ON THE LOCATION OF THE PEAKS OF LEAST-ENERGY SOLUTIONS TO SEMILINEAR DIRICHLET PROBLEMS WITH CRITICAL GROWTH

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We study the location of the peaks of solution for the critical growth problem $-\varepsilon^2\Delta u + u = f(u) + u^{2^*-1}$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain; $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent and f has a behavior like u^p, $1 < p < 2^* - 1$.

1. Introduction

In this paper, we will study the location of the peaks of *least-energy* solution for the problem

$$\begin{aligned} -\varepsilon^2\Delta u + u &= f(u) + u^{2^*-1} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\varepsilon > 0$, and f is a function satisfying some subcritical conditions. Here $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent.

By *least-energy* solution for problem (1.1) we mean a critical point at the *Mountain-Pass* level of the associated *energy* functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) dz - \int_\Omega \left[F(u) + \frac{1}{2^*}\left(u^+\right)^{2^*}\right] dz, \tag{1.2}$$

(where $u^+ = \max\{u, 0\}$), defined on the Hilbert space $H_o^1(\Omega)$ endowed with the norm

$$\|u\|_\varepsilon^2 = \int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) dz. \tag{1.3}$$

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The Mountain-Pass level of J_ε is defined by

$$c_\varepsilon = \inf_{g\in\Gamma} \max_{0\le t\le 1} J_\varepsilon(g(t)), \tag{1.4}$$

where Γ is the set of all continuous paths joining the origin and a fixed nonzero element e in $H_o^1(\Omega)$, such that $e \neq 0$ and $J_\varepsilon(e) \le 0$. Under suitable hypothesis (e.g., (f_1), (f_4), (f_5) below), it is not hard to check that $c_\varepsilon > 0$ does not depend on the element $0 \neq v \in H_o^1(\Omega)$ and u is a *least-energy* solution if and only if $J_\varepsilon(u) = c$ and $J_\varepsilon'(u) = 0$, and $J_\varepsilon(u) \le J_\varepsilon(v)$ for all $v \neq 0$ such that $J_\varepsilon'(v) = 0$.

The existence of least-energy solution of problem (1.1) was given in Brézis and Nirenberg in [3, Theorem 2.1] (see Lemma 2.4 in this paper).

In this paper, we will study some properties of the least-energy solution u_ε of problem (1.1) when ε is small. In order to describe these properties, we introduce the hypotheses on the function f.

Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a $C^{1,\alpha}$ function such that

(f_1) $f(0) = f'(0) = 0$;
(f_2) there is $q_1 \in (1, (N+2)/(N-2))$ such that

$$\lim_{s\to\infty} \frac{f(s)}{s^{q_1}} = 0; \tag{1.5}$$

(f_3) there are $q_2 \in (1, (N+2)/(N-2))$ and $\lambda > 0$ such that

$$f(s) \ge \lambda s^{q_2}, \quad \forall s > 0 \tag{1.6}$$

(when $N = 3$, we need $q_2 > 2$, otherwise we require a sufficiently large λ);
(f_4) if $F(s) = \int_o^s f(t)\,dt$, for some $\theta \in (2, q_1 + 1)$ we have

$$0 < \theta F(s) \le f(s)s, \quad \forall s > 0; \tag{1.7}$$

(f_5) the function $f(s)/s$ is increasing for $s > 0$.

Since our interest is on positive solutions we define $f(s) = 0$, in $s \le 0$.

Now we will state our main result.

Theorem 1.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^N$; f satisfies* (f_1), (f_2), (f_3), (f_4), (f_5); *and let u_ε be the least-energy solution of (1.1). Then, there is a $\varepsilon_o > 0$ such that*

(i) *u_ε attains only one local maximum at some $z_\varepsilon \in \Omega$ (hence global maximum), for all $\varepsilon \in (0, \varepsilon_o]$;*
(ii) *u_ε converges uniformly to zero over compact subsets of $\Omega \setminus \{z_\varepsilon\}$ as $\varepsilon \to 0$;*
(iii) $\operatorname{dist}(z_\varepsilon, \partial\Omega) \to \max_{z\in\Omega} \operatorname{dist}(z, \partial\Omega)$.

This statement is analogous to the one given by Ni and Wei in [8], in the subcritical case

$$\begin{aligned} -\varepsilon^2\Delta u + u = h(u), &\quad \text{in } \Omega, \\ u = 0, &\quad \text{on } \partial\Omega, \end{aligned} \tag{1.8}$$

where h satisfies the following hypothesis:

(i) (f_1), (f_2), (f_4), and (f_5) hold;

(ii) the global problem

$$-\Delta u + u = h(u), \quad \text{in } \mathbb{R}^N \tag{1.9}$$

has a unique positive solution in $H^1(\mathbb{R}^N)$;

(iii) this solution is *nondegenerate* in the sense that

$$-\Delta v + v = h'(u)v, \quad \text{in } \mathbb{R}^N \tag{1.10}$$

has no nontrivial spherically symmetric solution in $L^2(\mathbb{R}^N)$.

