# ON THE MANN AND ISHIKAWA ITERATION PROCESSES

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ABSTRACT. It is shown that a result of Chidume, involving the strong convergence of the Mann iteration process for continuous strongly accretive operators, is actually a corollary to a result by Nevanlinna and Reich. It is then shown that the Nevanlinna and Reich result can be extended to the case of an Ishikawa iteration process.

## 1. INTRODUCTION AND PRELIMINARIES

In [4, Theorem 1] Chidume gave a strong convergence theorem on the Mann iterative process for a class of continuous strongly accretive maps. We are going to show that Chidume's theorem is a corollary of a result by Nevanlinna and Reich [5, Theorem 3].

Recently, the authors have proved in [9, Theorem 2.1] a considerably more general strong convergence theorem for the Ishikawa iterative process for a class of strongly quasi-accretive operators. Theorem 2.1 of [9] is closely related to those strong convergence theorems of [5], [3]. We shall discuss the relations between Theorem 2.1 of [9] and the corresponding results in [5], [3].

Let $X$ be a real Banach space with a dual $X^*$, and let $J : X \to 2^{X^*}$ be the normalized duality mapping defined by

$$Jx = \{f \in X^* :< f, x >= \|f\|\|x\|, \ \|f\| = \|x\|\},$$

where $< \cdot, \cdot >$ denotes the generalized duality pairing.

It is well known that if $X^*$ is strictly convex, then $J$ is single-valued and such that $J(tx) = tJx$ for all $t \geq 0$, $x \in X$. If $X$ is uniformly smooth, then $J$ is uniformly continuous on bounded subsets of $X$.

An operator $T$ with domain $D(T)$ and range $R(T)$ in $X$ is said to be "accretive" if for every $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such

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that

$$< Tx - Ty, j(x-y) >\geq 0. \tag{1.1}$$

The operator $T$ is said to be "strongly accretive" if for each $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$< Tx - Ty, j(x-y) >\geq k\|x-y\|^2, \tag{1.2}$$

for some fixed real constant $k > 0$.

An accretive operator $T$ is "$m$-accretive" if $R(I+rT) = X$ for all $r > 0$, where $I$ denotes the identity operator.

We let $N(T) = \{x \in D(T) | Tx = 0\}$. If $N(T) \neq \phi$ and the inequality (1.1) ((1.2)) holds for all $x \in D(T)$ and $y \in N(T)$, then the corresponding operator $T$ is said to be "quasi-accretive" ("strongly quasi-accretive").

We denote the distance between a point $x \in X$ and a set $V \subset X$ by $d(x, V)$. Recall that a point $z \in V$ is said to be a "best approximation" to $x \in X$ if $\|x - z\| = d(x, V)$. A set $V \subset X$ is said to be a "sun" (see [5]) if: whenever $z \in V$ is a best approximation to $x \in X$, then $z$ is also a best approximation to $z + t(x-z)$ for all $t \geq 0$. It is well known that every convex set is a sun. If $V$ is a sun and $z \in V$ is a best approximation to $x \in X$, then there exists $j(x-z) \in J(x-z)$ such that $< y - z, j(x-z) >\leq 0$ for all $y \in V$. The set $V$ is said to be "proximinal" if for every $x \in X$ has at least one best approximation in $V$.

We need the following Lemmas.

**Lemma 1.1.** *Let $X$ be a reflexive Banach space and le $C$ be a closed convex subset of $X$. Then $C$ is proximinal.*

*Proof.* The proof is straightforward. ∎

**Lemma 1.2.** *Let $\{a_n\}$ be a nonnegative sequence satisfying*

$$a_{n+1} \leq a_n + \sigma_n \tag{1.3}$$

*with $\sum_{n=1}^{\infty} \sigma_n < +\infty$. Then $\lim\limits_{n\to\infty} a_n$ exists.*

*Proof.* Note that, for all $m \geq 1$,

$$a_{n+m} \leq a_n + \sum_{k=n}^{n+m-1} \sigma_k,$$

which implies $\lim\limits_{n\to\infty} \sup a_n \leq \lim\limits_{n\to\infty} \inf a_n$. ∎

**Lemma 1.3.** (Xu and Roach [8]) *Let $X$ be a real uniformly smooth Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, Jx > + K \max\{\|x\| + \|y\|, \frac{c}{2}\}\rho_X(\|y\|), \tag{1.4}$$

*for all $x, y \in X$, where $K$ and $c$ are positive constants, and $\rho(\tau)$ is the modulus of smoothness of $X$ (defined by*

$$\rho_X(\tau) = \sup\{\frac{1}{2}\|x+y\| + \frac{1}{2}\|x-y\| - 1 | \|x\| = 1, \|y\| \leq \tau\},$$

