\\&\qquad+(1-\varepsilon_0)\mu u(L)u_x(L)+\rho Lu_t^2(L)+\mu Lu_x^2(L)-(1-\varepsilon_0)\alpha v(0)v_x(0),\end{aligned}\tag{3.11}$$

where ε_0 is a small positive number.

Proof. We multiply first equation of (1.5) by $2xu_x+(1-\varepsilon_0)u$ to obtain

$$\begin{aligned}&\frac{d}{dt}\int_0^L(2xu_x+(1-\varepsilon_0)u)\rho u_tdx\\&\quad=\int_0^L(2xu_{tx}+(1-\varepsilon_0)u_t)\rho u_tdx+\int_0^L(2xu_x+(1-\varepsilon_0)u)(\mu u_{xx}+bv_x-\beta\theta_x)dx.\end{aligned}\tag{3.12}$$

Integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}\int_0^L (2xu_x+(1-\varepsilon_0)u)\rho u_t dx = {} & -\varepsilon_0\int_0^L \rho u_t^2 dx-(2-\varepsilon_0)\mu\int_0^L u_x^2 dx + b\int_0^L 2xu_x v_x dx \\ & - b(1-\varepsilon_0)\int_0^L u_x v dx + (1-\varepsilon_0)\mu u(L,t)u_x(L,t)+\rho L u_t^2(L,t) \\ & + \mu L u_x^2(L,t) - \beta\int_0^L (2xu_x+(1-\varepsilon_0)u)\theta_x dx.\end{aligned} \tag{3.13}$$

Similarly, we multiply second equation of (1.5) by $2xv_x+(1-\varepsilon_0)v$ and integrate over $]0,L[$, using integration by parts, to arrive at

$$\begin{aligned}\frac{d}{dt}\int_0^L (2xv_x+(1-\varepsilon_0)v)Jv_t dx = {} & -\varepsilon_0\int_0^L Jv_t^2 dx-(2-\varepsilon_0)\alpha\int_0^L v_x^2 dx + \varepsilon_0\int_0^L \xi v^2 dx + \alpha L v_t^2(L,t) \\ & + \mu L v_x^2(L,t) - \xi L v^2(L,t) - (1-\varepsilon_0)\alpha v(L,t)v_x(L,t) \\ & - \int_0^L b(2xv_x+(1-\varepsilon_0)v)u_x dx + m\int_0^L (2xv_x+(1-\varepsilon_0)v)\theta dx.\end{aligned} \tag{3.14}$$

Summing the above two identities and using Poincare's and Young's inequalities and taking ε_0 small, we deduce that

$$\begin{aligned}& \frac{d}{dt}\int_0^L (2xu_x+(1-\varepsilon_0)u)\rho u_t dx + \frac{d}{dt}\int_0^L (2xv_x+(1-\varepsilon_0)v)Jv_t dx \\ & \quad \le -\varepsilon_0\int_0^L \rho u_t^2 dx - \frac{(2-\varepsilon_0)}{2}\mu\int_0^L u_x^2 dx - 2b(1-\varepsilon_0)\int_0^L u_x v dx \\ & \qquad - \varepsilon_0\int_0^L Jv_t^2 dx - \frac{(2-\varepsilon_0)}{2}\alpha\int_0^L v_x^2 dx + C\int_0^L \theta_x^2 dx \\ & \qquad + (1-\varepsilon_0)\mu u(L,t)u_x(L,t) + \rho L u_t^2(L,t) + \mu L u_x^2(L,t) \\ & \qquad - \alpha L v_t^2(L,t) + \mu L v_x^2(L,t) - \xi L v^2(L,t) + (1-\varepsilon_0)\alpha v(L,t)v_x(L,t).\end{aligned} \tag{3.15}$$

The proof of Lemma 3.3 is completed. □

Now, we introduce the Lyapunov functional. So, for $N>0$ large enough, let

$$\mathcal{L}(t) = NE(t) + \int_0^L (2xu_x+(1-\varepsilon_0)u)\rho u_t dx + \int_0^L (2xv_x+(1-\varepsilon_0)v)Jv_t dx. \tag{3.16}$$