In [8], Ni and Wei also have described the asymptotic profile (in ε) of u_ε, giving a detailed description for ε small. Here in the critical case, the solutions have the same profile.

In this work we will show that a ground state solution of the critical problem (1.1) is also solution of a subcritical problem (1.8) by showing that for small ε we have a uniform bound for the L^∞ norm of u_ε.

The difficulty here lies in finding an upper bound for $\|u_\varepsilon\|_{L^\infty(\Omega)}$ by obtaining a bound for u_ε in $L^p(\Omega)$ norm, for all $p \geq 2$. In the subcritical case this boundedness is obtained since the family u_ε is bounded in $H^1(\Omega)$ but this argument does not work in the critical case. Here, we obtain an L^∞-bound for u_ε through the estimate below, which is based on Moser's iteration technique (see [11]) and is essentially due to Brézis and Kato [2].

Proposition 1.2. *Let Λ be an open subset and $q \in L^{N/2}(\Lambda)$. Suppose that $g : \Lambda \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying*

$$|g(x,s)| \leq (q(x) + C_g)|s|, \quad \forall s \in \mathbb{R},\ x \in \Lambda \text{ and for some } C_g > 0. \tag{1.11}$$

Then, if $v \in H_o^1(\Lambda)$ is such that

$$-\Delta v = g(x,v), \quad \text{in } \Lambda \tag{1.12}$$

we have $v \in L^p(\Lambda)$ for all $2 \leq p < \infty$. Moreover, there is a positive constant $C_p = C(p, C_g, q)$ such that

$$\|v\|_{L^{2^*(p+1)}(\Lambda)} \leq C_p \|v\|_{L^{2(p+1)}(\Lambda)}. \tag{1.13}$$

Remark 1.3. The dependence on q of C_p can be given uniformly on a family of functions $\{q_\varepsilon\}_{\varepsilon>0}$ such that q_ε converges in $L^{N/2}$ (see the appendix).

We have organized this paper as follows: the next section contains the proof of Theorem 1.1. This proof consists in a series of lemmas which show the L^∞-bound for u_ε, where these functions are solutions of a class of subcritical problems (1.8). The third section is an appendix proving Proposition 1.2, for the sake of completeness.

2. Proof of Theorem 1.1

Before proving Theorem 1.1, let us fix some notation and preliminaries.

Remark 2.1. Throughout this section, we use the equivalent characterization of c_ε, which is more adequate to our purposes, given by

$$c_\varepsilon = \inf_{v \in H_o^1(\Omega)\setminus\{0\}} \max_{t\geq 0} J_\varepsilon(tv). \tag{2.1}$$

(see Willem [13, Theorem 4.2]).

We denote by $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ the functional given by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left[F(u) + \frac{1}{2^*}(u_+)^{2^*}\right] dx, \tag{2.2}$$

where

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx, \tag{2.3}$$

associated with the problem

$$-\Delta u + u = f(u) + |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N. \tag{2.4}$$

It is known that under assumptions (f_1), (f_2), (f_3), (f_4), (f_5), and (2.4) possesses a *ground state* solution ω in the level

$$c = J(\omega) = \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} J(tv), \tag{2.5}$$

(see [1]).

Remark 2.2. It is easy to check that for each nonzero v in $H^1(\mathbb{R}^N)$, there is a unique $t_o = t(v)$ such that

$$J(t_o v) = \max_{t\geq 0} J(tv). \tag{2.6}$$

Indeed, since

$$J(tv) = \frac{t^2}{2}\|v\|^2 - \int_{\mathbb{R}^N} \left[F(tv) - \frac{t^{2^*}}{2^*}(v^+)^{2^*}\right] dx, \quad \text{for } t \geq 0, \tag{2.7}$$

the maximum point t_o of $J(tv)$ is given by

$$\|v\|^2 = \int_{\mathbb{R}^N} \left[t_o^{-1} v f(t_o v) + t_o^{2^*-2} (v^+)^{2^*} \right] dx. \tag{2.8}$$

We assume, without loss of generality that $0 \in \Omega$. Set $\Omega_\varepsilon = \{x \in \mathbb{R}^N;\ \varepsilon x \in \Omega\}$.