*and satisfying*

$$\lim_{\tau\to 0}\frac{\rho_X(\tau)}{\tau}=0).$$

**Lemma 1.4.** *If $X$ is uniformly smooth and $T : X \supset D(T) \to X$ is m-accretive, then $T$ is demiclosed, i.e., for any sequence $\{x_n\} \subset D(T)$, with $x_n \to x$ strongly and $Tx_n \to y$ weakly as $n \to \infty$, we have $x \in D(T)$ and $Tx = y$.*

*Proof.* See Barbu [1]. ∎

**Lemma 1.5.** *If $X$ is uniformly smooth and $T : X \to X$ is demi-continuous and accretive, then $T$ is m-accretive.*

*Proof.* See Browder [2]. ∎

## 2. Main results

Before we show our main results, we give a slight extension of [5, Theorem 3]. For the sake of simplicity, we only consider the following Mann iterative process

$$x_{n+1} = x_n - \lambda_n T x_n,\ n \geq 0, \tag{2.1}$$

where $x_0 \in X$ and $\{\lambda_n\}$ is a positive sequence. We shall study the convergence of $\{x_n\}$ under more general assumptions.

In the sequel, we always assume that $X$ is uniformly smooth and $N(T)$ has a nonempty convex subset $N_0(T)$.

**Theorem 2.1.** *Let $T$ be a quasi-accretive and demiclosed operator, and let $\{\lambda_n\}$ be a positive sequence such that $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\sum_{n=0}^{\infty}\rho_X(\lambda_n) < \infty$. Assume that $\{x_n\}$ satisfies (2.1) and $\{Tx_n\}$ is bounded. Let $P_0$ be an arbitrary selection of the nearest point mapping from $X$ onto $N_0(A)$ such that*

$$< y - P_0x, J(x - P_0x) > \leq 0 \text{ for all } y \in N_0(A).$$

*If there exists a strictly increasing function $\psi : R^+ \to R^+$, $\psi(0) = 0$, such that*

$$< Tx_n, J(x_n - P_0x_n) > \geq \psi(\|x_n - P_0x_n\|)\|Tx_n\|,\ n \geq 0, \tag{2.2}$$

*then $\{x_n\}$ converges strongly to a zero of $T$.*

*Proof.* Since $T$ is demiclosed, we know that $N_0(T)$ is closed. By Lemma 1.1 we see that $N_0(T)$ is proximinal. Thus we can choose a section $P_0 : X \to N_0(T)$ of the nearest point operator such that

$$< y - P_0x, J(x - P_0x) > \leq 0 \text{ for all } y \in N_0(T).$$

Let $j_n = J(x_n - P_0x_n)$ and $M = \sup\{\|Tx_n\| | n \geq 0\}$. By (2.1) and (1.4) we have

$$\begin{aligned}\|x_{n+1}-P_0x_{n+1}\|^2 &\leq \|x_{n+1} - P_0x_n\|^2 \\ &= \|x_n - P_0x_n - \lambda_n Tx_n\|^2 \\ &\leq \|x_n - P_0x_n\|^2 - 2\lambda_n < Tx_n, j_n > \\ &\quad + K \max\{\|x_n - P_0x_n\| + \lambda_n\|Tx_n\|, \frac{c}{2}\}\rho_X(\lambda_n\|Tx_n\|) \\ &\leq \|x_n - P_0x_n\|^2 - 2\lambda_n < Tx_n, j_n > \\ &\quad + M_1 \max\{\|x_n - P_0x_n\| + \lambda_n M, \frac{c}{2}\}\rho_X(\lambda_n),\end{aligned} \tag{2.3}$$

for some $M_1 > 0$. Here we have used the fact that $\rho_X(\tau)$ is nondecreasing and that there exists a constant $c_0 > 0$ such that $\frac{\rho_X(\eta)}{\eta^2} \leq c_0\frac{\rho_X(\tau)}{\tau^2}$ for any $\eta \geq \tau > 0$. The condition $\sum_{n=0}^{\infty} \rho_X(\lambda_n) < +\infty$ implies $\lambda_n \to 0$ as $n \to \infty$. We claim that $\|x_n - P_0x_n\|$ is bounded. Assume the contrary and let $d_n = \|x_n - P_0x_n\|$. We may also assume that $d_n + \lambda_n M \geq \frac{c}{2}$ for all $n \geq 0$. Then (2.2) leads to

