ON A CERTAIN FUNCTIONAL EQUATION IN THE ALGEBRA OF POLYNOMIALS WITH COMPLEX COEFFICIENTS

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Many analytical problems can be reduced to determining the number of roots of a polynomial in a given disc. In turn, the latter problem admits further reduction to the generalized Rauss-Hurwitz problem of determining the number of roots of a polynomial in a semiplane. However, this procedure requires complicated coefficient transformations. In the present paper we suggest a *direct* method to evaluate the number of roots of a polynomial with complex coefficients in a disc, based on studying a certain equation in the algebra of polynomials. An application for computing the rotation of plane polynomial vector fields is also given.

1. Functional equations: basic properties of solutions

Let

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0, \tag{1.1}$$

$$F(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}, \quad b_0 \neq 0,\ b_{n+1} \neq 0, \tag{1.2}$$

be polynomials with complex coefficients $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_{n+1}$ of degree n and $n+1$, respectively. Assume the polynomials f and F to satisfy the functional equation

$$(a + bz) f(z) + (c + dz) f^*(z) = F(z), \tag{1.3}$$

where a, b, c, d are certain complex numbers and the polynomial f^* is defined by

$$f^*(z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n = z^n \overline{f\left(\frac{1}{\bar{z}}\right)}. \tag{1.4}$$

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Consider along with (1.3) the following functional equation:

$$g(z)\cdot f(z) = (\bar{a}z+\bar{b})F(z) - (c+dz)F^*(z), \tag{1.5}$$

where

$$g(z) = \alpha z^2 + \beta z + \bar{\alpha}, \quad \alpha = \bar{a}b - \bar{c}d,\ \beta = |a|^2 + |b|^2 - |c|^2 - |d|^2, \tag{1.6}$$

$$F^*(z) = \bar{b}_0 z^{n+1} + \bar{b}_1 z^n + \cdots + \bar{b}_{n+1}. \tag{1.7}$$

Lemma 1.1. *If the polynomials f and F satisfy the functional equation (1.3), then they also satisfy the functional equation (1.5).*

Conversely, if at least one of the numbers α, β is different from zero and f, F satisfy (1.5), then they satisfy (1.3) as well.

Proof. Let polynomials (1.1) and (1.2) satisfy (1.3). By the definition of f^* and F^*, one has

$$(\bar{a}z+\bar{b})f^*(z) + (\bar{c}z+\bar{d})f(z) = F^*(z). \tag{1.8}$$

Multiplying (1.3) (resp., (1.8)) by $(\bar{a}z+\bar{b})$ (resp., by $(c+dz)$) and taking the difference of the obtained expressions, one obtains (1.5), where the coefficients α and β are defined by (1.6). Thus the first implication is established.

Conversely, assume that f and F satisfy (1.5), and at least one of α, β is different from zero. Since the coefficient β of g is real (cf. [4]), one has

$$z^2\overline{g\left(\frac{1}{\bar{z}}\right)} = g(z). \tag{1.9}$$

Therefore, it follows from (1.5) that

$$g(z)f^*(z) = (a+bz)F^*(z) - (\bar{c}z+\bar{d})F(z). \tag{1.10}$$

Multiplying (1.5) (resp., (1.10)) by $(a+bz)$ (resp., by $(c+dz)$) and summing up the obtained expressions, one arrives at the following equality:

$$g(z)\left[(a+bz)f(z) + (c+dz)f^*(z)\right] = g(z)F(z). \tag{1.11}$$

Since $g(z) \not\equiv 0$ and the algebra of polynomials does not contain zero divisors, it follows that f and F satisfy (1.3).

The lemma is completely proved. □

Assume that the polynomials f and F satisfy (1.3). It follows from (1.1)–(1.3) that (1.3) is equivalent to the following system:

$$aa_k + c\bar{a}_{n-k} + ba_{k-1} + d\bar{a}_{n-k+1} = b_k, \quad k = 0,1,\ldots,n+1, \tag{1.12}$$

where we put $a_k = 0$ for $k < 0$ and $k > n$.

Similarly, (1.5) is equivalent to the following system:

$$\bar{\alpha}a_k + \beta a_{k-1} + \alpha a_{k-2} = \bar{a}b_{k-1} - d\bar{b}_{n-k+2} + \bar{b}b_k - c\bar{b}_{n-k+1}, \quad k = 0,1,\dots,n+2, \tag{1.13}$$

where $a_k = b_k = 0$ for $k < 0$ and $a_k = b_{k+1} = 0$ for $k > n$.

Thus, under the assumption that a, b, c, and d satisfy the condition $|\alpha| + |\beta| > 0$, system (1.3) is equivalent to (1.12) as well as to (1.13).

Below we will list some properties of solutions to (1.3).

(1) (a) The coefficients a, b, c, d along with the polynomial f determine the polynomial F uniquely.
(b) If the polynomials f and F are defined by (1.1) and (1.2) and satisfy (1.3), then the coefficients a, b, c, d satisfy the conditions

$$|a| + |c| > 0, \qquad |b| + |d| > 0. \tag{1.14}$$

(c) If a collection (a,b,c,d,f,F) of numbers a, b, c, d, and polynomials f, F satisfy (1.3), then so is the collection $(\lambda a, \lambda b, \bar{\lambda}c, \bar{\lambda}d, f/\lambda, F)$ for any complex number $\lambda \neq 0$.