The restriction of J to $H_o^1(\Omega_\varepsilon)$ is the energy functional,

$$J(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2)\, dx - \int_{\Omega_\varepsilon} \left[F(u_+) + \frac{1}{2^*} u_+^{2^*} \right] dx, \quad u \in H_o^1(\Omega_\varepsilon), \tag{2.9}$$

associated with the problem

$$\begin{gathered} -\Delta u + u = f(u) + u^{2^*-1} \quad \text{in } \Omega_\varepsilon, \\ u = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{gathered} \tag{2.10}$$

If u_ε is a critical point of J_ε, the family

$$v_\varepsilon(x) = u_\varepsilon(z) = u_\varepsilon(\varepsilon x), \quad z = \varepsilon x \tag{2.11}$$

is such that each v_ε is a critical point of functional J restricted to $H_o^1(\Omega_\varepsilon)$ at the level

$$b_\varepsilon = J(v_\varepsilon) = \inf_{v \in H_o^1(\Omega_\varepsilon)\setminus\{0\}} \max_{t \geq 0} J(tv). \tag{2.12}$$

It is easy to check that $b_\varepsilon = \varepsilon^{-N} c_\varepsilon$ and from the definition of c it follows that $b_\varepsilon \geq c$ for all $\varepsilon > 0$.

We will start with the following property of $\{b_\varepsilon\}_{\varepsilon>0}$.

Lemma 2.3. *For* $\{b_\varepsilon\}_{\varepsilon>0}$, $\lim_{\varepsilon \to 0} b_\varepsilon = c$.

Proof. Fix ω a ground state solution of problem (2.4) and let $\psi_\varepsilon(x) = \varphi(\varepsilon x)\omega(x)$, where φ is a C^1-function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in B_1, \\ 0 & \text{if } x \notin B_2, \end{cases} \tag{2.13}$$

$B_1 = B_\rho(0)$, $B_2 = B_{2\rho}(0) \subset \Omega$. Observe that $\psi_\varepsilon \to \omega$ in $H^1(\mathbb{R}^N)$ and the support of ψ_ε is in Ω_ε. By definition of b_ε, we have $t_\varepsilon > 0$ such that

$$b_\varepsilon \leq \max_{t>0} J(t\psi_\varepsilon) = J(t_\varepsilon \psi_\varepsilon). \tag{2.14}$$

From (2.8) and condition (f_3) it follows that

$$\begin{aligned} \|\psi_\varepsilon\|^2 &= \int_{\mathbb{R}^N} \left[t_\varepsilon^{-1}\psi_\varepsilon f(t_\varepsilon\psi_\varepsilon) + t_\varepsilon^{2^*-2}\psi_\varepsilon^{2^*} \right] dx \\ &\geq \int_{\mathbb{R}^N} \left[\lambda t_\varepsilon^{q_2-1}\psi_\varepsilon^{q_2+1} + t_\varepsilon^{2^*-2}\psi_\varepsilon^{2^*} \right] dx, \end{aligned} \tag{2.15}$$

so that, t_ε is bounded. Equality (2.15) and Remark 2.2 show that $t_\varepsilon \to t(\omega) = 1$, as $\varepsilon \to 0$. Then we have $t_\varepsilon\psi_\varepsilon \to \omega$ in $H^1(\mathbb{R}^N)$ and

$$\lim_{\varepsilon\to 0} J(t_\varepsilon\psi_\varepsilon) = J(\omega) = c. \tag{2.16}$$

Combining (2.14), (2.16), and the inequality $b_\varepsilon \geq c$, for all $\varepsilon > 0$, we have proved this lemma. □

Lemma 2.4. *The inequality $c < (1/N)S^{N/2}$ holds, where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.*

Proof. For each $h > 0$, consider the function

$$\phi_h(x) = \frac{[N(N-2)h]^{(N-2)/4}}{(h+|x|^2)^{(N-2)/2}}. \tag{2.17}$$

We recall that ϕ_h satisfies the problem

$$\begin{gathered} -\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N, \\ u(x) > 0, \qquad \int_{\mathbb{R}^N} |\nabla u|^2 dx < \infty, \\ \int_{\mathbb{R}^N} |\nabla\phi_h|^2 dx = \int_{\mathbb{R}^N} \phi_h^{2^*} dx = S^{N/2} \quad \text{(see Talenti [12]).} \end{gathered} \tag{2.18}$$

Now, consider $\psi_h(x) = \varphi\phi_h(x)/\|\varphi\phi_h\|_{L^{2^*}(\mathbb{R}^N)}$, where φ is the function defined in the proof of Lemma 2.3. From condition (f_3) we have

$$J(t\psi_h) \leq \frac{t^2}{2}\int_{B_2} \left(|\nabla\psi_h|^2 + \psi_h^2 \right) dx - \frac{\lambda t^{q_2+1}}{q_2+1}\int_{B_2} \psi_h^{q_2+1} dx - \frac{t^{2^*}}{2^*}. \tag{2.19}$$

Using arguments as in [7], there exists $h > 0$ such that

$$\max_{t\geq 0}\left\{ \frac{t^2}{2}\int_{B_2} \left(|\nabla\psi_h|^2 + \psi_h^2 \right) dx - \frac{\lambda t^{q_2+1}}{q_2+1}\int_{B_2} \psi_h^{q_2+1} dx - \frac{t^{2^*}}{2^*} \right\} < \frac{1}{N}S^{N/2}. \tag{2.20}$$

Therefore, from (2.19) and (2.20) we have that

$$\max_{t\geq 0} J(t\psi_h) < \frac{1}{N} S^{N/2}, \tag{2.21}$$

and the proof of the lemma is completed. □

Notice that the same proof of Lemma 2.4 can be used to show that $b_\varepsilon < (1/N)S^{N/2}$, for all $\varepsilon > 0$. Using [3, Theorem 2.1], this inequality implies the existence of v_ε and then the existence of u_ε.

