# EXISTENCE RESULTS FOR GENERAL INEQUALITY PROBLEMS WITH CONSTRAINTS

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*To Professor Jean Mawhin on occasion of his 60th birthday*

This paper is concerned with existence results for inequality problems of type $F^0(u;v)+\Psi'(u;v)\geq 0$, for all $v\in X$, where $X$ is a Banach space, $F: X\to\mathbb{R}$ is locally Lipschitz, and $\Psi: X\to(-\infty+\infty]$ is proper, convex, and lower semicontinuous. Here $F^0$ stands for the generalized directional derivative of $F$ and $\Psi'$ denotes the directional derivative of $\Psi$. The applications we consider focus on the variational-hemivariational inequalities involving the $p$-Laplacian operator.

## 1. Introduction

The paper deals with nonlinear inequality problems of type

$$F^0(u;v-u)+h(v)-h(u)\geq 0, \quad \forall v\in C, \tag{1.1}$$

where $F^0$ stands for the generalized directional derivative of a locally Lipschitz functional $F$ (in the sense of Clarke [5]), $h$ is a convex, lower semicontinuous (in short, l.s.c.), and proper function, and $C$ is a nonempty, closed, and convex subset of a Banach space $X$. It is clear that in problem (1.1) we can put $h+I_C$ in place of $h$, where $I_C$ denotes the indicator function of the set $C$, to give the formulation with $v$ arbitrary in $X$. However, we keep the statement (1.1) for allowing various possible choices separately on the data $h$ and $C$.

The type of problem stated in (1.1) fits in the framework of the nonsmooth critical point theory developed by Motreanu and Panagiotopoulos [9], which is constructed for the nonsmooth functionals having the form

$$\Phi=\Psi+F \tag{1.2}$$

with $\Psi$ convex, l.s.c., and proper, and $F$ locally Lipschitz. Namely, a solution of


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(1.1) means, in fact, a critical point of the associated nonsmooth functional (1.2) with $\Psi = h + I_C$.

The existence results in the present paper extend different theorems in the smooth and nonsmooth variational analyses (see, for comparison, Ambrosetti and Rabinowitz [2], Chang [4], Dincă et al. [8], Motreanu and Panagiotopoulos [9], Rabinowitz [10], and Szulkin [11]). In this respect, we solve problems of type

$$F^0(u;v) + \Psi'(u;v) \geq 0, \quad \forall v \in X, \tag{1.3}$$

where $\Psi'$ stands for the directional derivative of a convex, proper, l.s.c. functional $\Psi$. Consequently, we are able to handle the abstract hemivariational inequality problem

$$F^0(u;v-u) + \langle d\varphi(u), v-u\rangle \geq 0, \quad \forall v \in C, \tag{1.4}$$

where $\varphi$ is a convex, Gâteaux differentiable functional and $d\varphi$ is its differential. In particular, this contains the differential inclusion problem

$$d\varphi(u) \in \partial(-F)(u) \tag{1.5}$$

which we considered in our previous paper [8].

The rest of the paper is organized as follows. In Section 2, we briefly recall several elements of nonsmooth critical point theory developed by Motreanu and Panagiotopoulos [9]. In Section 3, we study some general inequality problems in relation with the nonsmooth critical point theory. Section 4 presents applications for different discontinuous boundary value problems with $p$-Laplacian.

## 2. Notions and preliminary results

Let $X$ be a real Banach space and $X^*$ its dual. *The generalized directional derivative* of a locally Lipschitz function $F : X \to \mathbb{R}$ at $u \in X$ in the direction $v \in X$ is defined by

$$F^0(u;v) = \limsup_{w\to u,\, t\searrow 0} \frac{F(w+tv) - F(w)}{t}. \tag{2.1}$$

*The generalized gradient* (in the sense of Clarke [5]) of $F$ at $u \in X$ is defined to be the subset of $X^*$ given by

$$\partial F(u) = \{\eta \in X^* : F^0(u;v) \geq \langle \eta, v\rangle,\ \forall v \in X\}, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $X^*$ and $X$.

Let $\Psi : X \to (-\infty, +\infty]$ be a proper (i.e., $D(\Psi) := \{u \in X : \Psi(u) < +\infty\} \neq \emptyset$), convex, and l.s.c. function and let $F : X \to \mathbb{R}$ be locally Lipschitz.

We define the functional $\Phi : X \to (-\infty, +\infty]$ by $\Phi = \Psi + F$.

*Definition 2.1* Motreanu and Panagiotopoulos [9]. An element $u \in X$ is called critical point of the functional $\Phi$ if this inequality holds

$$F^0(u; v - u) + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in X. \tag{2.3}$$

*Definition 2.2* Motreanu and Panagiotopoulos [9]. The functional $\Phi$ is said to satisfy the Palais-Smale condition if every sequence $\{u_n\} \subset X$ for which $\Phi(u_n)$ is bounded and

$$F^0(u_n; v - u_n) + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X, \tag{2.4}$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$, contains a strongly convergent subsequence in $X$.

For the proof of the next theorem, we refer the reader to [8, Proposition 2.1] and [9, Corollary 3.2] (also see [8, Theorem 2.2]).

Theorem 2.3. (i) *If $u \in X$ is a local minimum for $\Phi$, then $u$ is a critical point of $\Phi$.*

(ii) *If $\Phi$ satisfies the Palais-Smale condition and there exist a number $\rho > 0$ and a point $e \in X$ with $\|e\| > \rho$ such that*

$$\inf_{\|v\|=\rho} \Phi(v) > \Phi(0) \geq \Phi(e), \tag{2.5}$$

*then $\Phi$ has a nontrivial critical point.*

*Remark 2.4.* Definitions 2.1 and 2.2 recover and unify the nonsmooth critical point theories (and a fortiori the smooth critical point theory, see, e.g., Ambrosetti and Rabinowitz [2] and Rabinowitz [10]) due to Chang [4] and Szulkin [11]. Precisely, if $\Psi = 0$, Definitions 2.1 and 2.2 reduce to the corresponding definitions of Chang [4], while if $F \in C^1(X, \mathbb{R})$, then Definitions 2.1 and 2.2 coincide with those in Szulkin [11].

