

REGULARITY OF MINIMIZERS FOR NONCONVEX VECTORIAL INTEGRALS WITH p - q GROWTH VIA RELAXATION METHODS

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Local Lipschitz continuity of local minimizers of vectorial integrals $\int_{\Omega} f(x, Du) dx$ is proved when f satisfies p - q growth condition and $\xi \mapsto f(x, \xi)$ is not convex. The uniform convexity and the radial structure condition with respect to the last variable are assumed only at infinity. In the proof, we use semicontinuity and relaxation results for functionals with nonstandard growth.

1. Introduction

In this paper, we study the regularity properties of the local minimizers of nonhomogeneous integral functionals of the form

$$I(u, \Omega) = \int_{\Omega} f(x, Du) dx, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and Du denotes the gradient matrix of a vector-valued function $u : \Omega \rightarrow \mathbb{R}^m$, $n, m > 1$.

We say that a function $u \in W_{loc}^{1,1}(\Omega, \mathbb{R}^m)$ is a local minimizer of I if $f(x, Du) \in L_{loc}^1(\Omega)$ and

$$I(u, \text{supp } \varphi) \leq I(u + \varphi, \text{supp } \varphi), \quad (1.2)$$

for any $\varphi \in W^{1,1}(\Omega, \mathbb{R}^m)$ with $\text{supp } \varphi \Subset \Omega$.

The main features of our functional I is the fact that the density energy $f = f(x, \xi)$ is not convex with respect to $\xi \in \mathbb{R}^{nm}$ and satisfies the so-called p - q growth condition, that is, there exist $1 < p < q$ and $c_0, c_1, L > 0$, such that

$$c_1 |\xi|^p - c_0 \leq f(x, \xi) \leq L(1 + |\xi|^q). \quad (1.3)$$

The regularity of minimizers in the p - q growth condition (1.3) has been extensively studied in the last years, starting from some papers of Marcellini, see for example [11, 12].

The study is motivated by several problems in different contexts of mathematical physics as, for example, in nonlinear elasticity and homogenization.

Most of the regularity results in the vectorial case $m > 1$ are obtained under the assumptions

$$f(x, \xi) = g(x, |\xi|) \quad x \in \Omega, \xi \in \mathbb{R}^{nm}, \tag{1.4}$$

with $g : \Omega \times [0, \infty[\rightarrow [0, \infty[$ and f convex with respect to ξ . For example, in a recent paper, Cupini et al. [2] proved that if (1.3), with $\xi \in \mathbb{R}^{nm}$, and (1.4), with $\xi \in \mathbb{R}^{nm} \setminus B_R(0)$, hold and, in addition, f is p -uniformly convex at infinity, the local minimizers of I are Lipschitz continuous when $1 < p \leq q < p(n+1)/n$. If $q > p(n+1)/n$, there are some counterexamples to the regularity. See also Esposito et al. [6] and Mascolo and Miglierini [13] for related results.

The regularity of minimizers of nonconvex functionals is achieved, usually, via relaxation methods. For instance, Fonseca et al. [7], in scalar case $m = 1$ and in standard growth $p = q$, observed that, by the known relaxation theorems, local minimizers of nonconvex functionals are also minimizers of

$$I^{**}(u, \Omega) = \int_{\Omega} f^{**}(x, Du) dx, \tag{1.5}$$

where f^{**} is the convex envelope of f with respect to ξ . Thus they can reduce to study the regularity of minimizers of convex functionals; see also Cupini and Miglierini [3].

However, in the vectorial case, the relaxation methods are not so well clarified when dealing with p - q growth condition.

The lower semicontinuity with respect to the weak topology $W^{1,p}(\Omega, \mathbb{R}^m)$ of quasi-convex functional, with integrand satisfying (1.3), has been studied by Marcellini [12] for $f = f(x, \xi)$ and $q < p(n+1)/n$ and by Fonseca and Malý [8] when $f = f(\xi)$ and $q < pn/(n-1)$. Further in [8], a representation of the relaxed functional is given when $Qf(\xi) = f^{**}(\xi)$, where $Qf(\xi)$ denotes the quasiconvex envelope of f .

Here we consider nonhomogeneous density energies $f = f(x, \xi)$, which are Caratheodory functions satisfying a local continuity condition with respect to x (see assumption (A2)). First, we prove a lower semicontinuity theorem when f is quasiconvex and $q < pn/(n-1)$ (see Theorem 2.2). Moreover, when

$$Qf(x, \xi) = f^{**}(x, \xi), \tag{1.6}$$

as in [8], we are able to prove that (see Theorem 2.6) for all $U \Subset \Omega$ and $u \in W^{1,p}(\Omega, \mathbb{R}^m)$,

$$\mathcal{F}^{p,q}(u, U) = I^{**}(u, U), \tag{1.7}$$

where

$$\mathcal{F}^{p,q}(u, \Omega) = \inf_{u_k} \left\{ \liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Du_k) dx, \right. \\ \left. (u_k) \in W^{1,q}(\Omega, \mathbb{R}^m), u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^m) \right\}. \tag{1.8}$$

Our main regularity result follows by equality (1.7).

Let f be not convex in ξ , p -uniformly convex at infinity, and satisfy the conditions (1.4) and (1.6). Then, in Theorem 3.2, we prove that all local minimizers of I are local Lipschitz continuous. We give a sketch of the proof.

Let u be a local minimizer of $I(u)$ and $U \Subset \Omega$. Consider

$$I(u, U) = \inf \{I(v, U) : v \in W_0^{1,p}(U, \mathbb{R}^m) + u\} \tag{1.9}$$

and the convex problem

$$\inf \{I^{**}(v, U) : v \in W_0^{1,p}(U, \mathbb{R}^m) + u\}. \tag{1.10}$$

For classical results, (1.10) has at least one solution \bar{u} and [2, Theorem 1.1] implies that $\bar{u} \in W_{\text{loc}}^{1,\infty}(U, \mathbb{R}^m)$. We prove that

$$I(u, U) = I^{**}(\bar{u}, U) = I^{**}(u, U). \tag{1.11}$$

The proof of the last equalities is based on the representation formula (1.7), on a method introduced by De Giorgi [5], and on the related ideas contained in Fusco [9] and in Marcellini [12]. Then, since (1.11) implies that u is a solution of the convex problem (1.10), we get $u \in W_{\text{loc}}^{1,\infty}(U, \mathbb{R}^m)$.

Moreover, we exhibit a class of density energies for which (1.6) is satisfied.

Finally, we study the relationship between (1.6) and the Lavrentiev phenomenon. In particular, we show that under (1.4) and (1.6), we do not have the occurrence of the phenomenon.

In conclusion, we observe that, in the scalar case, the regularity of minimizers for non-convex functionals is often the first step to prove the existence, see Mascolo and Schianchi [14] and Fonseca et al. [7], then it would be interesting to complete these researches in that direction.