Applying Young's inequality and Poincaré's inequality to the boundary terms, we have, for $\varepsilon > 0$,

$$\begin{aligned}
&(1-\varepsilon_0)\mu u(L,t)u_x(L,t) + (1-\varepsilon_0)\alpha v(L,t)v_x(L,t) \\
&\quad \le \varepsilon\left[\mu u^2(L,t) + \alpha v^2(L,t)\right] + C_\varepsilon\left[\mu u_x^2(L,t) + \alpha v_x^2(L,t)\right] \\
&\quad \le \varepsilon C\left(\int_0^L |u_x|^2 dx + \int_0^L |v_x|^2 dx\right) + C_\varepsilon\left[\mu u_x^2(L,t) + \alpha v_x^2(L,t)\right].
\end{aligned} \tag{3.17}$$

By rewriting the boundary conditions (2.5)-(2.6) as

$$\begin{aligned}
\mu u_x(L,t) &= -bv(L,t) - \eta_1\left(u_t + k_1(t)u - k_1(t)u^0 - k_1' \odot u\right)(L,t), \\
\alpha v_x(L,t) &= -\eta_2\left(v_t + k_2(t)v - k_2(t)v^0 - k_2' \odot v\right)(L,t),
\end{aligned} \tag{3.18}$$

and combining all above relations, using the fact that b is a small number, the condition 17, taking N large enough, ε_0 very small, and $\varepsilon \ll \varepsilon_0$, we obtain

$$\begin{aligned}
\frac{d\mathcal{L}}{dt}(t) \le& -\frac{\kappa N}{2}\int_0^L |\theta_x|^2 dx - N\frac{\eta_1}{8}u_t^2(L,t) + N\eta_1 k_1^2(t)\left|u^0(L)\right|^2 \\
&- \frac{N\eta_1}{2}k_1'' o u(L,t) - \frac{N\eta_2}{8}v_t^2(L,t) + N\eta_2 k_2^2(t)\left|v^0(L)\right|^2 \\
&- \frac{N\eta_2}{2}k_2'' o v(L,t) - \varepsilon_0\int_0^L \rho u_t^2 dx - \frac{(1-\varepsilon_0)}{8}\mu\int_0^L u_x^2 dx \\
&- \varepsilon_0\int_0^L J v_t^2 dx - \frac{(1-\varepsilon_0)}{8}\alpha\int_0^L v_x^2 dx \\
&+ C_\varepsilon\left(\left(-k_1' \odot u\right)^2 + \left(-k_2' \odot v\right)^2\right)(L,t).
\end{aligned} \tag{3.19}$$

Applying inequality (2.14) with $\mu = 1/2$, using the trace formula we have, for some positive constant c_0, the following estimate:

$$\frac{d}{dt}\mathcal{L}(t) \le -c_0 E(t) + C\left[k_1^2(t)\left|u^0(L)\right|^2 + k_2^2(t)\left|v^0(L)\right|^2\right]. \tag{3.20}$$

Also, by direct computations, it is easy to check that, for N large, we have

$$\frac{N}{2}E(t) \le \mathcal{L}(t) \le 2NE(t). \tag{3.21}$$

Therefore (3.20) becomes

$$\frac{d}{dt}\mathcal{L}(t) \leq -\omega\mathcal{L}(t) + C\left[k_1^2(t)|u^0(L)|^2 + k_2^2(t)|v^0(L)|^2\right], \tag{3.22}$$

for some positive constants ω. At this point we distinguish two cases.

Case 1. If $u^0(L) = v^0(L) = 0$, then (3.22) reduces

$$\frac{d}{dt}\mathcal{L}(t) \leq -\omega\mathcal{L}(t), \quad \forall t \geq 0. \tag{3.23}$$

A simple integration over $(0,t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\omega t}, \quad \forall t \geq 0. \tag{3.24}$$

By using (3.21), estimate (3.5) is proved.