Lemma 2.5. *There are* $\varepsilon_o > 0$; *a family* $\{y_\varepsilon\}_{\{0<\varepsilon\leq\varepsilon_o\}} \subset \mathbb{R}^N$, $y_\varepsilon \in \Omega_\varepsilon$; *constants* $R > 0$ *and* $\beta > 0$ *such that*

$$\int_{B_R(y_\varepsilon)} v_\varepsilon^2\, dx \geq \beta > 0, \quad \forall 0 < \varepsilon \leq \varepsilon_o, \tag{2.22}$$

$$\lim_{\varepsilon\to o} d(y_\varepsilon, \partial\Omega_\varepsilon) = \infty. \tag{2.23}$$

Proof. Start by showing that there is a family satisfying inequality (2.22). Arguing to the contrary, there is $\varepsilon_n \searrow 0$ such that for all $R > 0$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_R(x)} v_{\varepsilon_n}^2\, dx = 0. \tag{2.24}$$

Using (Lions [6, Lemma I.1]) we have

$$\int_{\mathbb{R}^N} v_{\varepsilon_n}^q\, dx = o_n(1), \quad \text{as } n \longrightarrow \infty,\ \forall 2 < q < 2^*, \tag{2.25}$$

and, from (f_1) and (f_2),

$$\int_{\mathbb{R}^N} F(v_{\varepsilon_n})\, dx = \int_{\mathbb{R}^N} v_{\varepsilon_n} f(v_{\varepsilon_n})\, dx = o_n(1). \tag{2.26}$$

Since $J'(v_{\varepsilon_n}) \cdot v_{\varepsilon_n} = 0$, we conclude from (2.26) that

$$||v_{\varepsilon_n}||^2 = \int_{\mathbb{R}^N} v_{\varepsilon_n}^{2^*}\, dx + o_n(1). \tag{2.27}$$

Let $\ell \geq 0$ be such that $\|v_{\varepsilon_n}\|^2 \to \ell$. Passing to the limit in $J(v_{\varepsilon_n}) = b_{\varepsilon_n}$ and using (2.26) we have

$$\ell = Nc \tag{2.28}$$

and hence $\ell > 0$. Now, using the definition of the constant S, we have

$$||v_{\varepsilon_n}||^2 \geq S\left(\int_{\mathbb{R}^N} v_{\varepsilon_n}^{2^*}\, dx\right)^{2/2^*}. \tag{2.29}$$

Taking the limit in the above inequalities, as $n \to \infty$, we achieve that

$$\ell \geq S\ell^{2/2^*}, \tag{2.30}$$

and by (2.28), that

$$c \geq \frac{1}{N} S^{N/2} \tag{2.31}$$

which contradicts Lemma 2.4 and then (2.22) holds.

Finally, to establish (2.23), suppose the contrary. That is, there exist $\varepsilon_n \to 0$ and $R > 0$ such that $\operatorname{dist}(y_{\varepsilon_n}, \partial\Omega_{\varepsilon_n}) \leq R$, hence $\operatorname{dist}(\varepsilon_n y_{\varepsilon_n}, \partial\Omega) \leq \varepsilon_n R$. Without loss of generality, we have $\varepsilon_n y_{\varepsilon_n} \to y_o$ for some $y_o \in \partial\Omega$. The arguments that follow can be found in [8].

Let ν be the unit interior normal to $\partial\Omega$ at y_o, and $\delta > 0$ such that $B_\delta(y_o + \delta\nu) \subset \Omega$ and $B_\delta(y_o - \delta\nu) \cap \Omega = \emptyset$. Let $\Omega_n = \{x \in \mathbb{R}^N : y_o + \varepsilon_n x \in \Omega\}$ and $w_n(x) = u_{\varepsilon_n}(y_o + \varepsilon_n x)$. This sequence w_n is bounded in $H^1(\mathbb{R}^N)$, $-\Delta w_n + w_n = f(w_n) + w_n^{2^*-1}$ in Ω_n,

$$\int_{B_{2R}(0)} w_n^2 \, dx \geq \int_{B_R(y_{\varepsilon_n})} v_{\varepsilon_n}^2 \, dx \geq \beta > 0, \quad \forall n, \tag{2.32}$$

and we have that w_n converges weakly to some w in $H^1(\mathbb{R}^N)$.