This paper is organized as follows. In Section 2, we give the proof of the semicontinuity theorem and the characterization of $\mathcal{F}^{p,q}$ and Section 3 is devoted to the study of the regularity of local minimizers.

2. Semicontinuity and relaxation

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $f : \Omega \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be a nonnegative Caratheodory function satisfying the following assumptions:

(A1) there exist $q > 1$ and $L > 0$ such that

$$0 \leq f(x, \xi) \leq L(1 + |\xi|^q), \tag{2.1}$$

for almost everywhere $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nm}$;

(A2) there exists a modulus of continuity $\lambda(t)$ (i.e., $\lambda(t)$ is a nonnegative increasing function that goes to zero as $t \rightarrow 0^+$) such that for every compact subset $\Omega_0 \subset \Omega$, there exists $x_0 \in \Omega_0$ such that

$$|f(x_0, \xi) - f(x, \xi)| \leq \lambda(|x - x_0|)(1 + f(x, \xi)), \tag{2.2}$$

for all $x \in \Omega_0$ and $\xi \in \mathbb{R}^{nm}$.

Remark 2.1. It is easy to check that (2.2) implies that there exists $\delta > 0$, with $\lambda(\delta) < 1$, such that when $\Omega_0 \subset \Omega$ is a compact set with $\text{diam}(\Omega_0) < \delta$, there exists a modulus of continuity $\mu(t)$ in $[0, \delta[$ such that

$$|f(x_0, \xi) - f(x, \xi)| \leq \mu(|x - x_0|)(1 + f(x_0, \xi)), \quad (2.3)$$

for all $x \in \Omega_0$ and for all $\xi \in \mathbb{R}^{nm}$.

In fact, fixing $\xi \in \mathbb{R}^{nm}$ for all $x \in \Omega$ such that $f(x_0, \xi) \geq f(x, \xi)$, the condition (2.2) implies (2.3) with $\mu = \lambda$.

Let $x \in \Omega_0$ be such that $f(x, \xi) > f(x_0, \xi)$, then (2.2) can be written as

$$f(x_0, \xi) \geq f(x, \xi)[1 - \lambda(|x - x_0|)] - \lambda(|x - x_0|). \quad (2.4)$$

Thus if δ is such that $\lambda(\delta) < 1$, then for all $\Omega_0 \Subset \Omega$ with $\text{diam}(\Omega_0) < \delta$, (2.3) holds with $\mu(t) = \lambda(t)/(1 - \lambda(t))$.

We say that a function $f(x, \xi)$ is quasiconvex with respect to ξ in the Morrey's sense if for every $x \in \Omega$ and $\xi \in \mathbb{R}^{nm}$,

$$\int_{\Omega} f(x, \xi + D\varphi(y)) dy \geq |\Omega| f(x, \xi) \quad \forall \varphi \in W_0^{1, \infty}(\Omega, \mathbb{R}^m). \quad (2.5)$$

First, we prove the following lower semicontinuity theorem in Sobolev spaces below the growth exponent for the density energy.

THEOREM 2.2. *Let f be a quasiconvex function satisfying (A1) and (A2) and let $1 < p < q < n/(n-1)p$. Then*

$$\liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Du_k) dx \geq \int_{\Omega} f(x, Du) dx, \quad (2.6)$$

for all $u \in W^{1,p}(\Omega, \mathbb{R}^m)$, with $f(x, Du) \in L^1(\Omega)$, and for all $u_k \in W^{1,q}(\Omega, \mathbb{R}^m)$ such that u_k converges to u in the weak topology of $W^{1,p}(\Omega, \mathbb{R}^m)$.

Proof. Let $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ and $u_k \rightharpoonup u$ in $W^{1,p}$.

Without loss of generality we may assume that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Du_k) dx < \infty. \quad (2.7)$$

Let Ω_0 be an open set compactly contained in Ω . For every integer ν such that $\lambda(1/\nu) < 1$, we consider a subdivision of Ω_0 in open sets Ω_i such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $\sum_i |\Omega_i| = |\Omega_0|$, and $\text{diam} \Omega_i < 1/\nu$. By (A2) and Remark 2.1, there exists $x_i \in \overline{\Omega_i}$ for which

$$\begin{aligned} |f(x_i, \xi) - f(x, \xi)| &\leq \lambda\left(\frac{1}{\nu}\right)(1 + f(x, \xi)), \\ |f(x_i, \xi) - f(x, \xi)| &\leq \mu\left(\frac{1}{\nu}\right)(1 + f(x_i, \xi)), \end{aligned} \quad (2.8)$$

for all $x \in \overline{\Omega}_i$ and $\xi \in \mathbb{R}^{nm}$. Then by using the second inequality of (2.8)

$$\begin{aligned} \int_{\Omega_0} f(x, Du) dx &\leq \sum_i \int_{\Omega_i} f(x, Du) dx \\ &\leq \sum_i \int_{\Omega_i} f(x_i, Du) dx + \mu \left(\frac{1}{\nu} \right) \sum_i \int_{\Omega_i} [1 + f(x_i, Du)] dx. \end{aligned} \quad (2.9)$$

We estimate the first integral in the right-hand side. Since $q < pn/(n-1)$, by the lower semicontinuity result for functionals with homogeneous density energy contained in [8, Theorem 4.1], we have

$$\int_{\Omega_i} f(x_i, Du) dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega_i} f(x_i, Du_k) dx. \quad (2.10)$$

The first inequality of (2.8) implies that

$$\begin{aligned} \sum_i \liminf_{k \rightarrow \infty} \int_{\Omega_i} f(x_i, Du_k) dx &\leq \sum_i \liminf_{k \rightarrow \infty} \int_{\Omega_i} \left\{ \left[1 + \lambda \left(\frac{1}{\nu} \right) \right] f(x, Du_k) + \lambda \left(\frac{1}{\nu} \right) \right\} dx \\ &\leq \left[1 + \lambda \left(\frac{1}{\nu} \right) \right] \liminf_{k \rightarrow \infty} \int_{\Omega_0} f(x, Du_k) dx + \lambda \left(\frac{1}{\nu} \right) |\Omega_0|. \end{aligned} \quad (2.11)$$

By (2.7) and (2.10), there exists $M > 0$ such that

$$\sum_i \int_{\Omega_i} f(x_i, Du) dx < M \quad (2.12)$$

and consequently we obtain

$$\begin{aligned} \int_{\Omega_0} f(x, Du) dx \\ \leq \left[1 + \lambda \left(\frac{1}{\nu} \right) \right] \liminf_{k \rightarrow \infty} \int_{\Omega_0} f(x, Du_k) dx + \lambda \left(\frac{1}{\nu} \right) |\Omega_0| + \mu \left(\frac{1}{\nu} \right) [M + |\Omega_0|]. \end{aligned} \quad (2.13)$$