(2) Given a polynomial F and numbers a, b, c, and d, satisfying $|\alpha| + |\beta| > 0$, there exists a unique f satisfying (1.3). Indeed, if $\alpha \neq 0$, then the first $n+1$ equations of system (1.13) completely determine the coefficients $a_0, a_1, \dots, a_n$ of the polynomial f. If, however, $\alpha = 0$ and $\beta \neq 0$, then all the coefficients $a_0, a_1, \dots, a_n$ of the polynomial f are completely determined by $n+1$ equations of system (1.13) starting with the second one.

(3) It follows from (1.5) that the roots of $g(z)$ turn out to be the roots of $G(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z)$. Also, if $\alpha \neq 0$ and $\beta^2 \neq 4|\alpha|^2$, then $z_0, z_1 = 1/\bar{z}_0$, where

$$z_0 = \frac{-\beta + \sqrt{\beta^2 - 4|\alpha|^2}}{2\alpha}, \tag{1.15}$$

are the roots of $g(z)$. Therefore, F and the coefficients a, b, c, d are connected by the following relations:

$$(\bar{a}z_j + \bar{b})F(z_j) - (c + dz_j)F^*(z_j) = 0, \quad j = 0,1. \tag{1.16}$$

If $\alpha \neq 0$ and $\beta^2 = 4|\alpha|^2$, then $z_1 = z_0$ is a multiple root of $g(z)$, and, therefore,

$$\begin{gathered} (\bar{a}z_0 + \bar{b})F(z_0) - (c + dz_0)F^*(z_0) = 0, \\ (\bar{a}z_0 + \bar{b})F'(z_0) + \bar{a}F(z_0) - (c + dz_0)F^{*'}(z_0) - dF^*(z_0) = 0. \end{gathered} \tag{1.17}$$

Finally, if $\alpha = 0$ and $\beta \neq 0$, then the linear function $g(z) = \beta z$ has the only root $z_0 = 0$. Hence it follows from (1.5) (see also (1.10)) that

$$\begin{gathered} \bar{b}F(0) - cF^*(0) = 0, \\ \bar{d}F(0) - aF^*(0) = 0. \end{gathered} \tag{1.18}$$

2. Functional equations: solubility conditions

Given the polynomial (1.2), consider the solubility problem for the functional equation (1.3) with respect to unknown coefficients a, b, c, d and a polynomial f. To treat the above problem, we will use the necessary conditions for the solubility of (1.3) given by (3.1)–(3.8) (depending on whether the root z_0 of $g(z)$ satisfies $0 < |z_0| < 1$, $|z_0| = 1$, or $z_0 = 0$).

Assume z_0 and z_1 to be given and consider system (1.16) with respect to unknown a, b, c, d. We will try to find a solution to (1.16) in such a way that (1.3) will have a solution with respect to f. Also, given z_0, we will follow the same way regarding system (1.17).

To describe the solubility conditions for the functional equation (1.3), it is convenient to introduce the notion of a regular point.

A point z is called *regular* with respect to the polynomial F if the following conditions are satisfied:

$$\begin{gathered} F(z)\cdot F^*(z)\neq 0, \quad \left|F(z)\right| \neq \left|F^*(z)\right|, \quad \text{for } |z|\neq 1, \\ (n+1)\left|F(z)\right|^2 \neq 2\Re\left[\overline{F(z)}F'(z)z\right], \quad \text{for } |z| = 1. \end{gathered} \tag{2.1}$$

Observe that the notion of a regular point is introduced with respect to the unit circle. It follows immediately from the definition of a regular point that if z_0 is regular, then so is $z_1 = 1/\bar{z}_0$ and vice versa.

According to the definition of polynomial F^*, the rational function

$$A(z) = \frac{F^*(z)}{F(z)} \tag{2.2}$$

satisfies the identity

$$\overline{A(z)}\cdot A\left(\frac{1}{\bar{z}}\right) \equiv 1. \tag{2.3}$$

In addition, $|A(z)| \neq 1$ for all regular points $z, |z| \neq 1$.

Assume that z_0 is a regular point of F, $A_0 = A(z_0)$, $A_1 = 1/\bar{A}_0$, and σ_0, σ_1 are arbitrary complex numbers. Consider the linear system

$$\begin{aligned} \bar{a}z_0 + \bar{b} &= \sigma_0 A_0, \qquad & \bar{a} + \bar{b}\bar{z}_0 &= \sigma_1 A_1, \\ c + dz_0 &= \sigma_0, \qquad & c\bar{z}_0 + d &= \sigma_1, \end{aligned} \tag{2.4}$$

with unknown a, b, c, and d. It should be pointed out that any solution to system (2.4) is also a solution to (1.16) for $z_0 \neq 0$, as well as a solution to (1.18) for $z_0 = 0$.