## 3. Critical points as solutions of inequality problems

Throughout this section, $(X, \|\cdot\|_X)$ is a real reflexive Banach space, compactly embedded in the real Banach space $(Z, \|\cdot\|_Z)$. Let $\mathcal{F} : Z \to \mathbb{R}$ be a locally Lipschitz function and let $\Psi : X \to (-\infty, +\infty]$ be convex, l.s.c., and proper.

We consider the inequality problem:

$$\text{Find } u \in D(\Psi) \text{ such that } (\mathcal{F}|_X)^0(u; v) + \Psi'(u; v) \geq 0, \quad \forall v \in X, \tag{3.1}$$

where $(\mathcal{F}|_X)^0$ denotes the generalized directional derivative of the restriction $\mathcal{F}|_X$ while $\Psi'(u; v)$ is the directional derivative of the convex function $\Psi$ at $u$ in the direction $v$ (which is known to exist). Note that if the Gâteaux differential $d\Psi(u)$ of $\Psi$ at $u \in D(\Psi)$ exists, then $\langle d\Psi(u), v\rangle = \Psi'(u; v)$, for all $v \in X$.

Proposition 3.1. *Each solution of problem (3.1) solves the problem:*

$$\textit{Find } u \in D(\Psi) \textit{ such that } \mathcal{F}^0(u;v) + \Psi'(u;v) \geq 0, \quad \forall v \in X. \tag{3.2}$$

*If, in addition to our assumptions, $X$ is densely embedded in $Z$, then problems (3.1) and (3.2) are equivalent.*

*Proof.* For $u, v \in X$, the inequality below holds

$$(\mathcal{F}|_X)^0(u;v) \leq \mathcal{F}^0(u;v). \tag{3.3}$$

This becomes an equality if $X$ is continuously and densely embedded in $Z$ (see [5, pages 46–47] and [9, pages 10–12]). □

Our approach for studying problem (3.1) is variational and relies on the use of the functional

$$\Phi = \Psi + \mathcal{F}|_X : X \longrightarrow (-\infty, +\infty] \tag{3.4}$$

which is clearly of the form required in the previous section with $F = \mathcal{F}|_X$.

The next result points out the relationship between the critical points of the functional $\Phi$ in (3.4) and the solutions of problem (3.1).

Proposition 3.2. (i) *If $u \in X$ is a critical point of the functional $\Phi$ in (3.4), that is,*

$$(\mathcal{F}|_X)^0(u;v-u) + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in X, \tag{3.5}$$

*then $u$ is a solution of problem (3.1).*

(ii) *Conversely, assume that $u \in X$ is a solution of problem (3.1). If either $\Psi$ is Gâteaux differentiable at $u$ or $\Psi$ is continuous at $u$, then $u$ is a critical point of $\Phi$, that is, relation (3.5) holds.*

*Proof.* (i) As $\Psi$ is proper, (3.5) obviously implies that $u \in D(\Psi)$. For an arbitrary $w \in X$, we set $v = u + tw$, $t > 0$, in (3.5). Dividing by $t$ and then letting $t \to 0^+$, we arrive at the conclusion that $u$ solves problem (3.1).

(ii) Let $u \in D(\Psi)$ be a solution of problem (3.1). If $\Psi$ is Gâteaux differentiable at $u$, then

$$\Psi(v) - \Psi(u) \geq \langle d\Psi(u), v-u \rangle = \Psi'(u;v-u), \quad \forall v \in X \tag{3.6}$$

which leads to (3.5).

If $\Psi$ is continuous at $u$, then a standard result of convex analysis (see Barbu and Precupanu [3, page 106]) allows to write

$$\Psi'(u;v) = \max\{\langle x^*, v\rangle : x^* \in \partial\Psi(u)\}, \quad \forall v \in X. \tag{3.7}$$

Using the definition of the subdifferential $\partial\Psi(u)$, we obtain (3.5). □

*Remark 3.3.* In view of Proposition 3.2(i), each result stating the existence of critical points for $\Phi$ in (3.4) asserts a fortiori existence of solutions to problem (3.1).

Theorem 3.4. *If $\Phi$ is coercive on $X$, that is,*

$$\Phi(u) \longrightarrow +\infty \quad \text{as } \|u\|_X \longrightarrow +\infty, \tag{3.8}$$

*then $\Phi$ has a critical point.*

*Proof.* The compact embedding of $X$ into $Z$ implies that $\mathcal{F}|_X$ is weakly continuous. We infer that $\Phi$ is sequentially weakly l.s.c. on $X$. Then, by standard theory, $\Phi$ is bounded from below and attains its infimum at some $u \in X$. From Theorem 2.3(i), $u$ is a critical point of $\Phi$. □

Towards the application of Theorem 2.3(ii) to the functional $\Phi$, we have to know when $\Phi$ satisfies the Palais-Smale condition. The following lemma provides a useful sufficient condition that improves the usual results based on the celebrated hypothesis $(p_5)$ in [2] or $(p_4)$ in [10].

Lemma 3.5. *Assume, in addition, that $\Psi$ and $\mathcal{F}$, entering the expression of $\Phi$ in (3.4), satisfy the following hypotheses:*

(H1) *$D(\Psi)$ is a cone and there exist constants $a_0, a_1, b_0, b_1 \geq 0$, $\alpha > 0$, and $\sigma \geq 1$ such that*

$$\Psi(u) - \alpha\Psi'(u;u) \geq a_0\|u\|_X^\sigma - a_1, \quad \forall u \in D(\Psi), \tag{3.9}$$

$$\mathcal{F}(u) - \alpha\left(\mathcal{F}|_X\right)^0(u;u) \geq -b_0\|u\|_X^\sigma - b_1, \quad \forall u \in D(\Psi), \tag{3.10}$$

$$a_0 > b_0 + \alpha \quad \text{if } \sigma = 1, \qquad a_0 > b_0 \quad \text{if } \sigma > 1; \tag{3.11}$$

(H2) *the following condition of $(S_+)$ type is satisfied: if $\{u_n\}$ is a sequence in $D(\Psi)$ provided $u_n \to u$ weakly in $X$ and $\limsup_{n\to\infty}(-\Psi'(u_n;u-u_n)) \leq 0$, then $u_n \to u$ strongly in $X$.*

*Then the functional $\Phi$ satisfies the Palais-Smale condition in the sense of Definition 2.2.*