Let $\mathbb{R}^N_{+,\nu}$ be the half space $\{x \in \mathbb{R}^N : x \cdot \nu > 0\}$. Notice that $B_{\varepsilon_n^{-1}\delta}(\varepsilon_n^{-1}\delta\nu) \subset \Omega_n$ and $B_{\varepsilon_n^{-1}\delta}(-\varepsilon_n^{-1}\delta\nu) \cap \Omega_n = \emptyset$ and then we can prove that for all compacts $K_+ \subset \mathbb{R}^N_{+,\nu}$ and $K_- \subset \mathbb{R}^N_{-,\nu} = \mathbb{R}^N \setminus \overline{\mathbb{R}^N_{+,\nu}}$, we have $K_+ \subset \Omega_n$ and $K_- \cap \Omega_n = \emptyset$, for n large.

Then for each $\phi \in C_o^\infty(\mathbb{R}^N_{+,\nu})$ such that $\operatorname{supp}\phi \subset \Omega_n$, we have

$$\int_{\mathbb{R}^N_{+,\nu}} (\nabla w_n \nabla \phi + w_n \phi) \, dx = \int_{\mathbb{R}^N_{+,\nu}} (f(w_n) + w_n^{2^*-1})\phi \, dx. \tag{2.33}$$

From (2.33), usual arguments show that $w \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N_+)$ and satisfies $-\Delta w + w = f(w) + w^{2^*-1}$, in $\mathbb{R}^N_{+,\nu}$, and $w \equiv 0$ in $\mathbb{R}^N_{-,\nu}$. Theorem I.1, due to Esteban and Lions in [4], shows that $w \equiv 0$ which contradicts

$$\int_{B_{2R}(0) \cap \mathbb{R}^N_{+,\nu}} w^2 \, dx \geq \beta > 0. \tag{2.34}$$

This completes the proof of the lemma. □

Now we will consider the translation of v_ε, defined by $\omega_\varepsilon(x) = v_\varepsilon(x + y_\varepsilon) = u_\varepsilon(\varepsilon y_\varepsilon + \varepsilon x)$ in $\tilde{\Omega}_\varepsilon = \{x \in \mathbb{R}^N; \varepsilon y_\varepsilon + \varepsilon x \in \Omega\}$ and $\omega_\varepsilon = 0$ outside $\tilde{\Omega}_\varepsilon$. From (2.23), any compact subset of $\mathbb{R}^N$ is contained in $\tilde{\Omega}_\varepsilon$, for ε sufficiently small.

From Lemma 2.5,

$$\int_{B_R(0)} \omega_\varepsilon^2 \, dx \geq \beta > 0, \quad \forall 0 < \varepsilon \leq \varepsilon_o. \tag{2.35}$$

Consider a sequence $\varepsilon_n \searrow 0$ and set $\widetilde{\Omega}_n = \widetilde{\Omega}_{\varepsilon_n}$, $\omega_n = \omega_{\varepsilon_n}$, $v_n = v_{\varepsilon_n}$, $y_\varepsilon = y_{\varepsilon_n}$.

We will prove that ω_n is bounded in the L^∞ norm. In that case, u_ε is also bounded in $L^\infty(\Omega)$ norm and the proof of Theorem 1.1 follows from the subcritical case, as Lemma 2.8 will show.

Since the sequence ω_n a translation of v_n, we have a uniform bound for $\|\omega_n\|$ and there is a $\omega_o \in H^1(\mathbb{R}^N)$ which is weak limit of ω_n in $H^1(\mathbb{R}^N)$. From (2.35) we have $\omega_o \neq 0$. We can write limit (2.23) in the following form

$$\lim_{n\to\infty} d(0, \partial\widetilde{\Omega}_n) = \infty. \tag{2.36}$$

Then for each $\phi \in C_o^\infty(\mathbb{R}^N)$ and large n such that $\operatorname{supp}\phi \subset \widetilde{\Omega}_n$, we have

$$\int_{\mathbb{R}^N} (\nabla\omega_n\nabla\phi + \omega_n\phi)\, dx = \int_{\mathbb{R}^N} (f(\omega_n) + \omega_n^{2^*-1})\phi\, dx, \quad \forall n. \tag{2.37}$$

From (2.37), usual arguments show that ω_o is a solution of problem (2.4), hence a critical point of J, and $J(\omega_o) \geq c$.

Lemma 2.6. *The sequence ω_n converges to ω_o in $H^1(\mathbb{R}^N)$ and $J(\omega_o) = c$.*

Proof. This fact comes from Lemma 2.5 and Fatou's lemma applied in the positive sequence $\omega_n f(\omega_n) - \theta F(\omega_n)$. Observe that

$$\begin{aligned}
b_{\varepsilon_n} &= J(v_n) - \frac{1}{\theta}J'(v_n)v_n \\
&= \left(\frac{\theta-2}{2\theta}\right)\|v_n\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}[v_n f(v_n) - \theta F(v_n)] + \left(\frac{2^*-\theta}{2^*\theta}\right)\int_{\mathbb{R}^N} v_n^{2^*} \\
&= \left(\frac{\theta-2}{2\theta}\right)\|\omega_n\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}[\omega_n f(\omega_n) - \theta F(\omega_n)] + \left(\frac{2^*-\theta}{2^*\theta}\right)\int_{\mathbb{R}^N} \omega_n^{2^*}.
\end{aligned} \tag{2.38}$$