For $|z_0| < 1$ system, (2.4) has the unique solution

$$\begin{aligned} \bar{a}\left(1 - |z_0|^2\right) &= \sigma_1 A_1 - \bar{z}_0\sigma_0 A_0, \qquad & \bar{b}\left(1 - |z_0|^2\right) &= \sigma_0 A_0 - \bar{z}_0\sigma_1 A_1, \\ c\left(1 - |z_0|^2\right) &= \sigma_0 - z_0\sigma_1, \qquad & d\left(1 - |z_0|^2\right) &= \sigma_1 - \bar{z}_0\sigma_0. \end{aligned} \tag{2.5}$$

With the above a, b, c, and d on hands, the coefficients α, β from formula (1.6) satisfy the equalities

$$\alpha\left(1-|z_0|^2\right)=\bar{z}_0\left(|A_1|^2-1\right)\left(|\sigma_0 A_0|^2-|\sigma_1|^2\right), \tag{2.6}$$

$$\beta\left(1-|z_0|^2\right)=-\left(|A_1|^2-1\right)\left(|\sigma_0 A_0|^2-|\sigma_1|^2\right). \tag{2.7}$$

Theorem 2.1. *Let z_0 with $|z_0|<1$ be a regular point of the polynomial F, and assume the numbers σ_0 and σ_1 to satisfy the condition*

$$|\sigma_0 A_0| \neq |\sigma_1|. \tag{2.8}$$

Let, further, a, b, c, and d be defined by (2.5). Then the functional equation (1.3) admits a solution $f(z)$.

Proof. To begin with, consider the case $z_0 \neq 0$. It follows from (2.8) and (2.6) that $\alpha \neq 0$. Formulae (2.4) and (2.3) provide that z_0 and $z_1 = 1/\bar{z}_0$ are the roots of the polynomial $g(z)$ (1.6):

$$\begin{gathered} g(z_0)=(a+bz_0)(\bar{a}z_0+\bar{b})-(c+dz_0)(\bar{c}z_0+\bar{d})=\bar{\sigma}_1\bar{A}_1\sigma_0A_0-\sigma_0\bar{\sigma}_1=\bar{\sigma}_1\sigma_0-\sigma_0\bar{\sigma}_1=0, \\ g(z_1)=z_1^2\overline{g(z_0)}=0. \end{gathered} \tag{2.9}$$

On the other hand, the numbers z_0 and z_1 are the roots of the polynomial

$$G(z)=(\bar{a}z+\bar{b})F(z)-(c+dz)F^*(z). \tag{2.10}$$

Therefore, by the Bezout theorem, the rational function

$$f(z)=\frac{G(z)}{g(z)} \tag{2.11}$$

is, in fact, a polynomial satisfying the functional equation (1.5). Now the statement of the theorem in the considered case follows from Lemma 1.1.

Assume now $z_0=0$. The condition (2.8) and equalities (2.6), (2.7) yield $\alpha=0$ and $\beta \neq 0$, that is, $g(z)=\beta z$ is a linear function. In addition, $z=0$ is a root of the polynomial $G(z)$. Therefore, $f(z)=G(z)/g(z)$ is a polynomial satisfying (1.5), and again the statement of the theorem in the considered case follows from Lemma 1.1.

Theorem 2.1 is completely proved. □

Assume now that a regular point z_0 of the polynomial F belongs to the unit circle $|z|=1$. Given numbers c and d, consider system (1.18) with unknown a, b. By solving system (1.18) one obtains

$$\bar{a}F^2=\Delta\cdot c+(FF^*+z\Delta)d, \qquad \bar{b}F^2=(FF^*-z\Delta)c-z^2\Delta d, \tag{2.12}$$

where

$$\begin{gathered}\Delta = z^n\Delta_0, \quad \Delta_0 = (n+1)|F|^2 - 2\Re[\bar{F}F'z], \\ F = F(z), \quad F' = F'(z), \quad F^* = F^*(z), \quad z = z_0.\end{gathered} \tag{2.13}$$

Take the coefficients a, b, c, and d satisfying (2.12) and define the polynomial $g(z)$ by means of formula (1.6). Formula (2.12) provides the following relations for α and β:

$$z\alpha|F|^4 = \Delta_0\big[|F|^2(|c|^2 - |d|^2) - \Delta_0|c+zd|^2\big], \tag{2.14}$$

$$\beta|F|^4 = 2\Delta_0\big[\Delta_0|c+zd|^2 - |F|^2(|c|^2 - |d|^2)\big]. \tag{2.15}$$

Theorem 2.2. *Let $z_0, |z_0| = 1$ be a regular point of the polynomial F and let the numbers c, d satisfy the condition:*

$$\Delta_0\left|c+z_0d\right|^2 \neq \left|F(z_0)\right|^2(|c|^2 - |d|^2). \tag{2.16}$$

If the coefficients a, b, c, and d satisfy relation (2.12), then the functional equation (1.3) has the unique solution (a,b,c,d,f).