Thus we go to the limit as $\nu \rightarrow \infty$ and then we get the result as $\Omega_0 \uparrow \Omega$. \square

Suppose that f satisfies (A1) and (A2) and let $Qf(x, \xi)$ be the quasiconvex envelope of f with respect to the second variable, that is,

$$Qf(x, \xi) = \sup \{ g \leq f : g \text{ quasiconvex with respect to } \xi \}. \quad (2.14)$$

By the results contained in Dacorogna [4] and in Giusti [10], we have that (A1) implies

$$Qf(x, \xi) = \inf \left\{ \frac{1}{|\Omega|} \int_{\Omega} f(x, \xi + D\varphi(y)) dy \mid \varphi \in W_0^{1,q} \right\}. \quad (2.15)$$

By the definition of Qf , we have that

$$0 \leq Qf(x, \xi) \leq L(1 + |\xi|^q). \quad (2.16)$$

Now we show that when f satisfies (A2), Qf satisfies the same property.

LEMMA 2.3. Let $f : \Omega \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfy (A1) and (A2); then there exists $\delta > 0$ such that for every compact subset Ω_0 of Ω with $\text{diam}(\Omega_0) < \delta$, there exists $x_0 \in \Omega_0$ such that

$$|Qf(x_0, \xi) - Qf(x, \xi)| \leq \mu(|x - x_0|)(1 + Qf(x, \xi)), \quad (2.17)$$

for all $x \in \Omega_0$ and for all $\xi \in \mathbb{R}^{nm}$, where $\mu(t) = \lambda(t)/(1 - \lambda(t))$, for $0 < t < \delta$.

Proof. By the characterization of the quasiconvex envelope (2.15), we have that, for $x \in \Omega$, there exists a sequence $(\varphi_{x_j}) \in W_0^{1,q}(\Omega, \mathbb{R}^m)$ such that

$$Qf(x, \xi) \leq \frac{1}{|\Omega|} \int_{\Omega} f(x, \xi + D\varphi_{x_j}(y)) dy \leq Qf(x, \xi) + \frac{1}{j}. \quad (2.18)$$

Let $\Omega_0 \Subset \Omega$ be a compact subset and let $x_0 \in \Omega_0$ satisfy (2.2):

$$\begin{aligned} & Qf(x_0, \xi) - Qf(x, \xi) \\ & \leq \frac{1}{|\Omega|} \int_{\Omega} f(x_0, \xi + D\varphi_{x_j}(y)) dy - \frac{1}{|\Omega|} \int_{\Omega} f(x, \xi + D\varphi_{x_j}(y)) dy + \frac{1}{j} \\ & \leq \lambda(|x - x_0|)(1 + Qf(x, \xi)) + \frac{1}{j}(\lambda(|x - x_0|) + 1); \end{aligned} \quad (2.19)$$

as $j \rightarrow \infty$, we obtain

$$Qf(x_0, \xi) - Qf(x, \xi) \leq \lambda(|x - x_0|)(1 + Qf(x, \xi)). \quad (2.20)$$

Now let $(\varphi_{x_0j}) \in W_0^{1,q}(\Omega, \mathbb{R}^m)$ satisfy (2.18) with x_0 in place of x . With the same previous arguments, by applying (2.3) instead of (2.2), for Ω_0 with $\text{diam}(\Omega_0) < \delta$ (with δ chosen such that $\lambda(\delta) < 1$), we get

$$Qf(x, \xi) - Qf(x_0, \xi) \leq \mu(|x - x_0|)(1 + Qf(x_0, \xi)), \quad (2.21)$$

so (2.20) and (2.21) imply (2.17). \square

Remark 2.4. Let $f(x, \xi) = a(x)g(\xi)$, where $g : \mathbb{R}^{nm} \rightarrow \mathbb{R}$ has q -growth:

$$0 < g(\xi) < L(1 + |\xi|^q), \quad (2.22)$$

where $L > 0$ is a constant and $a \in C^{0,\alpha}(\Omega)$. In this case, both f and Qf satisfy (A1) and (A2); in fact $Qf(x, \xi) = a(x)Qg(\xi)$.

We introduce the functional $\mathcal{F}^{p,q}$:

$$\begin{aligned} \mathcal{F}^{p,q}(u) = \inf_{u_k} \left\{ \liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Du_k) dx, \right. \\ \left. (u_k) \in W^{1,q}(\Omega, \mathbb{R}^m), u_k \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^m) \right\}. \end{aligned} \quad (2.23)$$

The following theorem holds.

THEOREM 2.5. *Let $1 < p < q < pn/(n - 1)$, let f satisfy (A1) and (A2), and $Qf(x, \xi) \in L^1(\Omega)$. Then for $u \in W^{1,p}(\Omega, \mathbb{R}^m)$,*

$$\mathcal{F}^{p,q} \geq \int_{\Omega} Qf(x, Du) dx. \tag{2.24}$$

When $u \in W^{1,q}(\Omega, \mathbb{R}^m)$, the equality in (2.24) holds.

Proof. It is not difficult to check that the arguments in the proof of [Theorem 2.2](#) can be applied to the quasiconvex functional $\int_{\Omega} Qf(x, Du) dx$ by choosing a decomposition of Ω_0 in open set Ω_i , with $\text{diam}(\Omega_i) < 1/\nu$, where ν is sufficiently big, so that Ω_i satisfies the condition of [Lemma 2.3](#). Then if $(u_k) \in W^{1,q}(\Omega, \mathbb{R}^m)$ and $u_k \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^m)$, we obtain

$$\liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Du_k) dx \geq \liminf_{k \rightarrow \infty} \int_{\Omega} Qf(x, Du_k) dx \geq \int_{\Omega} Qf(x, Du) dx; \tag{2.25}$$

taking the infimum over all such sequences, (2.24) follows.

When $u \in W^{1,q}(\Omega, \mathbb{R}^m)$, by the standard relaxation results, there exists a sequence $(w_j) \in W^{1,q}(\Omega, \mathbb{R}^m)$ such that $w_j \rightharpoonup u$ in $W^{1,q}(\Omega, \mathbb{R}^m)$ and

$$\liminf_{j \rightarrow \infty} \int_{\Omega} f(x, Dw_j) dx = \int_{\Omega} Qf(x, Du) dx, \tag{2.26}$$

which easily implies that (2.24) is an equality. □

Assume now a p -coercivity condition on f .

(A1') There exist $c_0, c_1, L > 0$ such that

$$c_1 |\xi|^p - c_0 \leq f(x, \xi) \leq L(1 + |\xi|^q). \tag{2.27}$$

In the following, $f^{**}(x, \xi)$ denotes the lower convex envelope of f with respect to the second variable.