*Proof.* Let $\{u_n\}$ be a sequence in $X$ for which there is a constant $M > 0$ with

$$\left|\Phi(u_n)\right| \leq M, \quad \forall n \geq 1, \tag{3.12}$$

and inequality (2.4) holds for $F = \mathcal{F}|_X$ and a sequence $\varepsilon_n \to 0^+$. By (3.12), each $u_n$ is in $D(\Psi)$. For $t > 0$, set $v = (1+t)u_n$ in (2.4) with $F = \mathcal{F}|_X$. Dividing by $t$ and then letting $t \searrow 0$, one obtains that

$$\Psi'(u_n;u_n) + \left(\mathcal{F}|_X\right)^0(u_n;u_n) \geq -\varepsilon_n\|u_n\|_X, \quad \forall n \geq 1. \tag{3.13}$$

Inequalities (3.12) and (3.13) ensure that for $n$ sufficiently large, one has

$$\begin{aligned} M+\alpha\|u_n\|_X &\geq \Psi(u_n)+\mathcal{F}(u_n)+\alpha\varepsilon_n\|u_n\|_X \\ &\geq \Psi(u_n)-\alpha\Psi'(u_n;u_n)+\left[\mathcal{F}(u_n)-\alpha(\mathcal{F}|_X)^0(u_n;u_n)\right]. \end{aligned} \tag{3.14}$$

Using (3.9) and (3.10), we find that

$$M+\alpha\|u_n\|_X \geq (a_0-b_0)\|u_n\|_X^\sigma - a_1 - b_1. \tag{3.15}$$

Then (3.11) and (3.15) show that $\{u_n\}$ is bounded in $X$. By the compactness of the embedding of $X$ into $Z$, the sequence $\{u_n\}$ contains a subsequence, again denoted by $\{u_n\}$ such that

$$u_n \longrightarrow u \quad \text{weakly in } X, \tag{3.16}$$

$$u_n \longrightarrow u \quad \text{strongly in } Z, \tag{3.17}$$

for some $u \in X$. Now put $v = u_n + t(u-u_n)$, $t>0$, in (2.4) with $F=\mathcal{F}|_X$. Similar to (3.13), we derive that

$$\Psi'(u_n;u-u_n)+(\mathcal{F}|_X)^0(u_n;u-u_n) \geq -\varepsilon_n\|u-u_n\|_X, \quad \forall n \geq 1. \tag{3.18}$$

This implies

$$\Psi'(u_n;u-u_n)+\mathcal{F}^0(u_n;u-u_n) \geq -\varepsilon_n\|u-u_n\|_X, \quad \forall n \geq 1. \tag{3.19}$$

As $\{u_n\}$ is bounded in $X$, we infer from (3.17) and the upper semicontinuity of $\mathcal{F}^0$ that

$$\liminf_{n\to\infty}\Psi'(u_n;u-u_n) \geq 0. \tag{3.20}$$

Taking into account (3.16) and (3.20), assumption (H2) completes the proof. □

*Remark 3.6.* If $\Psi'(u;\cdot)$ is homogeneous, for all $u \in D(\Psi)$, then (H2) becomes the usual form of the $(S_+)$ condition: if $\{u_n\}$ is a sequence in $D(\Psi)$ provided $u_n \to u$ weakly in $X$ and $\limsup_{n\to\infty}\Psi'(u_n;u_n-u) \leq 0$, *then* $u_n \to u$ *strongly in* $X$.

We can now state the following result.

Theorem 3.7. *Let* $\Phi$ *be defined in (3.4) and assume Lemma 3.5(H1) and (H2) together with the following hypotheses.*

(H3) *There exists an element* $\overline{u} \in D(\Psi)$ *such that*

$$a_1 + b_1 \leq (a_0 - b_0)\|\overline{u}\|_X^{\sigma}, \tag{3.21}$$

$$\Phi(\overline{u}) < 0. \tag{3.22}$$

(H4) *There exists a constant* $\rho > 0$ *such that*

$$\inf_{\|v\|_X=\rho} \Phi(v) > \Phi(0). \tag{3.23}$$

*Then* $\Phi$ *has a nontrivial critical point* $u \in X$. *In particular, problem (3.1) has a nontrivial solution.*

*Proof.* We apply Theorem 2.3(ii) to the functional $\Phi$ in (3.4). Lemma 3.5 guarantees that $\Phi$ satisfies the Palais-Smale condition. It remains to check that $\Phi$ verifies condition (2.5) with $\|e\|_X > \rho$. To this end, we prove that one can choose $e = t\overline{u}$ (with $\overline{u}$ entering (H3)) if $t > 0$ is sufficiently large.

First, note that $\overline{u} \neq 0$. Indeed, from (3.9), (3.10), and (3.21), we have

$$\Phi(\overline{u}) - \alpha\big[\Psi'(\overline{u};\overline{u}) + \left(\mathcal{F}|_X\right)^0(\overline{u};\overline{u})\big] \geq 0, \tag{3.24}$$

which leads to a contradiction with (3.22) if $\overline{u} = 0$.

We observe that, due to the fact that $\overline{u} \in D(\Psi)$ and since $D(\Psi)$ is a cone, the convex function $s \mapsto \Psi(s\overline{u})$ is locally Lipschitz on $(0, +\infty)$. A straightforward computation shows that

$$\begin{aligned}\partial_s\left(s^{-1/\alpha}\Phi(s\overline{u})\right) &= \partial_s\left(s^{-1/\alpha}\Psi(s\overline{u}) + s^{-1/\alpha}\mathcal{F}|_X(s\overline{u})\right)\\ &\subset -\frac{1}{\alpha}s^{-1/\alpha-1}\Psi(s\overline{u}) + s^{-1/\alpha}\partial_s\left(\Psi(s\overline{u})\right)\\ &\quad + \left(-\frac{1}{\alpha}s^{-1/\alpha-1}\mathcal{F}(s\overline{u}) + s^{-1/\alpha}\langle\partial(\mathcal{F}|_X)(s\overline{u}),\overline{u}\rangle\right), \quad \forall s > 0,\end{aligned} \tag{3.25}$$

where the notation $\partial_s$ stands for the generalized gradient with respect to $s$. For an arbitrary $t > 1$, Lebourg's mean value theorem yields some $\tau = \tau(t) \in (1,t)$ such that

$$t^{-1/\alpha}\Phi(t\overline{u}) - \Phi(\overline{u}) = \xi(t-1), \tag{3.26}$$

where $\xi \in \partial_s(s^{-1/\alpha}\Phi(s\overline{u}))|_{s=\tau}$. This implies

$$\begin{aligned}t^{-1/\alpha}\Phi(t\overline{u}) - \Phi(\overline{u}) \in \frac{1}{\alpha}(t-1)\tau^{-1/\alpha-1}\big[&\left(\alpha\tau\partial_s\left(\Psi(s\overline{u})\right)|_{s=\tau} - \Psi(\tau\overline{u})\right)\\ &+ \left(-\mathcal{F}(\tau\overline{u}) + \alpha\langle\partial(\mathcal{F}|_X)(\tau\overline{u}),\tau\overline{u}\rangle\right)\big].\end{aligned} \tag{3.27}$$