Proof. It follows from (1.6) and (2.15)–(3.1) that $\beta = -2\alpha z_0 \neq 0$. Since the coefficient β is real and $z_0\bar{z}_0 = 1$, one obtains the equality $\bar{\alpha} = \alpha z_0^2$, that is, $g(z) \equiv \alpha(z-z_0)^2$. At the same time, z_0 is a multiple root to the polynomial $G(z) = (\bar{a}z+\bar{b})F(z) - (c+dz)F^*(z)$. Therefore, the rational function $f(z) = G(z)/g(z)$ is, in fact, a polynomial satisfying the functional equation (1.5). To complete the proof of Theorem 2.1 it remains to apply Lemma 1.1. The theorem follows. □

Combining Theorems 2.1 and 2.2 with property (2) of solutions to (1.3) one can effectively compute the numbers a, b, c, d and the coefficients $a_0, a_1, \dots, a_n$ of the polynomial f. Indeed, assume, for instance, that $z = 0$ is a regular point of the polynomial F. Then the regularity condition for the point $z = 0$ along with condition (2.8) take the form

$$b_0 \cdot b_{n+1} \neq 0, \qquad \left|b_0\right| \neq \left|b_{n+1}\right|, \qquad \left|cb_{n+1}\right| \neq \left|db_0\right|; \tag{2.17}$$

also, the equalities (2.6) and (2.7) take the form

$$\bar{a}b_{n+1} = d\bar{b}_0, \qquad \bar{b}b_0 = c\bar{b}_{n+1}. \tag{2.18}$$

In this case the leading coefficient α of the polynomial $g(z)$, that is determined by a, b, c, d, is equal to zero. Hence, in order to determine unknown coefficients $a_0, a_1, \dots, a_n$ from system (1.13), one has

$$\begin{gathered} 0 = \bar{b}b_0 - c\bar{b}_{n+1}, \\ \beta a_0 = \bar{a}b_0 - d\bar{b}_{n+1} + \bar{b}b_1 - c\bar{b}_n, \\ \beta a_{k-1} = \bar{a}b_{k-1} - d\bar{b}_{n-k+2} + \bar{b}b_k - c\bar{b}_{n-k+1}, \quad k = 2,\dots,n, \\ \beta a_n = \bar{a}b_n - d\bar{b}_1 + \bar{b}b_{n+1} - c\bar{b}_0, \\ 0 = \bar{a}b_{n+1} - d\bar{b}_0. \end{gathered} \tag{2.19}$$

Observe that the first and the last equations in (2.19) coincide with (2.18). According to (2.17) and (2.18), one obtains the following relation for the coefficient β of the polynomial $g(z)$:

$$\beta = \left(\left| b_{n+1} \right|^2 - \left| b_0 \right|^2 \right) \left(\frac{|c|^2}{\left| b_0 \right|^2} - \frac{|d|^2}{\left| b_{n+1} \right|^2} \right) \neq 0. \tag{2.20}$$

Thus the coefficients $a_0, a_1, \dots, a_n$ of the polynomial f can be uniquely determined from system (2.19).

It follows from Theorems 2.1 and 2.2 that the existence of a regular point for the polynomial F is a sufficient condition for the existence of a solution to the functional equation (1.3).

It turns out that the existence of a regular point for the polynomial F is intimately connected to the linear (in)dependence of the polynomials F and F^* in the complex linear space of (complex) polynomials. To be more precise, *there exists a regular point for F if and only if F and F^* are linearly independent.* This statement is a direct consequence of the following lemma.

Lemma 2.3. *The following conditions are equivalent:*

(a) *F and F^* are linearly independent in the complex linear space of (complex) polynomials;*

(b) *the identity $|F(z)| \equiv |F^*(z)|$ is satisfied;*

(c) *the identity*

$$2\,\mathrm{Re}\left[\overline{F(w)}F'(w)w\right] \equiv (n+1)\left| F(w) \right|^2 \quad \forall |w| = 1 \tag{2.21}$$

is satisfied.

Proof. Assume (a) is satisfied: $F^* = C \cdot F$ for some nonzero complex number C. Then, according to the definition of the polynomial F^*, one has the following equality for the coefficients $b_0, b_{n+1} : \bar{b}_0 = Cb_{n+1}$, $\bar{b}_{n+1} = Cb_0$, from which it follows that $|C| = 1$, and therefore, $|F^*(z)| \equiv |F(z)|$.

Thus (a) implies (b).

Assume, further, $|F(z)| \equiv |F^*(z)|$. Applying to this identity the change of coordinates $z = rw$, $r \geq 0$, $|w| = 1$ and using the definition of F^*, one obtains

$$r^{2n+2}F(r^{-1}w)\overline{F(r^{-1}w)} \equiv F(rw)\overline{F(rw)}. \tag{2.22}$$

Differentiating the last identity with respect to the real argument r at the point $r = 1$ we obtain (c).

Thus (b) implies (c).

Finally, assume that (c) is satisfied and show that F and F^* are linearly dependent. Consider the function $\overline{F(w)}/F(w)$ on the unit circle $|w| = 1$, where $F(w) \neq 0$, and show that this function has a continuous extension over the unit circle. Assume that F vanishes at some point w_0 belonging to the unit circle. Then we have the following representation: $F(z) = (z - w_0)^k F_1(z)$, where $k \geq 1$ is an integer, $F_1(w_0) \neq 0$.

The conditions

$$\lim_{w\to w_0}\frac{\bar{w}-\bar{w}_0}{w-w_0}=-\frac{1}{w_0^2},\qquad |w|=1, \tag{2.23}$$

yield

$$\lim_{w\to w_0}\frac{\overline{F(w)}}{F(w)}=(-1)^k w_0^{-2k}\frac{\overline{F_1(w_0)}}{F_1(w_0)},\qquad |w|=1, \tag{2.24}$$

providing the continuity of the function $\overline{F(w)}/F(w)$.