We are able now to give a characterization of $\mathcal{F}^{p,q}$.

THEOREM 2.6. *Let f satisfy (A1') and (A2) and let $1 < p < q < p(n + 1)/n$. Assume that $Qf(x, \xi)$ is a convex function with respect to ξ , that is, $Qf(x, \xi) = f^{**}(x, \xi)$. Then, for all $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ such that $Qf(x, Du) \in L^1_{\text{loc}}(\Omega)$,*

$$\mathcal{F}^{p,q}(u, U) = \int_U Qf(x, Du) dx, \tag{2.28}$$

for all $U \Subset \Omega$, open set.

Proof. Consider a smooth kernel $\varphi \geq 0$ in \mathbb{R}^m with support in $\overline{B(0, 1)}$, $\int_{\mathbb{R}^n} \varphi(x) dx = 1$, and $\varphi_j(x) = j^n \varphi(jx)$. For each $j \in \mathbb{N}$, consider $\varphi_j * u \in W^{1,q}(U, \mathbb{R}^m)$. Again for the standard relaxation results, we can select a sequence $v_{jk} \in W^{1,q}(U, \mathbb{R}^m)$ such that $\lim_{k \rightarrow \infty} v_{jk} = \varphi_j * u$ weakly in $W^{1,q}(U, \mathbb{R}^m)$ and

$$\lim_{k \rightarrow \infty} \int_U f(x, Dv_{jk}) dx = \int_U Qf(x, D(\varphi_j * u)) dx. \tag{2.29}$$

We may extract a diagonal subsequence $u_j = v_{jk(j)}$ such that $u_j \rightharpoonup u$ in $W^{1,p}(U, \mathbb{R}^m)$ and

$$\left| \int_U f(x, Du_j) dx - \int_U Qf(x, D(\varphi_j * u)) dx \right| \leq \frac{1}{j}. \quad (2.30)$$

For every positive $\nu \in \mathbb{N}$ with $\nu \geq \bar{\nu} = \inf\{\nu : \lambda(1/\nu) < 1\}$, we consider a subdivision of U in the open sets U_i such that $\sum_i |U_i| = |U|$, $U_i \cap U_j = \emptyset$, for $i \neq j$ and $\text{diam}(U_i) < 1/\nu \forall i$. We have, from (2.21),

$$\begin{aligned} & \liminf_{j \rightarrow \infty} \int_U Qf(x, D(\varphi_j * u)) dx \\ &= \liminf_{j \rightarrow \infty} \sum_i \int_{U_i} Qf(x, D(\varphi_j * u)) dx \\ &\leq \liminf_{j \rightarrow \infty} \sum_i \mu\left(\frac{1}{\nu}\right) \int_{U_i} [1 + Qf(x_i, D(\varphi_j * u))] dx \\ &\quad + \liminf_{j \rightarrow \infty} \sum_i \int_{U_i} Qf(x_i, D(\varphi_j * u)) \\ &= \mu\left(\frac{1}{\nu}\right) |U| + \left(1 + \mu\left(\frac{1}{\nu}\right)\right) \liminf_{j \rightarrow \infty} \sum_i \int_{U_i} Qf(x_i, D(\varphi_j * u)) dx. \end{aligned} \quad (2.31)$$

Since Qf is convex and the measure μ_x^j , given by

$$\langle \mu_x^j, \nu \rangle := \int_U \varphi_j(x-y) \nu(y) dy, \quad (2.32)$$

is a probability measure, using Jensen inequality, we have, for fixed i ,

$$\int_{U_i} Qf(x_i, D(\varphi_j * u)) = \int_{U_i} Qf(x_i, \langle \mu_x^j, Du \rangle) dx \leq \int_{U_i} \langle \mu_x^j, Qf(x_i, Du) \rangle dx. \quad (2.33)$$

Since $Qf(x_i, Du) \in L^1(\Omega)$, by a known result, we get

$$\int_{U_i} \langle \mu_x^j, Qf(x_i, Du) \rangle dx \leq \int_{U_i} Qf(x_i, Du) dx, \quad (2.34)$$

then, by (2.20),

$$\begin{aligned} & \liminf_{j \rightarrow \infty} \int_U Qf(x, D(\varphi_j * u)) \\ &\leq \mu\left(\frac{1}{\nu}\right) |U| + \left(1 + \mu\left(\frac{1}{\nu}\right)\right) \sum_i \int_{U_i} Qf(x_i, Du) dx \\ &\leq \mu\left(\frac{1}{\nu}\right) |U| + \left(1 + \mu\left(\frac{1}{\nu}\right)\right) \lambda\left(\frac{1}{\nu}\right) \sum_i \int_{U_i} (1 + Qf(x, Du)) dx \\ &\quad + \left(1 + \mu\left(\frac{1}{\nu}\right)\right) \int_U Qf(x, Du) dx. \end{aligned} \quad (2.35)$$

As ν goes to ∞ , since $\int_U Qf(x, Du)dx < \infty$, we get

$$\liminf_{j \rightarrow \infty} \int_U Qf(x, D(\varphi_j * u))dx \leq \int_U Qf(x, Du)dx. \quad (2.36)$$

Then, by (2.30),

$$\begin{aligned} \mathcal{F}^{p,q}(u, U) &\leq \liminf_{j \rightarrow \infty} \int_U f(x, Du_j)dx \\ &= \liminf_{j \rightarrow \infty} \int_U Qf(x, D(\varphi_j * u))dx \\ &\leq \int_U Qf(x, Du)dx, \end{aligned} \quad (2.37)$$

so by (2.24), (2.28) holds. \square

Remark 2.7. Under the hypotheses of [Theorem 2.6](#), consider the following functional:

$$\begin{aligned} \mathcal{F}(u, \Omega) = \inf_{u_k} \left\{ \liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Du_k)dx, (u_k) \in W^{1,p}(\Omega, \mathbb{R}^m), \right. \\ \left. u_k \rightarrow u \text{ in } W^{1,p}(\Omega, \mathbb{R}^m), f(x, Du_k) \in L^1(\Omega) \right\}. \end{aligned} \quad (2.38)$$

For all $U \Subset \Omega$, we have

$$\mathcal{F}^{p,q}(u, U) = \mathcal{F}(u, U). \quad (2.39)$$

Indeed for all $u_k \in W^{1,p}(U, \mathbb{R}^m)$ with $f(x, Du_k) \in L^1(U)$ and $u_k \rightarrow u$ in $W^{1,p}(\Omega, \mathbb{R}^m)$, we have

$$\int_{\Omega} f^{**}(x, Du)dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} f^{**}(x, Du_k)dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Du_k)dx. \quad (2.40)$$

Therefore,

$$\mathcal{F}(u, U) \leq \mathcal{F}^{p,q}(u, U) = \int_U f^{**}(x, Du) \leq \mathcal{F}(u, U). \quad (2.41)$$

Now we exhibit a class of integrands $f : \Omega \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ such that $Qf(x, A) = f^{**}(x, A)$, for every $A \in \mathbb{R}^{nm}$.