Then, taking into account the convexity of $s \mapsto \Psi(s\overline{u})$, the regularity property of a convex function (see Clarke [5, pages 39–40]) and relations (3.9) and (3.10), we get that

$$\begin{aligned}\Phi(t\overline{u}) &\leq t^{1/\alpha}\Phi(\overline{u}) + \frac{1}{\alpha}t^{1/\alpha}(t-1)\tau^{-1/\alpha-1}\big[\big(\alpha\Psi'(\tau\overline{u};\tau\overline{u}) - \Psi(\tau\overline{u})\big)\\ &\qquad + \big(-\mathcal{F}(\tau\overline{u}) + \alpha(\mathcal{F}|_X)^0(\tau\overline{u};\tau\overline{u})\big)\big]\\ &\leq t^{1/\alpha}\Phi(\overline{u}) + \frac{1}{\alpha}t^{1/\alpha}(t-1)\tau^{-1/\alpha-1}\big[-(a_0-b_0)\tau^{\sigma}\|\overline{u}\|_X^{\sigma} + a_1 + b_1\big], \quad \forall t > 1.\end{aligned} \tag{3.28}$$

By (3.21) and because $\tau > 1$, we derive that

$$\Phi(t\overline{u}) \leq t^{1/\alpha}\Phi(\overline{u}), \qquad \forall t > 1. \tag{3.29}$$

Then (3.29) and assumption (3.22) imply

$$\lim_{t\to+\infty}\Phi(t\overline{u}) = -\infty. \tag{3.30}$$

Now, by means of (3.30), we can choose $\overline{t} > 0$ sufficiently large to satisfy

$$\overline{t}\|\overline{u}\|_X > \rho, \qquad \Phi(\overline{t}\overline{u}) \leq \Phi(0), \tag{3.31}$$

for $\rho > 0$ entering (H4). If we compare (3.23) and (3.31), it is seen that the requirement in (2.5) is achieved for $e = \overline{t}\overline{u}$. Theorem 2.3(ii) assures that $\Phi$ in (3.4) has a nontrivial critical point $u \in X$. Furthermore, Remark 3.3 shows that $u$ is a (nontrivial) solution of problem (3.1). The proof of Theorem 3.7 is thus complete. □

In the final part of this section, we are concerned with the case when

$$\Psi = \Psi_C := \varphi + I_C, \tag{3.32}$$

where $C$ is a nonempty, closed, and convex subset of $X$, $I_C$ denotes the indicator function of $C$, and $\varphi : X \to \mathbb{R}$ is a convex, Gâteaux differentiable functional. Note that $\Psi_C$ is convex, l.s.c., and proper and $D(\Psi_C) = C$. Therefore, the functional

$$\Phi = \Psi_C + \mathcal{F}|_X, \tag{3.33}$$

with $\mathcal{F}$ as at the beginning of this section, has the form required in (3.4).

Consider the following problem of variational-hemivariational inequality type:

$$\text{Find } u \in C \text{ such that } (\mathcal{F}|_X)^0(u;v-u) + \langle d\varphi(u), v-u\rangle \geq 0, \quad \forall v \in C. \tag{3.34}$$

*Remark 3.8.* (i) Taking into account that, for $u \in C$,

$$\Psi'_C(u;v) = \begin{cases} \langle d\varphi(u), v\rangle & \text{if } u+tv \in C \text{ for some } t \in (0,1], \\ +\infty & \text{otherwise,} \end{cases} \tag{3.35}$$

a straightforward computation shows that problem (3.34) is equivalent to the following problem of type (3.1):

$$\text{Find } u \in D(\Psi_C) = C \text{ such that } (\mathcal{F}|_X)^0(u;v) + \Psi'_C(u;v) \geq 0, \quad \forall v \in X. \tag{3.36}$$

(ii) If $C$ is a nonempty, closed, and convex cone, then each solution of problem (3.34) solves also the problem:

$$\text{Find } u \in C \text{ such that } (\mathcal{F}|_X)^0(u;v) + \langle d\varphi(u), v\rangle \geq 0, \quad \forall v \in C. \tag{3.37}$$

Proposition 3.9. *If $u \in X$ is a critical point of $\Phi$ in (3.33) and (3.32), then $u$ is a solution of problem (3.34).*

*Proof.* Viewing Remark 3.8(i), the conclusion follows from Proposition 3.2(i). □

Theorem 3.10. *If the functional $\Phi$ in (3.33) and (3.32) is coercive on $X$, then problem (3.34) has a solution.*

*Proof.* It is a direct consequence of Theorem 3.4 and Proposition 3.9. □

Theorem 3.11. *For the defining $\Phi$ data entering (3.33) and (3.32), we assume the following.*

(H1′) *The set $C$ is a nonempty, closed, and convex cone in $X$ and there exist constants $a_0, a_1, b_0, b_1 \geq 0$, $\alpha > 0$, and $\sigma \geq 1$ such that one has (3.11),*

$$\varphi(u) - \alpha\langle d\varphi(u), u\rangle \geq a_0\|u\|_X^\sigma - a_1, \quad \forall u \in C, \tag{3.38}$$

$$\mathcal{F}(u) - \alpha(\mathcal{F}|_X)^0(u;u) \geq -b_0\|u\|_X^\sigma - b_1, \quad \forall u \in C. \tag{3.39}$$