Differentiating the function $w^{n+1}\overline{F(w)}/F(w)$, $w=\exp(it)$ with respect to t at the points where $F(w)\neq 0$ and using condition (c), one obtains

$$\frac{d}{dt}\frac{w^{n+1}\overline{F(w)}}{F(w)}=iw^{n+1}\left\{\frac{(n+1)\overline{F(w)}-\overline{F'(w)w}}{F(w)}-\frac{\overline{F(w)}F'(w)w}{F^2(w)}\right\}=0. \tag{2.25}$$

The above equality along with the continuity of the function $w^{n+1}\overline{F(w)}/F(w)$ on the unit circle yield that the latter function is, in fact, constant, that is $w^{n+1}\overline{F(w)}=C\cdot F(w)$ or, equivalently, $F^*(w)=C\cdot F(w)$, $|w|=1$. Therefore, the coefficients of the polynomials $C\cdot F$ and F^* coincide. The linear dependence of the polynomials F and F^* is established and the proof of Lemma 2.3 is complete. □

3. An algorithm for computing the number of roots in the unit circle

In what follows we will be interested in the case when the coefficients a, b, c, d satisfy the following additional condition:

$$|a+bw|\geq|c+dw|,\qquad |w|=1. \tag{3.1}$$

LEMMA 3.1. *Linear functions $a+bz$, $c+dz$ satisfy condition (3.1) if and only if the numbers $\alpha=\bar{a}b-\bar{c}d$, $\beta=|a|^2+|b|^2-|c|^2-|d|^2$ satisfy the inequality*

$$2|\alpha|\leq\beta. \tag{3.2}$$

Proof. The equality

$$|a+bz|^2-|c+dz|^2=|a|^2+2\Re(\bar{a}bz)+|b|^2|z|^2-|c|^2-2\Re(\bar{c}dz)-|d|^2|z|^2 \tag{3.3}$$

yields, for $z=e^{it}$, $t\in[0,2\pi]$,

$$|a+bz|^2-|c+dz|^2=\beta+2\Re[\alpha e^{it}]. \tag{3.4}$$

Combining this with the equality

$$\min_t\Re[\alpha e^{it}]=-|\alpha|, \tag{3.5}$$

one obtains the equivalence of conditions (3.1) and (3.2). The lemma is proved. □

Observe that Lemma 3.1 allows one to verify effectively the validity of condition (3.1) for the coefficients a, b, c, d determined by regular points of the polynomial F.

Lemma 3.2. *Let $z_0, |z_0| \leq 1$, be a regular point of the polynomial F. Assume that the numbers a, b, c, d are determined by equalities (2.5) with*

$$(|A_0| - 1)(|\sigma_0 A_0| - |\sigma_1|) > 0, \quad |z_0| < 1, \tag{3.6}$$

or that they satisfy (2.12) with

$$\Delta_0\Big[\Delta_0 |c + z_0 d|^2 - |F(z_0)|^2 (|c|^2 - |d|^2)\Big] > 0, \quad |z_0| = 1. \tag{3.7}$$

Then a, b, c, d satisfy (3.1).

The statement following below provides an important property of solution (a,b,c,d,f) to the functional equation (1.3).

Theorem 3.3. *Assume that the polynomial F does not contain roots on the unit circle $|z| = 1$. Suppose, further, that the coefficients a, b, c, d satisfy condition (3.1) and*

$$|ad - bc| + \beta > 0, \quad \beta = |a|^2 + |b|^2 - |c|^2 - |d|^2. \tag{3.8}$$

Then the polynomial f as well as any polynomial of the parameterized family

$$G_\lambda(z) = (a + bz) f(z) + \lambda(c + dz) f^*(z), \quad 0 \leq \lambda \leq 1, \tag{3.9}$$

does not contain roots on the unit circle $|z| = 1$.

Proof. Arguing indirectly, one obtains the existence of numbers $w, |w| = 1$, and $\lambda \in [0,1]$ such that

$$G_\lambda(w) = (a + bw) f(w) + \lambda(c + dw) f^*(w) = 0. \tag{3.10}$$

Since $\bar{w}w = 1$, one has

$$f^*(w) = w^n \overline{f\left(\frac{1}{\bar{w}}\right)} = w^n \overline{f(w)}, \tag{3.11}$$

from which it follows that

$$(a + bw) f(w) + \lambda(c + dw) w^n \overline{f(w)} = 0. \tag{3.12}$$

By condition,

$$G_1(w) = (a + bw) f(w) + (c + dw) w^n \overline{f(w)} = F(w) \neq 0, \tag{3.13}$$

therefore, $f(w) \neq 0$ and $0 \leq \lambda < 1$. Now, using the equality (3.12), we obtain

$$|a + bw| = \lambda |c + dw|, \tag{3.14}$$

from which it follows (see Lemma 3.1) that $|a+bw| = 0$, $|c+dw| = 0$. The latter equalities yield $|a| = |b|$, $|c| = |d|$, $ad - bc = 0$ that contradicts condition (3.8) and the result follows. □

Denote by $\kappa(F)$ the number of roots of the polynomial F (counted according to their multiplicity) belonging to the open unit disc $|z| < 1$. As noted, the computing $\kappa(F)$ maybe a reduction to the generalized Rauss-Hurwitz problem of determining the number of roots of a polynomial in a semiplane (see, for instance, [1–3]). The above results allow us to construct an iterative process for computing $\kappa(F)$, namely, the following theorem.