$$d_{n+1}^2 \leq d_n^2 + M_1(d_n + \lambda_n M)\rho_X(\lambda_n)$$

and, consequently, for $\lambda_n \in (0, \frac{c}{4M})$,

$$d_{n+1} \leq d_n + M_1\rho_X(\lambda_n).$$

In view of Lemma 1.2 we see that $\lim_{n\to\infty} d_n$ exists, which contradicts with the assumption that $\{d_n\}$ is unbounded. Hence, from (2.3) we get

$$\|x_{n+1} - P_0x_{n+1}\|^2 \leq \|x_n - P_0x_n\|^2 + M_2\rho_X(\lambda_n), \tag{2.4}$$

for some constant $M_2 > 0$, since $\|x_n - P_0x_n\|$ is bounded. By Lemma 1.2 we see that $\lim_{n\to\infty} \|x_n - P_0x_n\|$ exists. Furthermore,

$$2\sum_{n=0}^{\infty} \lambda_n < Tx_n, j_n > \leq \|x_0 - P_0x_0\|^2 + M_2\sum_{n=0}^{\infty} \rho_X(\lambda_n) < +\infty. \tag{2.5}$$

Multiplying by $\lambda_n$ both sides of (2.2), we obtain

$$\lambda_n < Tx_n, j_n > \geq \psi(\|x_n - P_0x_n\|)\|\lambda_n Tx_n\|.$$

Using (2.1), we have

$$\lambda_n < Tx_n, j_n > \geq \psi(\|x_n - P_0x_n\|)\|x_{n+1} - x_n\|.$$

It follows that

$$\sum_{n=0}^{\infty} \lambda_n < Tx_n, j_n > \geq \sum_{n=0}^{\infty} \psi(\|x_n - P_0x_n\|)\|x_{n+1} - x_n\|.$$

Hence

$$\sum_{n=0}^{\infty} \psi(\|x_n - P_0x_n\|)\|x_{n+1} - x_n\| < \infty.$$

Now we consider the following two possible cases:

Case 1. $\lim_{n\to\infty}\inf \psi(\|x_n - P_0x_n\|) = 0$.

Since $\psi : R^+ \to R^+$ is strictly increasing, we have $\lim_{n\to\infty}\inf \|x_n - P_0x_n\| = 0$. Since $\lim_{n\to\infty} \|x_n - P_0x_n\|$ exists, we get $\lim_{n\to\infty} \|x_n - P_0x_n\| = 0$. On the other hand, by (2.3), we have

$$\|x_n - P_0x_k\|^2 \le \|x_k - P_0x_k\|^2 + M_2 \sum_{i=k}^{\infty} \rho_X(\lambda_i), \text{ for all } n > k.$$

Consequently, $\|x_n - x_k\| \le \|x_n - P_0x_k\| + \|P_0x_k - x_k\| \to 0$, as $k \to \infty$, $n \to \infty$. Hence $\{x_n\}$ must be a Cauchy sequence. Let $\lim_{n\to\infty} x_n = z$. Then $P_0x_n \to z$ as $n \to \infty$ because $\|x_n - P_0x_n\| \to 0$ as $n \to \infty$.

Note that the closedness of $N_0(T)$ and $\{P_0x_n\} \subset N_0(T)$ imply $z \in N_0(T)$.

Case 2. $\sum_{n=0}^{\infty} \|x_{n+1} - x_n\| < \infty$.

Observing that $\|x_{n+m} - x_n\| \le \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \to 0$ as $n, m \to \infty$, we assert that $\{x_n\}$ is a Cauchy sequence. Assume that $x_n \to x$ as $n \to \infty$. By (2.5) we know that $Tx_n \to 0$ as $n \to \infty$. Hence $x \in N_0(T)$, since $T$ is demiclosed.

The proof is complete. ∎

**Remark 1.** In [6, p. 89] Reich considered a continuous nondecreasing function $b : [0, \infty) \to [0, \infty)$ such that $b(0) = 0$, $b(ct) \le cb(t)$ for $c \ge 1$. In [7, p. 337] he established a relationship between the function $b$ and the modulus of smoothness of the Banach space $X$. Since any map satisfying the convergence condition introduced in [5] certainly satisfies condition (2.2), and the condition $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$ in [5] implies the condition $\sum_{n=0}^{\infty} \rho_X(\lambda_n) < \infty$, We see that our Theorem 2.1 generalizes the strong convergence results in [5, Theorem 3], [3, Theorem 3.1] and others.