(H2′) *The following condition of $(S_+)$ type is satisfied: if $\{u_n\}$ is a sequence in $C$ provided $u_n \to u$ weakly in $X$ and $\limsup_{n\to\infty}\langle d\varphi(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ strongly in $X$.*

(H3′) *There exists an element $\overline{u} \in C$ such that (3.21) holds with $a_0$, $a_1$, $b_0$, and $b_1$ from* (H1′) *together with*

$$\mathcal{F}(\overline{u}) + \varphi(\overline{u}) < 0. \tag{3.40}$$

(H4′) *There exists a constant $\rho > 0$ such that*

$$\inf_{\substack{\|v\|_X=\rho \\ v\in C}} (\mathcal{F}(v) + \varphi(v)) > \mathcal{F}(0) + \varphi(0). \tag{3.41}$$

*Then $\Phi$ in (3.33) and (3.32) has a nontrivial critical point $u \in C$. In particular, problem (3.34) has a nontrivial solution.*

*Proof.* Note that assumptions (H1′), (H2′), (H3′), and (H4′) are just (H1), (H2), (H3), and (H4), respectively, in the case where $D(\Psi) = C$ is a closed convex cone and $\Psi$ is given by (3.32). Thus it suffices to apply Theorem 3.7 and Proposition 3.9 to the functional $\Phi$ in (3.33) and (3.32). □

*Remark 3.12.* It is worth pointing out that if we take $C = X$, then problem (3.34) becomes

$$\text{Find } u \in X, \text{ such that } d\varphi(u) \in \partial\left(-\mathcal{F}|_X\right)(u). \tag{3.42}$$

Thus, [8, Theorems 3.2 and 3.4] are immediate consequences of Theorems 3.10 and 3.11, respectively.

## 4. Applications to nonsmooth boundary value problems

In order to illustrate how the abstract results of Section 3 can be applied, we consider a concrete problem of type (3.34). To this end, let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with Lipschitz-continuous boundary $\Gamma = \partial\Omega$ and let $\omega \subset \overline{\Omega}$ be a measurable set. Given $p \in (1,\infty)$, the Sobolev space $W^{1,p}(\Omega)$ is endowed with its usual norm (see [1, page 44]).

We denote

$$\begin{aligned} W_0 &= \{v \in W^{1,p}(\Omega) : v|_\Gamma = 0\}, \\ W_1 &= \left\{v \in W^{1,p}(\Omega) : \int_\Omega v = 0\right\}, \\ W_2 &= \{v \in W_1 : v|_\Gamma = \text{constant}\}. \end{aligned} \tag{4.1}$$

In the sequel, $W$ will stand for any of the above (closed) subspaces $W_0$, $W_1$, and $W_2$ of $W^{1,p}(\Omega)$. By the Poincaré-Wirtinger inequality, the functional

$$W \ni v \longmapsto \|v\|_{1,p} := \left(\int_\Omega |\nabla v|^p\right)^{1/p} \tag{4.2}$$

is a norm on $W$, equivalent to the induced norm from $W^{1,p}(\Omega)$. The dual space $W^*$ is considered endowed with the dual norm of $\|\cdot\|_{1,p}$.

Now, we define the $p$-Laplacian operator $-\Delta_p : W \to W^*$ by

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v, \quad \forall u, v \in W. \tag{4.3}$$

Arguments similar to those in [7] show that the convex functional $\varphi : W \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p}\|u\|_{1,p}^p, \quad \forall u \in W, \tag{4.4}$$

is continuously differentiable on $W$ and its differential is $-\Delta_p$, that is,

$$\langle d\varphi(u), v\rangle = \langle -\Delta_p u, v\rangle, \quad \forall u, v \in W. \tag{4.5}$$

Moreover, as $d\varphi$ is the duality mapping on $W$, corresponding to the gauge function $t \mapsto t^{p-1}$ and because $W$ is uniformly convex, $d\varphi$ satisfies condition $(S_+)$ (see Remark 3.6).

If $p^*$ stands for the Sobolev critical exponent, that is,

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases} \tag{4.6}$$

then, for any fixed $q \in (1, p^*)$, by the Rellich-Kondrachov theorem, the embedding $W \hookrightarrow L^q(\Omega)$ is compact (the space $L^q(\Omega)$ is understood with its usual norm $\|\cdot\|_{0,q}$).

The results in Section 3 will be applied by taking $X = W$, $Z = L^q(\Omega)$, and $\varphi$ defined in (4.4).

Further, to complete the setting, let a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable and satisfy the growth condition

$$|g(x,s)| \leq c_1 |s|^{q-1} + c_2 \quad \text{for a.e. } x \in \Omega, \ \forall s \in \mathbb{R}, \tag{4.7}$$

where $c_1, c_2 \geq 0$ are constants. For a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, we put

$$\begin{aligned} \underline{g}(x,s) &= \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|t-s|<\delta} g(x,t), \\ \overline{g}(x,s) &= \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|t-s|<\delta} g(x,t). \end{aligned} \tag{4.8}$$

The following condition will be invoked below:

$$\underline{g} \text{ and } \overline{g} \text{ are } N\text{-measurable} \tag{4.9}$$

(recall that a function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is called *$N$-measurable* if $h(\cdot, u(\cdot)) : \Omega \to \mathbb{R}$ is measurable whenever $u : \Omega \to \mathbb{R}$ is measurable).

By (4.7), the primitive $G : \Omega \times \mathbb{R} \to \mathbb{R}$ of function $g$:

$$G(x,s) = \int_0^s g(x,t)\,dt \quad \text{for a.e. } x \in \Omega, \ \forall s \in \mathbb{R}, \tag{4.10}$$

satisfies

$$|G(x,s)| \leq \frac{c_1}{q}|s|^q + c_2|s| \quad \text{for a.e. } x \in \Omega, \ \forall s \in \mathbb{R}. \tag{4.11}$$

Taking into account (4.11), we define the functional $\mathcal{G} : L^q(\Omega) \to \mathbb{R}$ by putting

$$\mathcal{G}(u) = -\int_\Omega G(x,u), \qquad \forall u \in L^q(\Omega). \tag{4.12}$$