We say that $f : \Omega \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is rank-one convex if, for every $x \in \Omega$,

$$f(x, \lambda A + (1 - \lambda)B) \leq \lambda f(x, A) + (1 - \lambda)f(x, B), \quad (2.42)$$

for all $\lambda \in [0, 1]$ and $A, B \in \mathbb{R}^{nm}$ with $\text{rank}(A - B) \leq 1$.

For $f : \Omega \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$, we define the rank-one convex envelope as

$$Rf(x, A) = \sup\{g \leq f : g \text{ rank-one convex with respect to } A\}. \quad (2.43)$$

We prove the following result.

PROPOSITION 2.8. Let Ω be a bounded open set of \mathbb{R}^n and let $f : \Omega \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$. Assume that there exists a function $g : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$\begin{aligned} f(x, A) &= g(x, |A|), \\ g(x, t) &= g(x, -t), \end{aligned} \tag{2.44}$$

for every $x \in \Omega$ and for every $t > 0$.

Moreover, assume that there exists a measurable, nonnegative function $\alpha : \Omega \rightarrow \mathbb{R}_+$ such that

$$g^{**}(x, t) = g(x, \alpha(x)), \tag{2.45}$$

for all t with $|t| < \alpha(x)$ and

$$g^{**}(x, t) = g(x, t), \tag{2.46}$$

for all $t \geq \alpha(x)$.

Then,

$$g^{**}(x, |A|) = f^{**}(x, |A|) = Qf(x, |A|) = Rf(x, |A|). \tag{2.47}$$

Proof. We use the same techniques contained in Dacorogna [4]. First, we prove that $g^{**}(x, |A|) \geq f^{**}(x, |A|)$.

Let $x \in \Omega$ and let ε be fixed, then by the characterization of the convex envelope, there exist $\lambda \in [0, 1]$ and $b, c \in \mathbb{R}_+$ such that

$$\begin{aligned} \varepsilon + g^{**}(x, |A|) &\geq \lambda g(x, b) + (1 - \lambda)g(x, c), \\ |A| &= \lambda b + (1 - \lambda)c. \end{aligned} \tag{2.48}$$

Choosing $B = bA/|A|$ and $C = cA/|A|$, we get

$$\varepsilon + g^{**}(x, |A|) \geq \lambda f(x, B) + (1 - \lambda)f(x, C) \geq f^{**}(x, |A|). \tag{2.49}$$

From the arbitrariness of ε , we have the claimed result. To prove the reverse inequality, we first show that (2.45) implies that $g^{**}(x, t)$ is not decreasing with respect to t .

Fixing $x \in \Omega$, we have

$$g^{**}(x, \nu) \geq g^{**}(x, \alpha(x)), \tag{2.50}$$

for all $\nu \geq \alpha(x)$.

In fact, if $\nu > \alpha(x)$ is such that $g^{**}(x, \nu) < g^{**}(x, \alpha(x))$, then we can choose $w < \alpha(x)$ such that $\lambda w + (1 - \lambda)\nu = \alpha(x)$. Since $g^{**}(x, w) = g^{**}(x, \alpha(x))$, we get

$$\lambda g^{**}(x, w) + (1 - \lambda)g^{**}(x, \nu) < g^{**}(x, \alpha(x)) = g^{**}(x, \lambda w + (1 - \lambda)\nu) \tag{2.51}$$

in contradiction with the convexity of the function $t \rightarrow g^{**}(x, t)$.

Now let $\nu, w > 0$ with $\nu > w$. If $0 < w < \nu < \alpha(x)$, we have

$$g^{**}(x, \nu) = g^{**}(x, \alpha(x)) = g^{**}(x, w); \tag{2.52}$$

if $0 < w < \alpha(x) < v$, we have

$$g^{**}(x, v) \geq g^{**}(x, \alpha(x)) = g^{**}(x, w); \tag{2.53}$$

assume now that $\alpha(x) < w < v$ and $g^{**}(x, v) < g^{**}(x, w)$, then there exists $\lambda \in (0, 1)$ such that $w = \lambda v + (1 - \lambda)\alpha(x)$, so we get

$$\begin{aligned} \lambda g^{**}(x, v) + (1 - \lambda)g^{**}(x, \alpha(x)) &< \lambda g^{**}(x, w) + (1 - \lambda)g^{**}(x, w) \\ &= g^{**}(x, w) \\ &= g^{**}(x, \lambda v + (1 - \lambda)\alpha(x)), \end{aligned} \tag{2.54}$$

again in contradiction with the convexity of the function g^{**} .

Since $g^{**}(x, t)$ is not decreasing, $g^{**}(x, |A|)$ is a convex function less than f , so we obtain $g^{**}(x, |A|) < f^{**}(x, A)$.

Therefore, we have proved that $g^{**}(x, |A|) = f^{**}(x, A)$. We are going now to prove that $Rf = f^{**}$.

We observe that

$$g^{**}(x, |A|) = f^{**}(x, A) \leq Rf(x, A). \tag{2.55}$$

When $|A| \geq \alpha(x)$, by (2.46),

$$g^{**}(x, |A|) = g(x, |A|) = f(x, A) \geq Rf(x, A). \tag{2.56}$$

So we can reduce to the case $|A| < \alpha(x)$ when $g^{**}(x, |A|) = g^{**}(x, \alpha(x))$.

Let $A = (A_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$, with $A_1^1 \neq 0$, and define

$$\lambda = \frac{1}{2} \left(1 + \frac{|A_1^1|}{(\alpha(x)^2 - |A|^2 + |A_1^1|^2)^{1/2}} \right), \tag{2.57}$$

we have $1/2 < \lambda < 1$. Let $E = (E_j^i)_{1 \leq i \leq m, 1 \leq j \leq n}$ be such that

$$E_1^1 = \frac{2A_1^1}{1 - 2\lambda}, \quad E_j^i = 0, \quad i > 1, j > 1 \tag{2.58}$$

and let

$$B = A - (1 - \lambda)E, \quad C = A + \lambda E. \tag{2.59}$$

Therefore, we have that

$$A = \lambda B + (1 - \lambda)C, \quad |B| = |C| = \alpha(x), \quad \text{rank}(B - C) \leq 1, \tag{2.60}$$

so by the characterization of the rank-one convex envelope,

$$Rf(x, A) \leq \lambda f(x, B) + (1 - \lambda)f(x, C) = g^{**}(x, \alpha(x)) = f^{**}(x, A). \tag{2.61}$$

□

As a final remark, we point out that condition (1.6), that is, $Qf(x, \xi) = f^{**}(x, \xi)$, is connected with the Lavrentiev phenomenon. In an abstract framework, let X and Y be two topological spaces of weakly differentiable functions, with $Y \subset X$ and Y dense in X . We say that there is the Lavrentiev phenomenon when

$$\inf_{u \in X} \int f(x, Du) dx < \inf_{u \in Y} \int f(x, Du) dx. \quad (2.62)$$

The following result holds.