From (2.38)

$$\begin{aligned}
c \leq J(\omega_o) &= J(\omega_o) - \frac{1}{\theta}J'(\omega_o)\omega_o \\
&= \left(\frac{\theta-2}{2\theta}\right)\|\omega_o\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}[\omega_o f(\omega_o) - \theta F(\omega_o)] + \left(\frac{2^*-\theta}{2^*\theta}\right)\int_{\mathbb{R}^N} \omega_o^{2^*} \\
&\leq \liminf\left(\frac{\theta-2}{2\theta}\right)\|\omega_n\|^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}[\omega_n f(\omega_n) - \theta F(\omega_n)] + \left(\frac{2^*-\theta}{2^*\theta}\right)\int_{\mathbb{R}^N} \omega_n^{2^*} \\
&= \lim_{n\to\infty} b_{\varepsilon_n} = c.
\end{aligned} \tag{2.39}$$

We have proved that $J(\omega_o) = c$ and then (2.39) becomes an equality.

Combining (2.39) with the three following inequalities:

$$\begin{gathered}\|\omega_o\|^2 \le \liminf \|\omega_n\|^2, \\ \int_{\mathbb{R}^N} [\omega_o f(\omega_o) - \theta F(\omega_o)]\,dx \le \liminf \int_{\mathbb{R}^N} [\omega_n f(\omega_n) - \theta F(\omega_n)]\,dx, \\ \int_{\mathbb{R}^N} \omega_o^{2^*}\,dx \le \liminf \int_{\mathbb{R}^N} \omega_n^{2^*}\,dx,\end{gathered} \tag{2.40}$$

we conclude that $\|\omega_n\| \to \|\omega_o\|$ and then $\omega_n \to \omega_o$ in $H^1(\mathbb{R}^N)$. □

We are ready to conclude the proof of our main result. From Proposition 1.2 and Remark 1.3 with $q(x) = \omega_n^{2^*-2} \in L^{N/2}$; $g(x,s) = f(s) + s^{2^*} - s$, we have $\omega_n \in L^t$ for all $t \ge 2$ and

$$\|\omega_n\|_{L^t} \le C_t, \tag{2.41}$$

where C_t does not depend on n.

Now we will make use of a very particular version of [5, Theorem 8.17], due to Trudinger.

Proposition 2.7. *Suppose that $t > N$, $g \in L^{t/2}(\Lambda)$, and $u \in H_o^1(\Lambda)$ satisfies (in the weak sense)*

$$-\Delta u + u \le \tilde{g}(x), \tag{2.42}$$

where Λ is an open subset of $\mathbb{R}^N$. Then for any ball $B_{2R}(y) \subset \Lambda$,

$$\sup_{B_R(y)} u \le C\left(\|u^+\|_{L^2(B_{2R}(y))} + \|g\|_{L^{t/2}(B_{2R}(y))}\right), \tag{2.43}$$

where C depends on N, t, and R.

We know that each ω_n satisfies

$$-\Delta \omega_n + \omega_n = \omega_n^{2^*-1} + f(\omega_n), \quad \text{in } \tilde{\Omega}_n \tag{2.44}$$

and this implies that

$$-\Delta \omega_n + \omega_n \le g_n(x) = \omega_n^{2^*-1} + f(\omega_n), \quad \text{in } \mathbb{R}^N \tag{2.45}$$

in the weak sense.

Since (2.41) holds, $\|g_n\|_{L^t}$ is bounded from above for some $t > N$. Using Proposition 2.7 in (2.45) we have

$$\sup_{B_1(y)} \omega_n \le C\left(\|\omega_n\|_{L^2(B_{2R}(y))} + \|g_n\|_{L^t(B_{2R}(y))}\right) \tag{2.46}$$

for all $y \in \mathbb{R}^N$, which implies that there is a constant $a > 0$, independent of n, such that

$$\omega_n(x) \le a, \quad \forall x \in \mathbb{R}^N. \tag{2.47}$$

It follows that there is a $\varepsilon_o > 0$ such that

$$u_\varepsilon(z) \le a, \quad \forall z \in \Omega,\ \forall \varepsilon < \varepsilon_o. \tag{2.48}$$