It is known (see, e.g., Chang [4]) that $\mathcal{G}$ is Lipschitz continuous on the bounded subsets of $L^q(\Omega)$. At this stage, we introduce the closed convex cone $K$ in $W$:

$$K = \{u \in W : u(x) \geq 0 \text{ for a.e. } x \in \omega\} \tag{4.13}$$

and we formulate the problem:

$$\text{Find } u \in K \text{ such that } \left(\mathcal{G}|_W\right)^0(u; v-u) + \langle -\Delta_p u, v-u\rangle \geq 0, \qquad \forall v \in K. \tag{4.14}$$

Thus, the functional framework in Section 3 is now accomplished by taking $\mathcal{F} = \mathcal{G}$ and $C = K$. Clearly, problem (4.14) is of the same type as (3.34). Before passing on to obtaining existence results for problem (4.14), it should be noticed that the nonsmooth functional $\Phi = \Phi_K : W \to (-\infty, +\infty]$, defined by

$$\Phi_K = \mathcal{G}|_W + \varphi + I_K \tag{4.15}$$

with $\varphi$ in (4.4), $I_K$ the indicator function of the cone $K$ in (4.13), has the form required in (3.33) and (3.32).

We also need to invoke the following constant, depending on the cone $K$ in the Banach space $W$:

$$\lambda_1 = \lambda_{1,K} := \inf\left\{\frac{\|v\|_{1,p}^p}{\|v\|_{0,p}^p} : v \in K \setminus \{0\}\right\}. \tag{4.16}$$

Note that

$$\|v\|_{0,p} \leq \lambda_1^{-1/p}\|v\|_{1,p}, \qquad \forall v \in K. \tag{4.17}$$

Theorem 4.1. *Assume (4.7) together with*

(i) $\limsup_{s\to-\infty} pG(x,s)/|s|^p < \lambda_1$ *uniformly for a.e.* $x \in \Omega \setminus \omega$;
(ii) $\limsup_{s\to+\infty} pG(x,s)/s^p < \lambda_1$ *uniformly for a.e.* $x \in \Omega$.

*Then problem (4.14) has a solution.*

*Proof.* By Theorem 3.10, it suffices to show that the functional $\Phi_K$ in (4.15) is coercive on $W$.

From (i) and (ii), there are numbers $\varepsilon \in (0,\lambda_1)$ and $s_0 > 0$ such that

$$G(x,s) \leq \frac{\lambda_1 - \varepsilon}{p}|s|^p \quad \text{for a.e. } x \in \Omega\setminus\omega,\ \forall s < -s_0, \tag{4.18}$$

$$G(x,s) \leq \frac{\lambda_1 - \varepsilon}{p}s^p \quad \text{for a.e. } x \in \Omega,\ \forall s > s_0. \tag{4.19}$$

Using (4.11), we can find a positive constant $k = k(s_0)$ with

$$|G(x,s)| \le k \quad \text{for a.e. } x \in \Omega,\ \forall s \in [-s_0, s_0]. \tag{4.20}$$

For $u \in K$, we put

$$\Omega_- := \{x \in \Omega : u < 0\}, \qquad \Omega_+ := \Omega \setminus \Omega_-. \tag{4.21}$$

Notice that by (4.13) we have $\Omega_- \subset \Omega \setminus \omega$. Then by (4.18) and (4.20), it follows that

$$\begin{aligned}\int_{\Omega_-} G(x,u) &= \int_{[u<-s_0]} G(x,u) + \int_{[-s_0 \le u < 0]} G(x,u) \\ &\le \frac{\lambda_1 - \varepsilon}{p} \int_{\Omega_-} |u|^p + k|\Omega|.\end{aligned} \tag{4.22}$$

On the other hand, by (4.19) and (4.20), one sees that

$$\begin{aligned}\int_{\Omega_+} G(x,u) &= \int_{[u>s_0]} G(x,u) + \int_{[0 \le u \le s_0]} G(x,u) \\ &\le \frac{\lambda_1 - \varepsilon}{p} \int_{\Omega_+} |u|^p + k|\Omega|.\end{aligned} \tag{4.23}$$

Combining (4.22) and (4.23), the following estimate holds:

$$\int_\Omega G(x,u) \le 2k|\Omega| + \frac{\lambda_1 - \varepsilon}{p} \|u\|_{0,p}^p, \quad \forall u \in K. \tag{4.24}$$

Then, from (4.15), it follows that

$$\Phi_K(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega G(x,u) \ge \frac{1}{p}\|u\|_{1,p}^p - \frac{\lambda_1 - \varepsilon}{p}\|u\|_{0,p}^p - 2k|\Omega|, \quad \forall u \in W. \tag{4.25}$$

By (4.17), we infer

$$\Phi_K(u) \ge \frac{\varepsilon}{p\lambda_1}\|u\|_{1,p}^p - 2k|\Omega|, \quad \forall u \in W, \tag{4.26}$$

showing that

$$\lim_{\|u\|_{1,p} \to \infty} \Phi_K(u) = +\infty. \tag{4.27}$$

□

Theorem 4.2. *Assume (4.7), (4.9), and* $\operatorname{int}(\Omega \setminus \omega) \neq \emptyset$ *if* $W = W_1$ *or* $W = W_2$, *together with*

(i) $\limsup_{s \nearrow 0} pG(x,s)/|s|^p < \lambda_1$ *uniformly for a.e.* $x \in \Omega \setminus \omega$;
(ii) $\limsup_{s \searrow 0} pG(x,s)/s^p < \lambda_1$ *uniformly for a.e.* $x \in \Omega$;

*and there are numbers $\theta > p$ and $s_0 > 0$ such that*

(iii) $0 < \theta G(x,s) \leq sg(x,s)$ *for a.e.* $x \in \Omega \setminus \omega$, $\forall s \leq -s_0$,

(iv) $0 < \theta G(x,s) \leq sg(x,s)$ *for a.e.* $x \in \Omega$, $\forall s \geq s_0$.

*Then problem (4.14) has a nontrivial solution.*

*Proof.* We will apply Theorem 3.11. Without loss of generality, we may suppose in (4.7) that $q \in (p, p^*)$. For $u \in K$ (see (4.13)), the sets $\Omega_-$ and $\Omega_+$ will be considered as being defined by (4.21), and recall that $\Omega_- \subset \Omega \setminus \omega$.