COROLLARY 2.9. *Under the hypotheses of Theorem 2.6, the Lavrentiev phenomenon (2.62) does not occur.*

Proof. Define, as in Buttazzo and Mizel [1], the Lavrentiev gap:

$$\mathcal{L}(u) = \mathcal{F}_Y(u) - \mathcal{F}_X(u), \quad \mathcal{L}(u) = 0 \quad \text{if } \mathcal{F}_X(u) = +\infty, \quad (2.63)$$

where

$$\begin{aligned} \mathcal{F}_X(u) &= \sup \{G : X \rightarrow [0, +\infty] : G \text{ l.s.c, } G \leq I \text{ on } X\}, \\ \mathcal{F}_Y(u) &= \sup \{G : X \rightarrow [0, +\infty] : G \text{ l.s.c, } G \leq I \text{ on } Y\}. \end{aligned} \quad (2.64)$$

Since in our case $X = W^{1,p}(U, \mathbb{R}^m)$ and $Y = W^{1,q}(U, \mathbb{R}^m)$, $U \Subset \Omega$ and

$$\mathcal{F}_Y(u, U) = \mathcal{F}^{p,q}(u, U) = I^{**}(u, U) \leq \mathcal{F}_X(u, U), \quad (2.65)$$

we conclude that $\mathcal{L}(u) = 0$. □

Therefore, Proposition 2.8 exhibits a class of functions satisfying the condition (B1) (i.e., $f(x, \xi) = g(x, |\xi|)$) for which the Lavrentiev phenomenon does not occur.

3. Relaxation and regularity

In this section, we apply the relaxation equality contained in Theorem 2.6 to get the $W^{1,\infty}$ regularity for local minimizers of nonconvex functionals.

Let $f : \Omega \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be a Caratheodory function satisfying (A1) and (A2) and the following assumptions:

(B1) there exist $R > 0$ and a function g such that, for almost everywhere $x \in \Omega$ and every $\xi \in \mathbb{R}^{nm} \setminus B_R(0)$,

$$f(x, \xi) = g(x, |\xi|); \quad (3.1)$$

(B2) f is p -uniformly convex at infinity, with $p \leq q$, that is, there exist $p > 1$ and $\nu > 0$ such that, for almost everywhere $x \in \Omega$ and for every $\xi_1, \xi_2 \in \mathbb{R}^{nm} \setminus B_R(0)$ end-points of a segment contained in the complement of $B_R(0)$,

$$\frac{1}{2}[f(x, \xi_1) + f(x, \xi_2)] \geq f\left(x, \frac{\xi_1 + \xi_2}{2}\right) + \nu(1 + |\xi_1|^2 + |\xi_2|^2)^{(p-2)/2} |\xi_1 - \xi_2|^2; \quad (3.2)$$

(B3) for almost everywhere $x \in \Omega$ and every $\xi \in \mathbb{R}^{nm} \setminus B_R(0)$, let $D_t^+g(x, |\xi|)$ be the right-side derivative of g with respect to t and denote

$$D_{\xi_i^\alpha}^+ f(x, \xi) = D_t^+ g(x, |\xi|) \frac{\xi_i^\alpha}{|\xi|}. \tag{3.3}$$

Then for every $\xi \in \mathbb{R}^{nm} \setminus B_R(0)$, the vector field $x \mapsto D_\xi^+ f(x, \xi)$ is weakly differentiable and

$$|D_x D_\xi^+ f(x, \xi)| \leq L_1 (1 + |\xi|)^{q-1}. \tag{3.4}$$

Let

$$I(v, \Omega) = \int_\Omega f(x, Dv) dx. \tag{3.5}$$

Cupini, et al. in [2, Theorem 1.1] proved the following regularity result for the local minimizer of I when f is convex with respect to ξ .

THEOREM 3.1. *Let u be a local minimizer of (3.5) whose integrand f is convex with respect to ξ and satisfies the assumptions (A1), (B1), (B2), and (B3), $1 < p \leq q < p(n + 1)/n$. Then u is locally Lipschitz continuous, and, for all $B_r(x_0) \Subset \Omega$,*

$$\sup_{B_{r/4}(x_0)} |Du| \leq c \left[\int_{B_r(x_0)} (1 + f(x, Du)) dx \right]^\beta, \tag{3.6}$$

where $c = c(n, p, q, L, L_1, R, \nu)$ and $\beta = \beta(n, p, q)$.

In the sequel, we need some properties of functions satisfying (A1), (A2), and the p -uniformly convexity at infinity. In [2, Lemma 2.2], it is proved that assumption (B2) implies that f is p -coercive, that is, there exist $c_0, c_1 > 0$ such that

$$c_1 |\xi|^p - c_0 \leq f(x, \xi). \tag{3.7}$$

By [2, Theorem 2.5(iv)], it follows that if f satisfies the condition (B2), there exists R_0 depending on ν, p, q, L such that

$$f^{**}(x, \xi) = f(x, \xi), \tag{3.8}$$

for almost everywhere $x \in \Omega$ and $\xi \in \mathbb{R}^{nm} \setminus B_{R_0}(0)$.

Moreover, by assumption (A2) and Remark 2.1, it is easy to check that there exists $\delta > 0$ such that, for every $\xi \in B_{R_0}(0)$ and $x \in U$, with $U \Subset \Omega$ and $\text{diam } U < \delta$, there exists $x_0 \in U$ such that

$$f(x, \xi) \leq \max_{\xi \in B_{R_0}(0)} f(x_0, \xi) [1 + \mu(\delta)] + \mu(\delta), \tag{3.9}$$

then, for every $\xi_1, \xi_2 \in B_{R_0}(0)$,

$$|f(x, \xi_1) - f(x, \xi_2)| \leq c \left[\max_{\xi \in B_{R_0}(0)} f(x_0, \xi) + 1 \right] = c_2(R_0, U), \tag{3.10}$$

where c, c_2 are positive constants.

We now consider $\xi_1, \xi_2 \in \mathbb{R}^{nm} \setminus B_{R_0}(0)$. By (3.8) and the growth condition, we obtain

$$|f(x, \xi_1) - f(x, \xi_2)| \leq c_3 [1 + |\xi_1| + |\xi_2|]^{q-1} |\xi_1 - \xi_2|, \quad (3.11)$$

with $c_3 = c_3(L, q)$.