Case 2. If $u^0(L) \neq 0$ or $v^0(L) \neq 0$, then (3.22) gives

$$\frac{d}{dt}\mathcal{L}(t) \leq -\omega\mathcal{L}(t) + C_1k_1^2(t) + C_2k_2^2(t), \tag{3.25}$$

where

$$C_1 = C|u^0(L)|^2, \qquad C_2 = C|v^0(L)|^2. \tag{3.26}$$

In this case we introduce the following functional:

$$F(t) := \mathcal{L}(t) - C_1e^{-\omega t}\int_0^t k_1^2(t)e^{\omega s}ds - C_2e^{-\omega t}\int_0^t k_2^2(t)e^{\omega s}ds. \tag{3.27}$$

A simple differentiation of F, using (3.25), leads to

$$\frac{dF}{dt}(t) \leq -\omega F(t). \tag{3.28}$$

Again a simple integration over $(0,t)$ yields

$$F(t) \leq F(0)e^{-\omega t}, \quad \forall t \geq 0. \tag{3.29}$$

A combination of (3.21), (3.27), and (3.29) then yields the estimate (3.6). This completes the proof of Theorem 3.1.

4. Polynomial Decay

In this section we study the asymptotic behavior of the solutions of system (1.5)–(1.9) when the resolvent kernels k_i $(i = 1, 2)$ satisfy

$$k_i(0) > 0, \qquad k_i(t) \geq 0, \qquad k_i'(t) \leq 0, \qquad k_i''(t) \geq \gamma_i(-k_i'(t))^{1+q_i}, \tag{4.1}$$

for $(q_1, q_2) \neq (0, 0)$, $0 \leq q_i < 1/2$, and some positive constants γ_i. These assumptions imply that k_i' decays polynomially to 0 if $q_i > 0$. That is,

$$0 \leq -k_i'(t) \leq C(1 + t)^{-1/q_i}. \tag{4.2}$$

The following lemmas will play an important role in the sequel.

Lemma 4.1. *(see [13]) Let $p > 1$, $0 \leq r < 1$, and $t \geq 0$. Then for $0 < r$,*

$$\begin{aligned}
&(|k'|o\phi(L,t))^{1+1/(1-r)(1+p)} \\
&\quad\leq 2\left(\|\phi\|^2_{L^\infty(0,T,H^1(0,L))}\int_0^t |k'(s)|^r ds\right)^{1/(1-r)(1+p)}\left(|k'|^{1+1/(1+p)}o\phi(L,t)\right),
\end{aligned} \tag{4.3}$$

and for $r = 0$,

$$\begin{aligned}
&(|k'|o\phi(L,t))^{1+1/(1+p)} \\
&\quad\leq 2\left(\int_0^t \|\phi(\cdot,s)\|^2_{H^1(0,L)} ds + t\|\phi(\cdot,t)\|^2_{H^1(0,L)}\right)^{1/(1+p)}\left(|k'|^{1+1/(1+p)}o\phi(L,t)\right),
\end{aligned} \tag{4.4}$$

where

$$p = \frac{1}{q} - 1. \tag{4.5}$$

Theorem 4.2. *Given that $((u^0, v^0), (u^1, v^1, \theta^0)) \in (V^2 \times (L^2(0,L))^3)$, assume that b small number and (4.1) hold. Then there exists a positive constant $\lambda > 0$, for which the energy E satisfies, for all $t \geq 0$, the following decay estimates:*

$$E(t) \leq \frac{\lambda}{(1+t)^{1/q}}, \quad if\, u^0(L) = v^0(L) = 0 \; on. \tag{4.6}$$

Otherwise,

$$E(t) \leq \frac{\lambda}{(1+t)^{(1-r)/q}}\left[1 + \int_0^t (k_1^2(s) + k_2^2(s))(1+s)^{(1-r)(p+1)} ds\right], \tag{4.7}$$

where

$$q = \max\{q_1, q_2\}, \qquad \frac{1}{p+1} < r < \frac{p}{p+1}, \quad p = \min\{p_1, p_2\}. \tag{4.8}$$