First we check (H4′). By (i) and (ii), one can find numbers $\varepsilon \in (0, \lambda_1)$ and $\delta_0 > 0$ such that

$$G(x,s) \leq \frac{\lambda_1 - \varepsilon}{p}|s|^p \quad \text{for a.e. } x \in \Omega \setminus \omega,\ \forall s \in [-\delta_0, 0), \tag{4.28}$$

$$G(x,s) \leq \frac{\lambda_1 - \varepsilon}{p}|s|^p \quad \text{for a.e. } x \in \Omega,\ \forall s \in (0, \delta_0]. \tag{4.29}$$

From (4.11), there exists a constant $c = c(\delta_0)$ with

$$G(x,s) \leq c|s|^q \quad \text{for a.e. } x \in \Omega,\ \forall |s| > \delta_0. \tag{4.30}$$

For an arbitrary $u \in K$, by (4.28) and (4.30) we have

$$\begin{aligned}\int_{\Omega_-} G(x,u) &= \int_{\Omega_- \cap [-\delta_0 \leq u]} G(x,u) + \int_{[u<-\delta_0]} G(x,u)\\ &\leq \frac{\lambda_1 - \varepsilon}{p}\int_{\Omega_-}|u|^p + c\int_{\Omega_-}|u|^q.\end{aligned} \tag{4.31}$$

Similarly, (4.29) and (4.30) imply

$$\begin{aligned}\int_{\Omega_+} G(x,u) &= \int_{\Omega_+ \cap [u \leq \delta_0]} G(x,u) + \int_{[u>\delta_0]} G(x,u)\\ &\leq \frac{\lambda_1 - \varepsilon}{p}\int_{\Omega_+}|u|^p + c\int_{\Omega_+}|u|^q.\end{aligned} \tag{4.32}$$

Then, combining (4.31) and (4.32), we infer

$$\int_\Omega G(x,u) \leq \frac{\lambda_1 - \varepsilon}{p}\|u\|_{0,p}^p + c\|u\|_{0,q}^q. \tag{4.33}$$

Taking into account the continuity of the embedding $W \hookrightarrow L^q(\Omega)$, from (4.33) and (4.17) we get, for a constant $\tilde{c}$, the relations

$$\begin{aligned}\mathcal{G}(u) + \varphi(u) &= -\int_\Omega G(x,u) + \frac{1}{p}\|u\|_{1,p}^p \geq \frac{\varepsilon}{\lambda_1 p}\|u\|_{1,p}^p - \tilde{c}\|u\|_{1,p}^q > 0\\ &= \mathcal{G}(0) + \varphi(0)\end{aligned} \tag{4.34}$$

provided $u \in K$ and $\|u\|_{1,p} = \rho > 0$ is sufficiently small. Therefore, Theorem 3.11(H4$'$) is satisfied.

To check hypothesis (H1$'$), we proceed as follows. From (iv), we have

$$\frac{G(x,s)}{s} \leq \frac{1}{\theta} g(x,s) \quad \text{for a.e. } x \in \Omega,\ \forall s \geq s_0. \tag{4.35}$$

For a.e. $x \in \Omega$, the primitive $G(x,s)$ as a function of $s$ being continuous (even locally Lipschitz), (4.35) implies

$$\frac{G(x,s)}{s} \leq \frac{1}{\theta} \underline{g}(x,s) \quad \text{for a.e. } x \in \Omega,\ \forall s > s_0. \tag{4.36}$$

Similarly, by (iii), we get

$$G(x,s) \leq \frac{1}{\theta} s\overline{g}(x,s) \quad \text{for a.e. } x \in \Omega \setminus \omega,\ \forall s < -s_0. \tag{4.37}$$

Recall that under the assumptions (4.7) and (4.9), for $u \in L^q(\Omega)$, the following inclusion holds (see [4, Theorem 2.1]):

$$\partial(-\mathcal{G})(u) \subset \left[\underline{g}(x,u), \overline{g}(x,u)\right] \quad \text{for a.e. } x \in \Omega. \tag{4.38}$$

Then, from (4.20), (4.36), (4.37), (4.38), and (4.7), for an arbitrary $u \in K$, we obtain

$$\begin{aligned}
-\mathcal{G}(u) &= \int_\Omega G(x,u) = \int_{[u<-s_0]} G(x,u) + \int_{[u>s_0]} G(x,u) + \int_{[-s_0 \leq u \leq s_0]} G(x,u) \\
&\leq \frac{1}{\theta}\left[\int_{[u<-s_0]} u\overline{g}(x,u) + \int_{[u>s_0]} u\underline{g}(x,u)\right] + k|\Omega| \\
&\leq \frac{1}{\theta}\left[\int_{[u<-s_0]} uw + \int_{[u>s_0]} uw\right] + k|\Omega| \\
&= \frac{1}{\theta}\left[\int_\Omega uw - \int_{[|u|\leq s_0]} uw\right] + k|\Omega| \\
&\leq \frac{1}{\theta}\int_\Omega uw + k_0, \quad \forall w \in \partial(-\mathcal{G})(u),
\end{aligned} \tag{4.39}$$

for a constant $k_0 > 0$. As $\partial(-\mathcal{G})(u) = -\partial\mathcal{G}(u)$, it follows that

$$\mathcal{G}(u) \geq \frac{1}{\theta}\int_\Omega uw - k_0, \quad \forall w \in \partial\mathcal{G}(u). \tag{4.40}$$

Taking the supremum over $w \in \partial\mathcal{G}(u)$ in (4.40), we deduce

$$\mathcal{G}(u) - \frac{1}{\theta}\left(\mathcal{G}|_W\right)^0(u;u) \ge -k_0, \quad \forall u \in K. \tag{4.41}$$

By virtue of (4.4) and (4.5), one has

$$\varphi(u) - \frac{1}{\theta}\langle d\varphi(u), u\rangle = \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u\|_{1,p}^p, \quad \forall u \in W. \tag{4.42}$$

From (4.41) and (4.42), it turns out that Theorem 3.11(H1$'$) is fulfilled with

$$\alpha = \frac{1}{\theta}, \qquad a_0 = \frac{1}{p} - \frac{1}{\theta}, \qquad a_1 = 0, \qquad \sigma = p, \qquad b_0 = 0, \qquad b_1 = k_0. \tag{4.43}$$