Therefore, it is easy to check that there exist $c_2 = c_2(R_0, U)$ and $c_3(L, q)$ such that

$$|f(x, \xi_1) - f(x, \xi_2)| \leq c_2 + c_3 [1 + |\xi_1| + |\xi_2|]^{q-1} |\xi_1 - \xi_2|. \quad (3.12)$$

Finally, for every $x \in U$, where $U \Subset \Omega$, and for every $\xi \in \mathbb{R}^{nm}$, (3.8) implies that

$$f(x, t\xi) \leq c_4(1 + f(x, \xi)), \quad (3.13)$$

for all $t \in [0, 1]$, where c_4 depends on ν and R_0 .

Our main result is the following regularity theorem.

THEOREM 3.2. *Let f satisfy (A1), (A2), (B1), (B2), and (B3), with $1 < p < q < p(n+1)/n$ and $Qf(x, \xi) = f^{**}(x, \xi)$. Let u be a local minimizer of the functional I in (3.5), then $u \in W_{\text{loc}}^{1, \infty}(\Omega, \mathbb{R}^m)$ and estimate (3.6) holds.*

Proof. We suppose that for every $U \Subset \Omega$, u is a local minimizer and then u is a solution of the following boundary value problem:

$$I(u) = \inf \{I(\nu, U), \nu \in W_0^{1,p}(\Omega, \mathbb{R}^m) + u\}. \quad (3.14)$$

Consider the relaxed problem of (3.14):

$$I^{**}(u) = \inf \{I^{**}(\nu, U), \nu \in W_0^{1,p}(\Omega, \mathbb{R}^m) + u\}, \quad (3.15)$$

where, as usual,

$$I^{**}(\nu, U) = \int_U f^{**}(x, D\nu) dx. \quad (3.16)$$

By the convexity of I^{**} , the problem (3.15) has at least a solution $\bar{u} \in W_0^{1,p}(\Omega, \mathbb{R}^m) + u$. Since \bar{u} is also a local minimizer of I^{**} and f^{**} satisfies (A1), (B1), (B2), and (B3) in $\Omega \times \mathbb{R}^{nm} \setminus B_{R_0}(0)$, then we can apply [Theorem 3.1](#) and so we obtain $\bar{u} \in W_{\text{loc}}^{1, \infty}(U, \mathbb{R}^m)$.

Observe that $f(x, D\bar{u}) \in L^1(U)$. In fact, we can suppose, without loss of generality, that $\text{diam } U < \delta$; then, for $\xi \in B_{R_0}(0)$, by (3.9),

$$f(x, \xi) \leq c_2(R_0, U); \quad (3.17)$$

and then we get

$$\begin{aligned} \int_U f(x, D\bar{u}) dx &= \int_{U \cap \{x: |D\bar{u}| \leq R_0\}} f(x, D\bar{u}) dx + \int_{U \cap \{x: |D\bar{u}| > R_0\}} f^{**}(x, D\bar{u}) dx \\ &\leq c_2(R_0, U) |U| + \int_U f^{**}(x, D\bar{u}) dx < \infty. \end{aligned} \quad (3.18)$$

Let $(u_k) \subset W^{1,q}(U, \mathbb{R}^m)$, with $u_k \rightharpoonup w - W^{1,p}(U, \mathbb{R}^m)$; for $\varepsilon > 0$, let $\Sigma \Subset U$, Σ open, such that

$$I(\bar{u}, U \setminus \Sigma) \leq \varepsilon; \quad (3.19)$$

and let $M > 0$ be such that

$$\|D\bar{u}\|_{\infty, \Sigma} \leq M. \quad (3.20)$$

Moreover, let $\Sigma_0 \Subset \Sigma$ and let $\nu \in \mathbb{N}$. For $i = 1, \dots, \nu$, we define

$$\Sigma_i = \left\{ x \in \Sigma : \text{dist}(x, \Sigma_0) < \frac{i}{\nu} R \right\}, \quad (3.21)$$

where $R = \text{dist}(\Sigma_0, \partial\Sigma)$. For $i = 1, \dots, \nu$, consider the scalar functions $\varphi_i \in C_0^1(\Sigma_i)$ such that

$$0 \leq \varphi_i \leq 1, \quad \varphi_i(x) = \begin{cases} 1 & \text{in } \Sigma_{i-1}, \\ 0 & \text{in } \Sigma \setminus \Sigma_i, \end{cases} \quad |D\varphi_i| \leq \frac{\nu+1}{R}. \quad (3.22)$$

For all $k \in \mathbb{N}$, define

$$v_{ki} = (1 - \varphi_i)\bar{u} + \varphi_i u_k. \quad (3.23)$$

We have $v_{ki} = u_k$ in Σ_{i-1} and $v_{ki} = \bar{u}$ in $\Sigma \setminus \Sigma_{i-1}$. Define $\tilde{v}_{ki} = v_{ki}$ in Σ and $\tilde{v}_{ki} = \bar{u}$ in $U \setminus \Sigma$.

Consider

$$\begin{aligned} \int_U f(x, D\tilde{v}_{ki}) dx &= \int_{U \setminus \Sigma} f(x, D\bar{u}) dx + \int_{\Sigma \setminus \Sigma_i} f(x, D\bar{u}) dx \\ &\quad + \int_{\Sigma_{i-1}} f(x, Du_k) dx + \int_{\Sigma_i \setminus \Sigma_{i-1}} f(x, Dv_{ki}) dx, \end{aligned} \quad (3.24)$$

then we obtain

$$\begin{aligned} \int_U f(x, D\tilde{v}_{ki}) dx &\leq \varepsilon + \int_{\Sigma \setminus \Sigma_i} f(x, D\bar{u}) dx + \int_{\Sigma_{i-1}} f(x, Du_k) dx \\ &\quad + \int_{\Sigma_i \setminus \Sigma_{i-1}} [f(x, Dv_{ki}) - f(x, \varphi_i Du_k)] dx \\ &\quad + \int_{\Sigma_i \setminus \Sigma_{i-1}} f(x, \varphi_i Du_k) dx. \end{aligned} \quad (3.25)$$

Inequality (3.12) gives

$$\begin{aligned} J_k^i &= \int_{\Sigma_i \setminus \Sigma_{i-1}} [f(x, Dv_{ki}) - f(x, \varphi_i Du_k)] dx \\ &\leq c_2 |\Sigma_i \setminus \Sigma_{i-1}| + c_3 \int_{\Sigma_i \setminus \Sigma_{i-1}} [1 + |Dv_{ki}| + |\varphi_i Du_k|]^{q-1} |Dv_{ki} - \varphi_i Du_k| dx \\ &\leq c_2 |\Sigma_i \setminus \Sigma_{i-1}| + c_3 \int_{\Sigma_i \setminus \Sigma_{i-1}} [1 + |D\bar{u}| + 2|Du_k| + |D\varphi_i| |u_k - \bar{u}|]^{q-1} \\ &\quad \times [(1 - \varphi_i)|D\bar{u}| + |D\varphi_i| |u_k - \bar{u}|] dx. \end{aligned} \quad (3.26)$$