Moreover, if

$$\int_0^{\infty} \left(k_1^2(s) + k_2^2(s)\right)(1+s)^{(1-r)(p+1)} ds < +\infty \tag{4.9}$$

for some r*, satisfying* (4.8)*, then* (4.7) *reduces to* (4.6)*.*

Proof. By using (4.1) in (3.7), we easily see that

$$\begin{aligned}
\frac{dE}{dt} \leq & -\kappa \int_0^L |\theta_x|^2 dx - \frac{\eta_1}{2} u_t^2(L,t) + \frac{\eta_1}{2} k_1^2(t) \left|u^0(L)\right|^2 \\
& + \frac{\eta_1}{2} k_1'(t) |u(L,t)|^2 - \frac{\eta_1 \gamma_1}{2} \left(-k_1'(t)\right)^{1+q_1} o u(L,t) \\
& - \frac{\eta_2}{2} v_t^2(L) + \frac{\eta_2}{2} k_2^2(t) \left|v^0(L)\right|^2 \\
& + \frac{\eta_2}{2} k_2'(t) |v(L)|^2 - \frac{\eta_2 \gamma_2}{2} \left(-k_2'(t)\right)^{1+q_2} o v(L,t).
\end{aligned} \tag{4.10}$$

By defining the functional $\mathcal{L}(t)$ as in (3.16), we get

$$\begin{aligned}
\frac{d\mathcal{L}}{dt}(t) \leq & -\frac{\kappa N}{2} \int_0^L |\theta_x|^2 dx - \frac{N\eta_1}{8} u_t^2(L,t) + N\eta_1 k_1^2(t) \left|u^0(L)\right|^2 \\
& - \frac{N\eta_1\gamma_1}{2} \left(-k_1'(t)\right)^{1+q_1} o u(L,t) - \frac{N\eta_2}{8} v_t^2(L,t) + N\eta_2 k_2^2(t) \left|v^0(L)\right|^2 \\
& - \frac{N\eta_2\gamma_2}{2} \left(-k_2'(t)\right)^{1+q_2} o v(L,t) - \varepsilon_0 \int_0^L \rho u_t^2 dx - \frac{(1-\varepsilon_0)}{8} \mu \int_0^L u_x^2 dx \\
& - \varepsilon_0 \int_0^L J v_t^2 dx - \frac{(1-\varepsilon_0)}{8} \alpha \int_0^L v_x^2 dx \\
& + C_\varepsilon \left(\left(-k_1' \odot u\right)^2 + \left(-k_2' \odot v\right)^2 \right)(L,t).
\end{aligned} \tag{4.11}$$

Applying inequality (2.14) for k_i' with $\mu = (p_1+2)/2(p_1+1)$ if $i=1$ and $\mu = (p_2+2)/2(p_2+1)$ if $i=2$, we get

$$\left|k_1' \odot u\right|^2 \leq C\left(-k_1'\right)^{1+q_1} o u, \qquad \left|k_2' \odot v\right|^2 \leq C\left(-k_2'\right)^{1+q_2} o v\,. \tag{4.12}$$

Using the above inequalities and taking N large, then for some positive constant c_1, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \leq &- c_1\bigg[\int_0^L |\theta|^2 dx + \int_0^L \rho u_t^2 dx + \int_0^L \mu u_x^2 dx + \int_0^L J v_t^2 dx + \int_0^L \alpha v_x^2 dx \\ &+ (-k_1'(t))^{1+q_1} ou(L,t) + (-k_2'(t))^{1+q_2} ov(L,t)\bigg] \\ &+ C_\varepsilon\left(k_1^2(t)|u^0(L)|^2 + k_2^2(t)|v^0(L)|^2\right) - \frac{N\eta_1}{8}u_t^2(L,t) - \frac{N\eta_2}{8}v_t^2(L,t).\end{aligned} \tag{4.13}$$