To check condition Theorem 3.11(H3$'$), we first note that, on the basis of (i), (ii) and arguing as in the proof of [7, Proposition 7], one has

$$G(x,t) \ge \gamma_1(x)t^\theta \quad \text{for a.e. } x \in \Omega,\ \forall t > s_0, \tag{4.44}$$

$$G(x,t) \ge \gamma_2(x)|t|^\theta \quad \text{for a.e. } x \in \Omega \setminus \omega,\ \forall t < -s_0, \tag{4.45}$$

where $\gamma_1, \gamma_2 \in L^\infty(\Omega)$, $\gamma_1(x) > 0$ for a.e. $x \in \Omega$, and $\gamma_2(x) > 0$ for a.e. $x \in \Omega \setminus \omega$. Since, by assumption, $\operatorname{int}(\Omega \setminus \omega) \neq \varnothing$ if $W = W_1$ or $W = W_2$, there is some $\overline{u} \in K$ such that $|\Omega(\overline{u})| > 0$, where $\Omega(\overline{u}) = \{x \in \Omega : \overline{u} > s_0\}$. For $t \ge 1$, using (4.20), (4.44), (4.45), and the inclusion $[t\overline{u} < -s_0] \subset \Omega \setminus \omega$, we estimate $-\mathcal{G}$ as follows:

$$\begin{aligned}
-\mathcal{G}(t\overline{u}) &= \int_{[t|\overline{u}|>s_0]} G(x,t\overline{u}) + \int_{[t|\overline{u}|\le s_0]} G(x,t\overline{u}) \\
&\ge \int_{[t|\overline{u}|>s_0]} G(x,t\overline{u}) - k|\Omega| \\
&= \int_{[t\overline{u}>s_0]} G(x,t\overline{u}) + \int_{[t\overline{u}<-s_0]} G(x,t\overline{u}) - k|\Omega| \\
&\ge t^\theta\left[\int_{\Omega(\overline{u})} \gamma_1(x)\overline{u}^\theta + \int_{[t\overline{u}<-s_0]} \gamma_2(x)|\overline{u}|^\theta\right] - k|\Omega| \\
&\ge t^\theta \int_{\Omega(\overline{u})} \gamma_1(x)\overline{u}^\theta - k|\Omega|.
\end{aligned} \tag{4.46}$$

Therefore,

$$\mathcal{G}(t\overline{u}) + \varphi(t\overline{u}) \le -t^\theta \int_{\Omega(\overline{u})} \gamma_1(x)\overline{u}^\theta + \frac{t^p}{p}\|\overline{u}\|_{1,p}^p + k|\Omega|, \quad \forall t \ge 1. \tag{4.47}$$

Taking into account $\theta > p$, it follows that $\Phi_K(t\overline{u}) \to -\infty$ as $t \to +\infty$. This establishes (H3$'$) with $\overline{u}$ replaced by $t\overline{u}$, for some $t \ge 1$ sufficiently large.

Finally, hypothesis (H2′) is also satisfied because, as we have already noted, the duality mapping $d\varphi$ verifies condition $(S_+)$.

The application of Theorem 3.11 concludes the proof. □

*Remark 4.3.* If $\omega = \emptyset$, then $K = W$. Taking into account Remark 3.12, in this case, problem (4.14) becomes

$$\text{Find } u \in W \text{ such that } -\Delta_p u \in \partial(-\mathcal{G}|_W)(u). \tag{4.48}$$

This means that for $u \in W$, it corresponds $h \in \partial(-\mathcal{G}|_W)(u) \subset \partial(-\mathcal{G})(u) \subset L^{q'}(\Omega)$, with $1/q + 1/q' = 1$, such that $u$ satisfies the variational equality

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + hv\right) = 0, \quad \forall v \in W. \tag{4.49}$$

Assuming (4.7) and (4.9), inclusion (4.38) and equality (4.49) show that each solution of problem (4.48) for $W = W_0$ also solves the differential inclusion problem:

$$\text{Find } u \in W_0 = W_0^{1,p}(\Omega) \text{ such that } -\Delta_p u \in \left[\underline{g}(x,u), \overline{g}(x,u)\right] \quad \text{for a.e. } x \in \Omega. \tag{4.50}$$

In the case $W = W_1$, denoting by $\hat{w} = (1/|\Omega|)\int_\Omega w$ the mean value of any $w \in L^1(\Omega)$, relation (4.49) is expressed as follows:

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla w + h(w - \hat{w})\right) = 0, \quad \forall w \in W^{1,p}(\Omega), \tag{4.51}$$

or, equivalently,

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u \nabla w + (h - \hat{h})w\right) = 0, \quad \forall w \in W^{1,p}(\Omega). \tag{4.52}$$

Thus, if $W = W_1$, with $u \in W$ in (4.48), the following problem is solved:

$$\begin{aligned} \text{Find } u \in W_1 \text{ such that } -\Delta_p u \in \Big[\underline{g}(x,u) - \widehat{\overline{g}(\cdot,u)},\ & \overline{g}(x,u) \\ & - \widehat{\underline{g}(\cdot,u)}\Big] \quad \text{for a.e. } x \in \Omega. \end{aligned} \tag{4.53}$$

A problem similar to (4.53) is solved when $W = W_2$ in (4.48).

Corollary 4.4 (see [8, Theorem 5.1]). *Assume (4.7), (4.9), and*

$$\limsup_{|s|\to+\infty} \frac{pG(x,s)}{|s|^p} < \lambda_{1,W_0} \quad \textit{uniformly for a.e. } x \in \Omega. \tag{4.54}$$

*Then problem (4.50) has a solution.*

*Proof.* Theorem 4.1 applies with $\omega = \emptyset$. □

COROLLARY 4.5 (see [6, Theorem 3.6] and [8, Theorem 5.2]). *Assume (4.7) and (4.9) together with*

$$\limsup_{s\to 0} \frac{pG(x,s)}{|s|^p} < \lambda_{1,W_0} \quad \text{uniformly for a.e. } x \in \Omega. \tag{4.55}$$

*If there are numbers $\theta > p$ and $s_0 > 0$ such that*

$$0 < \theta G(x,s) \le sg(x,s) \quad \text{for a.e. } x \in \Omega,\ \forall |s| \ge s_0, \tag{4.56}$$

*then problem (4.50) has a nontrivial solution.*

*Proof.* We apply Theorem 4.2 with $\omega = \emptyset$. □

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