To conclude the proof observe that u_ε becomes a solution of the subcritical case (1.8) with h given by

$$h(s) = \begin{cases} f(s) + s^{2^*-1}, & \text{if } s \le a, \\ f(s) + \dfrac{(2^*-1)}{(\theta-1)} a^{2^*-\theta} s^{\theta-1} - \dfrac{(2^*-\theta)}{(\theta-1)} a^{2^*-1}, & \text{if } s > a, \end{cases} \tag{2.49}$$

where $\theta > 2$ is that one fixed in condition (f_4). It is easy to check that h is a $C^{1,\alpha}$ function, h and $H(s) = \int_o^s h(\tau)\,d\tau$ satisfy (f_1), (f_2), (f_3), (f_4), and (f_5). Let $\tilde{J}_\varepsilon$ be the C^1-functional on $H_o^1(\Omega)$ given by

$$\tilde{J}_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2 |\nabla u|^2 + u^2\right) dz - \int_\Omega H(u)\,dz. \tag{2.50}$$

Since $f(s) + s^{2^*-1} \ge h(s)$ for all $s > 0$, we have that

$$J_\varepsilon(u) \le \tilde{J}_\varepsilon(u), \quad \forall u \in H_o^1(\Omega), \tag{2.51}$$

$J_\varepsilon(u_\varepsilon) = \tilde{J}_\varepsilon(u_\varepsilon)$, $J'_\varepsilon(u_\varepsilon) = \tilde{J}'_\varepsilon(u_\varepsilon) = 0$. We conclude that u_ε is a least-energy solution of the subcritical problem (1.8).

Lemma 2.8. (i) *If $\tilde{c}_\varepsilon$ is the minimax level of $\tilde{J}_\varepsilon$, then $\tilde{c}_\varepsilon = c_\varepsilon$;*
(ii) *each u_ε is a critical point of $\tilde{J}_\varepsilon$ in the minimax level and satisfies (1.8).*

Since global problem (1.9) has a unique nondegenerate positive solution (cf. [9, 10]), Theorem 1.1 comes from [8, Theorem 2.2] applied to the functional $\tilde{J}_\varepsilon$, and the asymptotic profile comes from [8, Theorem 2.3].

Appendix

Let Λ be some general domain in $\mathbb{R}^N$ (bounded or unbounded). We will start with the following lemma due to Brézis and Kato [2].

Lemma A.1. *Let $q \in L^{N/2}(\Lambda)$ be a nonnegative function. Then, for every $\varepsilon > 0$, there is a constant $\sigma_\varepsilon = \sigma(\varepsilon, q) > 0$ such that*

$$\int_\Lambda q(x) u^2\,dx \le \varepsilon \int_\Lambda |\nabla u|^2\,dx + \sigma_\varepsilon \int_\Lambda u^2\,dx, \quad \forall u \in H_o^1(\Lambda). \tag{A.1}$$

Remark A.2. If $q_k \to q$ in $L^{N/2}(\Lambda)$, we can choose a constant σ_ε independent of k. That is, $\sigma(\varepsilon, q_k) = \sigma_\varepsilon$ and

$$\int_\Lambda q_k(x)u^2\,dx \le \varepsilon \int_\Lambda |\nabla u|^2\,dx + \sigma_\varepsilon \int_\Lambda u^2\,dx, \quad \forall u \in H_o^1(\Lambda),\ k \in \mathbb{N}. \tag{A.2}$$

Proof. Let $\sigma_\varepsilon = \sigma(\varepsilon, q) > 0$ be such that

$$\|q\|_{L^{N/2}(\{q\ge\sigma_\varepsilon\})} \le \varepsilon S, \tag{A.3}$$

where S is a best constant in the Sobolev immersion $H_o^1(\Lambda) \hookrightarrow L^{2^*}(\Lambda)$, where $2^* = 2N/(N-2)$. For all $u \in H_o^1(\Lambda)$, we have

$$\begin{aligned}
\int_\Lambda q(x)u^2\,dx &= \int_{\{q\ge\sigma_\varepsilon\}} q(x)u^2\,dx + \int_{\{q\le\sigma_\varepsilon\}} q(x)u^2\,dx \\
&\le \sigma_\varepsilon \int_{\{q\le\sigma_\varepsilon\}} u^2\,dx + \int_{\{q\ge\sigma_\varepsilon\}} q(x)u^2\,dx \\
&\le \sigma_\varepsilon \int_\Lambda u^2\,dx + \|q\|_{L^{N/2}(\{q\ge\sigma_\varepsilon\})}\|u\|^2_{L^{2^*}(\{q\ge\sigma_\varepsilon\})}.
\end{aligned} \tag{A.4}$$

Inequality (A.1) follows from Sobolev estimate and the choice of σ_ε. □

Remark 1.3 follows from the proof of Lemma A.1 and the inequality

$$\int_\Lambda q_k(x)u^2\,dx \le \int_\Lambda q(x)u^2\,dx + \|q_k - q\|_{L^{N/2}(\Lambda)}\|u\|^2_{L^{2^*}(\Lambda)}. \tag{A.5}$$

Proof of Proposition 1.2. For any $n \in \mathbb{N}$ and $p > 0$, consider $A_n = \{x \in \Lambda : |v|^p \le n\}$, $B_n = \Lambda \setminus A_n$, and define v_n by

$$v_n = v|v|^{2p} \quad \text{in } A_n, \qquad v_n = n^2 v \quad \text{in } B_n. \tag{A.6}$$