Now, we fix $0 < r < 1$ such that $1/(p+1) < r < p/(p+1)$ and $p = \min\{p_1, p_2\}$. From (4.1) we get

$$\begin{gathered}\int_0^t |k_i'(s)|^r ds \leq C\textstyle\int_0^\infty (1+t)^{-r/q_i} ds < \infty, \quad i = 1,2. \\ [-k_1'(t) ou(L)]^{1+1/(1-r)(p+1)} \leq C[(-k_1'(t))^{1+1/(p_1+1)} ou(L)] \\ [-k_2'(t) ov(L)]^{1+1/(1-r)(p+1)} \leq C[(-k_2'(t))^{1+1/(p_2+1)} ov(L)].\end{gathered} \tag{4.14}$$

Consequently we have

$$\begin{aligned}CE(t)^{1+1/(1-r)(p+1)} \leq &\int_0^L \left[\rho|u_t|^2 + J|v_t|^2 + c|\theta|^2 + \mu|u_x|^2 + [\alpha|v_x|^2\right] dx \\ &+ [(-k_1'(t))^{1+q_1} ou(L)] + [(-k_2'(t))^{1+q_2} ov(L)].\end{aligned} \tag{4.15}$$

Inserting (4.15) into (4.13) and using (3.21), we deduce that

$$\frac{d}{dt}\mathcal{L}(t) \leq -C(\mathcal{L}(t))^{1+1/(1-r)(p+1)} + C_\varepsilon\left(k_1^2(t)|u^0(L)|^2 + k_2^2(t)|v^0(L)|^2\right). \tag{4.16}$$

Here, we distinguish two cases.

Case 1 ($u^0(L) = v^0(L) = 0$). In this case (4.16) reduces to

$$\frac{d}{dt}\mathcal{L}(t) \leq -C(\mathcal{L}(t))^{1+1/(1-r)(p+1)}. \tag{4.17}$$

A simple integration over $(0,t)$ gives

$$\mathcal{L}(t) \leq \frac{C}{(1+t)^{(1-r)(p+1)}}, \quad \forall t \geq 0. \tag{4.18}$$

As a consequence of (4.18) and the fact that $(1-r)(p+1) > 1$, we easily verify that

$$\int_0^\infty \mathcal{L}(t)dt + \sup_{t\geq 0} t\mathcal{L}(t) < \infty. \tag{4.19}$$

Therefore, by using Lemma 4.1, (3.3), and (3.21), we have

$$\begin{aligned} [|k_1'|ou(L)]^{1+1/(1+p)} &\leq C\left[|k_1'|^{1+1/(1+p_1)}ou(L)\right] \\ [|k_2'|ov(L)]^{1+1/(1+p)} &\leq C\left[|k_2'|^{1+1/(1+p_2)}ov(L)\right], \end{aligned} \tag{4.20}$$

which implies that

$$\begin{aligned} \left[|k_1'|^{1+1/(1+p_1)}ou(L)\right]^{(p+1)/(p+2)} &\geq C[|k_1'|ou(L)] \\ \left[|k_2'|^{1+1/(1+p_2)}ov(L)\right]^{(p+1)/(p+2)} &\geq C[|k_2'|ov(L)]. \end{aligned} \tag{4.21}$$

Consequently we have

$$\begin{aligned} CE(t)^{1+1/(p+1)} \leq \int_0^L &\rho|u_t|^2 + J|v_t|^2 + c|\theta|^2 + \mu|u_x|^2 + [\alpha|v_x|^2]dx \\ &+ [(-k_1'(t))^{1+q_1}ou(L)] + [(-k_2'(t))^{1+q_2}ov(L)]. \end{aligned} \tag{4.22}$$

Inserting (4.22) into (4.13), with $u^0(L) = v^0(L) = 0$, we deduce that

$$\frac{d}{dt}\mathcal{L}(t) \leq -C(\mathcal{L}(t))^{1+1/(p+1)}. \tag{4.23}$$

A simple integration then leads to

$$\mathcal{L}(t) \leq \frac{C}{(1+t)^{p+1}} = \frac{C}{(1+t)^{1/q}}, \quad \forall t \geq 0. \tag{4.24}$$

where $q = \max\{q_1, q_2\}$. By using (3.21), estimate (4.6) is established.