Theorem 3.4. *Assume that the polynomial F does not have roots on the unit circle $|z| = 1$. If the polynomials F and F^* are linearly dependent, then*

$$\kappa(F) = \frac{n+1}{2}, \tag{3.15}$$

otherwise

$$\kappa(F) = \kappa(G_0), \tag{3.16}$$

where $G_0(z) = (a+bz)f(z)$ and the collection (a,b,c,d,f) satisfying (3.1), (3.8) is a solution to the functional equation (1.3).

Proof. Assume the polynomials F and F^* to be linearly dependent, that is $F^*(z) \equiv CF(z)$. Then the following presentation takes place:

$$F(z) = b_{n+1}(z-z_1)^{\alpha_1}\cdots(z-z_m)^{\alpha_m}\left(z-\frac{1}{\bar{z}_1}\right)^{\alpha_1}\cdots\left(z-\frac{1}{\bar{z}_m}\right)^{\alpha_m}, \tag{3.17}$$

where

$$|z_s| < 1, \quad s = 1,\dots,m, \qquad 2(\alpha_1+\cdots+\alpha_m) = n+1. \tag{3.18}$$

From this it follows that the number $n+1$ is even and

$$\kappa(F) = \alpha_1+\cdots+\alpha_m = \frac{(n+1)}{2}. \tag{3.19}$$

Assume now the polynomials F and F^* to be linearly independent. Then, by Lemma 2.3, F admits a regular point $z_0, |z_0| \le 1$. Therefore, by Lemmas 1.1, 2.3, and 3.2, there exists a collection (a,b,c,d,f) satisfying conditions (3.1), (3.8) and being a solution to the functional equation (1.3). Combining Theorem 3.3 and the Rouche theorem one obtains $\kappa(F) = \kappa(G_0)$, where $G_0(z) = (a+bz)f(z)$.

The proof of Theorem 3.4 is complete. □

It is easy to see that the number $\kappa(G_0)$ satisfies the equality

$$\kappa(G_0) = \kappa(f) + \varepsilon, \tag{3.20}$$

where $\varepsilon = 1$ for $|a| < |b|$ and $\varepsilon = 0$, otherwise. A simple argument shows that if the numbers a, b, c, d are determined by a regular point z_0, then ε can be evaluated according to the formulae

$$\varepsilon = \frac{1+\operatorname{sign}\left(|F^*(z_0)| - |F(z_0)|\right)}{2}, \quad |z_0| < 1, \qquad \varepsilon = \frac{1-\operatorname{sign}(\Delta_0)}{2}, \quad |z_0| = 1. \tag{3.21}$$

Thus, under the assumptions of Theorems 2.1 and 2.2, one has

$$\kappa(F) = \frac{1+\operatorname{sign}\left(|F^*(z_0)| - |F(z_0)|\right)}{2} + \kappa(f), \quad |z_0| < 1, \tag{3.22}$$

$$\kappa(F) = \frac{1-\operatorname{sign}(\Delta_0)}{2} + \kappa(f), \quad |z_0| = 1. \tag{3.23}$$

Formulae (3.22) and (3.23) give rise to a recurrent procedure for the computation of κ. Indeed, they allow one to compute $\kappa(F)$, where F is a polynomial of degree $n+1$, based on $\kappa(f)$, where f is a polynomial of degree n and its coefficients are completely determined by coefficients of F.

Observe that if $z_0 = 0$, then formula (3.22) takes the form

$$\kappa(F) = \frac{1+\operatorname{sign}\left(|b_{n+1}| - |b_0|\right)}{2} + \kappa(f). \tag{3.24}$$

Here the coefficients of f can be determined from system (2.19).

4. Criterion for the absence of roots on the unit circle

Given the coefficients of the polynomials F and F^*, one can construct the following $(2n+2)\times(2n+2)$ matrix:

$$M_F = \begin{pmatrix} b_0 & b_1 & \cdot & b_{n+1} & \cdot & 0 \\ \bar{b}_{n+1} & \bar{b}_n & \cdot & \bar{b}_0 & \cdot & 0 \\ 0 & b_0 & \cdot & b_n & \cdot & 0 \\ 0 & \bar{b}_{n+1} & \cdot & \bar{b}_0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & b_1 & \cdot & b_{n+1} \\ 0 & 0 & \cdot & \bar{b}_{n+1} & \cdot & \bar{b}_0 \end{pmatrix}. \tag{4.1}$$

The matrix M_F coincides with the Sylvester matrix of F and F^* (up to a permutation of its lines). Therefore, $\det(M_F)$ coincides (up to a sign) with the resultant $R(F,F^*)$ of the polynomials F and F^*. Let z_s, $s = 1,\dots,n+1$ be all the roots of F (counted according to their multiplicities). By condition, $F(0) = b_0 \neq 0$, therefore, all the roots are different from zero. By definition of the polynomial F^*, the numbers $\bar{z}_s^{-1}$, $s = 1,\dots,n+1$, are roots of F^*. Hence (cf. [5]), $R(F,F^*)$ can be represented as follows:

$$R(F,F^*) = \bar{b}_0^{n+1} b_{n+1}^{n+1} \prod_{s,t} \left(z_s - \frac{1}{\bar{z}_t} \right). \tag{4.2}$$

Formula (4.2) gives rise to the following criteria for the polynomial F to have no roots on the unit circle.