Observe that $v_n \in H_o^1(\Lambda)$, $v_n \le |v|^{2p+1}$ and

$$\nabla v_n = (2p+1)|v|^{2p}\nabla v \quad \text{in } A_n, \qquad \nabla v_n = n^2 \nabla v \quad \text{in } B_n. \tag{A.7}$$

So, using v_n as a test function

$$\int_\Lambda \nabla v \nabla v_n\,dx = \int_\Lambda g(x,v)v_n\,dx. \tag{A.8}$$

Using (A.7), we have

$$\begin{aligned}
(2p+1)\int_{A_n} |v|^{2p}|\nabla v|^2\,dx + n^2 \int_{B_n} |\nabla v|^2\,dx \\
\le \int_\Lambda |g(x,v)v_n|\,dx \le \int_\Lambda (q(x)+C_g)\,|vv_n|\,dx.
\end{aligned} \tag{A.9}$$

Now consider

$$\omega_n = v|v|^p \quad \text{in } A_n, \qquad \omega_n = nv \quad \text{in } B_n. \tag{A.10}$$

Notice that $\omega_n^2 = vv_n \leq |v|^{2(p+1)}$ and

$$\nabla\omega_n = (p+1)|v|^p\nabla v \quad \text{in } A_n, \qquad \nabla v_n = n\nabla v \quad \text{in } B_n. \tag{A.11}$$

Therefore,

$$\int_\Lambda |\nabla\omega_n|^2\,dx = (p+1)^2\int_{A_n}|v|^{2p}|\nabla v|^2\,dx + n^2\int_{B_n}|\nabla v|^2\,dx. \tag{A.12}$$

Combining (A.9) and (A.12), we obtain

$$\frac{2p+1}{(p+1)^2}\int_\Lambda |\nabla\omega_n|^2\,dx \leq \int_\Lambda (q(x)+C_g)\omega_n^2\,dx. \tag{A.13}$$

Let σ_p be given by Lemma A.1 with $\varepsilon = (2p+1)/2(p+1)^2$. Then

$$\int_\Lambda |\nabla\omega_n|^2\,dx \leq \tilde{C}_p\int_\Lambda \omega_n^2\,dx, \tag{A.14}$$

where $\tilde{C}_n = (2(p+1)^2/(2p+1))(C_g+\sigma_p)$. Suppose that $v \in L^{2(p+1)}(\Lambda)$ for some $p \geq 2$. Applying Sobolev immersion in inequality (A.14) we have

$$\left[\int_{A_n}\omega_n^{2^*}\,dx\right]^{2/2^*} \leq \left[\int_\Lambda \omega_n^{2^*}\,dx\right]^{2/2^*} \leq S\tilde{C}_p\int_\Lambda |v|^{2(p+1)}\,dx \tag{A.15}$$

that is,

$$\left[\int_{A_n}|v|^{2^*(p+1)}\,dx\right]^{2/2^*}dx \leq C_p\int_\Lambda |v|^{2(p+1)}\,dx, \tag{A.16}$$

where

$$C_p = \frac{2(p+1)^2}{2p+1}S(C_g+\sigma_p). \tag{A.17}$$

Now, passing to the limit in (A.16) we have $v \in L^{2^*(p+1)}(\Lambda)$ and

$$\|v\|_{L^{2^*(p+1)}(\Lambda)} \leq C_p\|v\|_{L^{2(p+1)}(\Lambda)}. \tag{A.18}$$

The proof follows from the following iteration argument: let p_1 a positive such that $2(p_1+1) = 2^*$. It is easy to see that $0 < p_1$ and $v \in L^{2(p_1+1)}(\Lambda)$. Using inequality (A.18) we have

$$v \in L^{2^*(p_1+1)}(\Lambda). \tag{A.19}$$

Now choose p_2 such that $2(p_2+1) = 2^*(p_1+1)$. It is easy to see that $0 < p_1 < p_2$ and $v \in L^{p_2+1}(\Lambda)$. Using inequality (A.18) we have

$$v \in L^{2^*(p_2+1)}(\Lambda). \tag{A.20}$$

Continuing with this iteration we obtain an increasing sequence p_k given by $2(p_{k+1}+1) = 2^*(p_k+1)$ such that $v \in L^{2(p_{k+1}+1)}(\Lambda)$ for all $k \in \mathbb{N}$. From

$$p_{k+1}+1 = \frac{N}{N-2}(p_k+1), \tag{A.21}$$

it follows that

$$p_{k+1}+1 = \left[\frac{N}{N-2}\right]^k 2^*. \tag{A.22}$$

This shows that p_k goes to ∞ and therefore,

$$v \in L^p(\Lambda), \quad \forall p \geq 2. \tag{A.23}$$

□

Remark A.3. Proposition 1.2 is valid for positive subsolutions of problem (1.12) as we can check in its proof.

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