Case 2 ($u^0(L) \neq 0$ or $v^0(L) \neq 0$). In this case (4.16) gives that

$$\frac{d}{dt}\mathcal{L}(t) \leq -C(\mathcal{L}(t))^{1+1/(1-r)(p+1)} + C_1k_1^2(t) + C_2k_2^2(t), \tag{4.25}$$

where

$$C_1 = C_\varepsilon|u^0(L)|^2, \qquad C_2 = C_\varepsilon|v^0(L)|^2. \tag{4.26}$$

We then introduce the following functional $H(t) := \mathcal{L}(t) - g(t)$, where

$$\begin{aligned} g(t) &:= C_1(1+t)^{-(1-r)(p+1)}\int_0^t k_1^2(s)(1+s)^{(1-r)(p+1)}ds \\ &\quad + C_2(1+t)^{-(1-r)(p+1)}\int_0^t k_2^2(s)(1+s)^{(1-r)(p+1)}ds. \end{aligned} \tag{4.27}$$

By using (4.1), it is easy to show that, for some $t_0 > 0$, we have

$$\begin{aligned} g(t)^{1+1/(1-r)(p+1)} &= (1+t)^{-1-(1-r)(p+1)} \\ &\quad \times \left[\int_0^t (C_1k_1^2(s) + C_2k_2^2(s))(1+s)^{(1-r)(p+1)}ds\right]^{1+1/(1-r)(p+1)} \\ &\geq \gamma_0(1+t)^{-1-(1-r)(p+1)} \\ &\quad \times \left[\int_0^t (C_1k_1^2(s) + C_2k_2^2(s))(1+s)^{(1-r)(p+1)}ds\right], \quad \forall t \geq t_0, \end{aligned} \tag{4.28}$$

where

$$\gamma_0 = \left[\int_0^{t_0} (C_1k_1^2(s) + C_2k_2^2(s))(1+s)^{(1-r)(p+1)}ds\right]^{1/(1-r)(p+1)}. \tag{4.29}$$

Hence, simple calculations give

$$g'(t) + Cg(t)^{1+1/(1-r)(p+1)} \geq C_1k_1^2(t) + C_2k_2^2(t), \quad \forall t \geq t_0. \tag{4.30}$$

Thanks to (4.25)–(4.30), we have

$$H'(t) \leq -C[\mathcal{L}(t)^{1+1/(1-r)(p+1)} - g(t)^{1+1/(1-r)(p+1)}]. \tag{4.31}$$

By using the fact that

$$\mathcal{L}(t)^{1+1/(1-r)(p+1)} = [H(t) + g(t)]^{1+1/(1-r)(p+1)} \geq [H(t)^{1+1/(1-r)(p+1)} + g(t)^{1+1/(1-r)(p+1)}] \tag{4.32}$$

estimate (4.31) gives

$$H'(t) \leq -CH(t)^{1+1/(1-r)(p+1)}, \quad \forall t \geq t_0. \tag{4.33}$$

A simple integration of (4.33) over (t_0, t) then leads to

$$H(t) \leq \frac{C}{(1+t)^{(1-r)(p+1)}}, \quad \forall t \geq t_0. \tag{4.34}$$

Therefore, using (4.27) we get, for all, $t \geq t_0$,

$$\mathcal{L}(t) \leq \frac{C}{(1+t)^{(1-r)(p+1)}} \left[1 + \int_0^t \left(k_1^2(s) + k_2^2(s)\right)(1+s)^{(1-r)(p+1)} ds \right]. \tag{4.35}$$

Again, recalling (3.21) and using the continuity and the boundedness of E, estimate (4.7) is established.

If, in addition,

$$\int_0^\infty \left(k_1^2(s) + k_2^2(s)\right)(1+s)^{(1-r)(p+1)} ds < +\infty \tag{4.36}$$

for some r, satisfying (4.8) then (4.35) takes the form

$$\mathcal{L}(t) \leq \frac{C}{(1+t)^{(1-r)(p+1)}}. \tag{4.37}$$

By repeating (4.18)–(4.24), the desired estimate is established. □

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