Theorem 4.1. *Let* $\det(M_F)$ *be different from zero. Then* F *does not have roots on the unit circle* $|z| = 1$.

Assume that a collection $(a,b,1,1,f)$ of the numbers $a,b,c=d=1$ and a polynomial f satisfy the functional equation (1.3) and the condition $\alpha \equiv \bar{a}b - 1 = 0$. Using a, b define the following square matrix of order $2n+2$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ -\bar{b} & 1 & -\bar{a} & 1 & \cdot & 0 & 0 & 0 & 0 \\ 1 & -a & 1 & -b & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{b} & 1 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & -\bar{b} & 1 & -\bar{a} & 1 \\ 0 & 0 & 0 & 0 & \cdot & 1 & -a & 1 & -b \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

It is easy to see that $\det(P)$ satisfies the following condition:

$$\det P = (-1)^{n-1}\beta^{n-1}\left(1 - |a|^2\right), \tag{4.4}$$

where $\beta = |a|^2 + |b|^2 - 2$.

Let M_f be a square matrix of order $2n$ determined by the polynomial f. Using the matrices M_F, M_f, and P one can express a connection between the coefficients of polynomials F and f, given in system (2.19), by the following matrix equality:

$$PM_F = \begin{pmatrix} b_0 & b_1 & b_2 & & \cdot & & & 0 \\ 0 & & & & & & & 0 \\ 0 & & & & \beta M_f & & & 0 \\ \cdot & & & & & & & \cdot \\ 0 & & & & \cdot & \bar{b}_2 & \bar{b}_1 & \bar{b}_0 \end{pmatrix}. \tag{4.5}$$

Consider the matrix written in the right-hand side of (4.5) more intently: (i) its first and last lines coincide with the corresponding lines of the matrix M_F; (ii) the remaining entries (except for zeros related to the first and the last column) are filled out by the elements of the matrix $\beta \cdot M_F$. Therefore, equality (4.5) connects $\det(M_F)$ and $\det(M_f)$ as follows.

Theorem 4.2. *Let* $c = d = 1$ *and assume that the numbers* a, b *along with coefficients* $a_0,\dots,a_n$ *of the polynomial* f *satisfy (2.19),* $\beta \neq 0$. *Then one has*

$$\det M_F = (-1)^{n+1}\left(\left|b_{n+1}\right|^2 - \left|b_0\right|^2\right)\beta^n \det M_f. \tag{4.6}$$

Proof. Combining equality (4.5), the above formula for $\det(P)$ with the standard determinant properties yields

$$\det M_F = \frac{\left|b_0\right|^2 \det\left(\beta M_f\right)}{\det P} = (-1)^{n+1}\left(\left|b_{n+1}\right|^2 - \left|b_0\right|^2\right)\beta^n \det M_f. \tag{4.7}$$

Assume a collection $(a,b,1,1,f,F)$ of the numbers $a,b,c=d=1$ and polynomials f,F to satisfy the functional equation (1.3) and the condition $\alpha \equiv \bar{a}b-1=0$. Consider a sequence of collections $(a^k,b^k,1,1,F^k)$ of numbers $a^k,b^k,c^k=d^k=1$ and polynomials

$$F^k(z)=b_{0k}+b_{1k}z+\cdots+b_{kk}z^k, \quad b_{0k}\neq 0,\ b_{kk}\neq 0, \tag{4.8}$$

of degree k, satisfying the functional equation

$$(a^k+b^kz)F^k(z)+(1+z)\left(F^k\right)^*(z)=F^{k+1}(z), \quad k=n,n-1,\dots,1, \tag{4.9}$$

and the condition

$$\alpha^k \equiv \overline{a^k}b^k-1=0, \tag{4.10}$$

where $a^n=a$, $b^n=b$ $F^n=f$, $F^{n+1}=F$. By Theorem 2.1, if $z=0$ is a regular point of the polynomial

$$F^{k+1}(z)=b_{0k+1}+b_{1k+1}z+\cdots+b_{k+1,k+1}z^{k+1}, \tag{4.11}$$

that is,

$$b_{0k+1}\cdot b_{k+1,k+1}\neq 0, \quad |b_{0k+1}|\neq|b_{k+1,k+1}|, \tag{4.12}$$

then the system of (4.9), (4.10) has the unique solution (a^k,b^k,F^k). Moreover,

$$\overline{a^k}=\frac{\overline{b_{0k+1}}}{b_{k+1,k+1}}, \qquad \overline{b^k}=\frac{1}{a^k}. \tag{4.13}$$

This along with formula (3.24) justify the following relation:

$$\kappa(F)=\sum_{k=1}^{n+1}\frac{1+\operatorname{sign}\left(|b_{kk}|-|b_{0k}|\right)}{2}. \tag{4.14}$$

□

5. Application for computing the rotation of a plane vector field

(1) Consider a vector field $\Phi(x,y)=\{p(x,y),q(x,y)\}$, where

$$p(x,y)=\sum_{k,j}a_{kj}x^ky^j, \qquad q(x,y)=\sum_{k,j}b_{kj}x^ky^j \tag{5.1}$$

are polynomials in real variables x and y with real coefficients. Assume that $\Phi(x,y)\neq 0$, $(x,y)\in S=\{(x,y):x^2+y^2=1\}$. We are interested in computing the rotation $\gamma(\Phi,S)$.