By applying the Holder inequality with the exponents $p/(q-1)$ and $p/(p-q+1)$, we obtain

$$\begin{aligned} \sum_{i=1}^{\nu} J_k^i &\leq c_2 |\Sigma \setminus \Sigma_0| + c_3 \left\| 1 + |D\bar{u}| + 2 |Du_k| + \frac{\nu+1}{R} \|u_k - \bar{u}\| \right\|_{L^p}^{q-1} \\ &\quad \times \left(M |\Sigma \setminus \Sigma_0|^{(p-q+1)/p} + \frac{\nu+1}{R} \|u_k - \bar{u}\|_{L^{p/(p-q+1)}(\Sigma_\nu)} \right). \end{aligned} \quad (3.27)$$

Taking in account that the sequence (Du_k) has a bounded norm in $L^p(\Omega, \mathbb{R}^{nm})$ and that, since the assumption $q < p(n+1)/n$ implies $p/(p-q+1) < p^*$ and (u_k) converges $s - L^{p/(p-q+1)}(\Omega, \mathbb{R}^m)$ to \bar{u} , we can conclude that there exists a constant c_5 independent of k and ν such that

$$\limsup_{k \rightarrow \infty} \sum_{i=1}^{\nu} J_k^i \leq c_5. \quad (3.28)$$

Observe that, since $v_{ki} \in W_0^{1,p}(U, \mathbb{R}^m) + u$ and $\bar{u} = u$ on ∂U , by the minimality of u for problem (3.14), we get $I(u, U) \leq I(\tilde{v}_{ki}, U)$, for all i for all k .

Therefore, summing up with respect to $i = 1, \dots, \nu$ in (3.25), we get

$$\begin{aligned} \nu \int_U f(x, Du) dx &\leq \nu \varepsilon + \sum_{i=1}^{\nu} \int_{\Sigma \setminus \Sigma_i} f(x, D\bar{u}) dx + \nu \int_{\Sigma} f(x, Du_k) dx \\ &\quad + c_5 + \sum_{i=1}^{\nu} \int_{\Sigma_i \setminus \Sigma_{i-1}} f(x, \varphi_i Du_k) dx. \end{aligned} \quad (3.29)$$

Moreover, since (3.13) implies that

$$f(x, \varphi_i Du_k) \leq c_4 [1 + f(x, Du_k)], \quad (3.30)$$

we obtain

$$\int_U f(x, Du) dx \leq \varepsilon + \frac{\nu + c_6}{\nu} \int_U f(x, Du_k) dx + \frac{c_5}{\nu} + \frac{1}{\nu} \int_{\Sigma \setminus \Sigma_0} f(x, D\bar{u}) dx. \quad (3.31)$$

As k goes to infinity, we get

$$\int_U f(x, Du) dx \leq \varepsilon + \frac{\nu + c_6}{\nu} \liminf_{k \rightarrow \infty} \int_U f(x, Du_k) dx + \frac{c_5}{\nu} + \frac{1}{\nu} \int_{\Sigma \setminus \Sigma_0} f(x, D\bar{u}) dx. \quad (3.32)$$

Then passing to the limit $\nu \rightarrow \infty$,

$$\int_U f(x, Du) dx \leq \varepsilon + \liminf_{k \rightarrow \infty} \int_U f(x, Du_k) dx. \quad (3.33)$$

Taking the infimum over the sequences (u_k) ,

$$\int_U f(x, Du) dx \leq \mathcal{F}^{p,q}(\bar{u}, U). \quad (3.34)$$

Since $Qf(x, \xi) = f^{**}(x, \xi)$, by [Theorem 2.6](#), we have

$$\int_U f(x, Du) dx \leq \int_U f^{**}(x, D\bar{u}) dx. \tag{3.35}$$

On the other hand, \bar{u} is a solution of [\(3.15\)](#), then

$$\int_U f(x, Du) dx = \int_U f^{**}(x, D\bar{u}) dx = \int_U f^{**}(x, Du) dx, \tag{3.36}$$

which implies that u is also a solution of [\(3.15\)](#). [Theorem 2.2](#) ensures that $u \in W_{loc}^{1,\infty}(U, \mathbb{R}^m)$; for the arbitrariness of U , we get $u \in W_{loc}^{1,\infty}(\Omega, \mathbb{R}^m)$. Estimate [\(3.6\)](#) follows easily. \square

Remark 3.3. We observe that the arguments of the proof of [Theorem 3.2](#) show that, for all $\bar{u} \in W_{loc}^{1,\infty}(\Omega, \mathbb{R}^m)$ and for all $U \Subset \Omega$, we have

$$\mathcal{F}_0(\bar{u}, U) = \mathcal{F}^{p,q}(\bar{u}, U) = \int_U f^{**}(x, D\bar{u}) dx, \tag{3.37}$$

where

$$\mathcal{F}_0(\bar{u}, U) = \inf_{u_k} \left\{ \liminf_{k \rightarrow \infty} \int_{\Omega} f(x, Du_k) dx, (u_k) \in W_0^{1,p}(\Omega, \mathbb{R}^m) + \bar{u}, \right. \\ \left. u_k - \bar{u} \text{ in } W^{1,p}(\Omega, \mathbb{R}^m), f(x, Du_k) \in L^1(\Omega) \right\}. \tag{3.38}$$

Indeed, back to the proof of [Theorem 3.2](#), the sequence $\tilde{v}_{ki} \in W_0^{1,p}(U, \mathbb{R}^m) + \bar{u}$ is such that $f(x, D\tilde{v}_{ki}) \in L^1(U)$ and $\tilde{v}_{ki} \rightarrow \bar{u}$ in $W^{1,p}(\Omega, \mathbb{R}^m)$. Therefore, taking the lower bound in [\(3.25\)](#) and summing up with respect to $i = 1, \dots, \gamma$ as $k \rightarrow \infty$, we have

$$\mathcal{F}_0(\bar{u}, U) \leq \varepsilon + \frac{\nu + c_6}{\nu} \liminf_{k \rightarrow \infty} \int_U f(x, Du_k) dx + \frac{c_5}{\nu} + \frac{1}{\nu} \int_{\Sigma \setminus \Sigma_0} f(x, D\bar{u}) dx. \tag{3.39}$$

As $\nu \rightarrow \infty$, we get

$$\mathcal{F}_0(\bar{u}, U) \leq \mathcal{F}^{p,q}(\bar{u}, U) \leq \mathcal{F}_0(\bar{u}, U). \tag{3.40}$$

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