Recall the definition of rotation. Consider the complex presentation of the field Φ:

$$p+iq=\exp\left(i\theta(t)\right)|p+iq|, \quad p+iq=p(\cot t,\sin t)+iq(\cos t,\sin t), \quad t\in[0,2\pi], \tag{5.2}$$

where $\theta(t)$ is a continuous function. Then (cf. [4])

$$\gamma(\Phi,S) := \frac{1}{2\pi}\big(\theta(2\pi)-\theta(0)\big). \tag{5.3}$$

The following statement reduces the computation of rotation of a plane vector field to the computation of the number of roots in the unit disc of some polynomial.

Lemma 5.1. *Given a (polynomial) plane vector field Φ, there exists a unique pair (m,F), where m is an integer and F is a polynomial in complex variable with complex coefficients, $F(0)\neq 0$, satisfying the following condition:*

$$F(z) = z^m\big(p(x,y)+iq(x,y)\big), \quad z = x+iy,\ |z| = 1. \tag{5.4}$$

Proof. Take a polynomial $P(x,y) = p(x,y)+iq(x,y)$ in real variables x, y. The change of variables

$$(x,y) \longrightarrow \left(\frac{1+z^2}{2z}, \frac{i(1-z^2)}{2z}\right) \tag{5.5}$$

determines the rational function in complex variable z:

$$R(z) = P\left(\frac{1+z^2}{2z}, \frac{i(1-z^2)}{2z}\right). \tag{5.6}$$

The function $R(z)$ can be represented in the form

$$R(z) = \frac{F(z)}{z^m}, \tag{5.7}$$

where $F(z)$ is a polynomial satisfying the condition $F(0)\neq 0$ and m is an integer. Since

$$x = \frac{1+z^2}{2z}, \quad y = \frac{i(1-z^2)}{2z}, \quad z = x+iy,\ |z| = 1, \tag{5.8}$$

the pair (m,F) satisfies (5.4).

To complete the proof of Lemma 5.1, it remains to establish the uniqueness of the pair satisfying (5.4). Suppose that (m_1,F_1) is another pair satisfying

$$F_1(z) = z^{m_1}\big(p(x,y)+iq(x,y)\big), \quad z = x+iy,\ |z| = 1, \tag{5.9}$$

and $F_1(0)\neq 0$. Assuming, without loss of generality, that $m_1 \geq m$, one obtains the following equalities for F_1 and $z^{m_1-m}F$:

$$F_1(z) = z^{m_1}P(x,y) = z^{m_1-m}z^mP(x,y) = z^{m_1-m}F(z) \tag{5.10}$$

for $z = x+iy$, $|z| = 1$. From this it follows that F_1 and $z^{m_1-m}F$ coincide. Further, by assumption, $F(0)\neq 0$, $F_1(0)\neq 0$, hence $m_1 = m$. Thus $F_1 = F$ and Lemma 5.1 is completely proved. $\square$

Let Φ and F be as in Lemma 5.1. Then $\gamma(\Phi, S)$ and $\kappa(F)$ satisfy the following relation:

$$\kappa(F) = m + \gamma(\Phi, S). \tag{5.11}$$

(2) Assume a field Φ to be given in a parametric form: $\Phi(t) = \{p(t), q(t)\}$, where $p(t)$, $q(t)$ are real continuous 2π-periodic functions. Suppose $\Phi(t) \neq 0$, $t \in [0, 2\pi]$. The field Φ may be considered as the one defined on the unit circle S, by assigning to each point $x = \cos t$, $y = \sin t$ the vector $\{p(t), q(t)\}$. Therefore, the rotation $\gamma(\Phi, S)$ is correctly defined on S. Assuming the functions $p(t)$, $q(t)$ to be smooth enough, one can assign to the field Φ the Fourier series of the complex function $P(t) = p(t) + iq(t)$:

$$P(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ikt). \tag{5.12}$$

Since the series (5.12) converges uniformly and Φ does not vanish, there exists an integer N such that for all $t \in [0, 2\pi]$ the following estimate is true:

$$\left| P(t) - \sum_{k=-N}^{N} c_k \exp(ikt) \right| < |P(t)|. \tag{5.13}$$

Set $m := \max\{-k : |c_k| > 0,\ |k| \leq N\}$ and consider the polynomial

$$F(z) = \sum_{k=0}^{N+m} c_{k-m} z^k. \tag{5.14}$$

Using the same arguments as in Section 1 it is easy to check that $\gamma(\Phi, S)$ and $\kappa(F)$, where F is defined by (5.14), satisfy (5.11).

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