Chidume [4] proved the following theorem:

**Theorem 1 of [4].** *Let $X$ be a real Banach space with a uniformly convex dual space, $X^*$. Suppose that $T : X \to X$ is a continuous strongly accretive map such that $(I - T)$ has bounded range. For a given $f \in X$, define $S : X \to X$ by $Sx = f - Tx + x$ for each $x \in X$. Consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by $x_0 \in X$ and*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Sx_n,$$

*for $n \ge 0$, where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence satisfying the following:*

(i) $0 < \lambda_n \le 1$ *for all* $n \ge 0$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$.

*Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the solution of $Tx = f$.*

We have the following theorem:

**Theorem 2.2.** *Theorem 1 of Chidume [4] is a corollary of Theorem 2.1 above.*

*Proof.* Set $A = T - f$, for any given $f \in X$. Under the assumptions of Chidume [4, Theorem 1], $N_0(A) = \{q\}$, where $q$ is the unique solution to $Tx = f$. Observe that

$$\begin{aligned} x_{n+1} &= (1-\lambda_n)x_n + \lambda_n(f - Tx_n + x_n) \\ &= x_n - \lambda_n x_n + \lambda_n f - \lambda_n T x_n + \lambda_n x_n \\ &= x_n - \lambda_n A x_n, \end{aligned} \tag{2.6}$$

and

$$Ax_n = Tx_n - f = x_n - (x_n - Tx_n + f).$$

Since $\{x_n - Tx_n\} \subset R(I-T)$ is bounded, the only thing we need to do is to verify the boundedness of $\{x_n\}$. We consider the two possible cases:

Case 1. There exists an $n_0 \geq 0$ such that $\|x_{n_0} - q\| \leq 1$.

We let $M_3 = \sup\{\|f + x_n - Tx_n\| | n \geq 0\},\ M_4 = \max\{1, 2M_3\}$. By (2.6) we have

$$\|x_{n+1} - q\| \leq (1-\lambda_{n_0}) + 2\lambda_{n_0} M_3 \leq M_4,$$

and, by induction, we find

$$\|x_{n_0+m} - q\| \leq M_4 \text{ for all } m \geq 1.$$

This shows that $\{x_n\}$ is bounded.

Case 2. For all $n \geq 0$, $\|x_n - q\| > 1$.

We shall show that this case is impossible.

Since $T$ is strongly accretive, so is $A$. Thus there exists some constant $k \in (0,1)$ such that

$$< Ax_n, J(x_n - q) > \geq k\|x_n - q\|^2. \tag{2.7}$$

By using [10, Lemma 1.1] and (2.7) we have

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\leq \|x_n - q\|^2 - 2\lambda_n < Ax_n, J(x_{n+1} - q) > \\ &= \|x_n - q\|^2 - 2\lambda_n < Ax_n, J(x_n - q) > \\ &\quad - 2\lambda_n < Ax_n, J(x_{n+1} - q) - J(x_n - q) > \\ &\leq \|x_n - q\|^2 - 2\lambda_n k \|x_n - q\|^2 \\ &\quad - 2\lambda_n < \frac{Ax_n}{\|x_n - q\|}, J\frac{x_{n+1} - q}{\|x_n - q\|} - J\frac{x_n - q}{\|x_n - q\|} > \|x_n - q\|^2 \\ &= ((1 - 2\lambda_n k) - 2\lambda_n a_n)\|x_n - q\|^2, \end{aligned} \tag{2.8}$$

where $a_n =< \frac{Ax_n}{\|x_n - q\|}, J\frac{x_{n+1}-q}{\|x_n-q\|} - J\frac{x_n - q}{\|x_n - q\|} >$.

Now, we want to show that $a_n \to 0$ as $n \to \infty$. It follows from $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$ that $\lambda_n \to 0$ as $n \to \infty$, $\frac{\|Ax_n\|}{\|x_n - q\|} \leq 1 + M_3 + \|q\|$ and

$$\frac{x_{n+1} - q}{\|x_n - q\|} - \frac{x_n - q}{\|x_n - q\|} = -\lambda_n \frac{Ax_n}{\|x_n - q\|} \to 0 \text{ as } n \to \infty.$$

Hence we have

$$J\frac{x_{n+1}-q}{\|x_n-q\|}-J\frac{x_n-q}{\|x_n-q\|}\to 0 \text{ as } n\to\infty,$$

since $J$ is uniformly continuous on bounded subset of $X$. Consequently, $a_n \to 0$ as $n \to \infty$.

Now, we may choose $n_1 \geq 0$ such that for every $n \geq n_1$, $k+2a_n > 0$. Thus we have

$$\|x_{n+1}-q\|^2 \leq (1-k\lambda_n)\|x_n-q\|^2$$

$$\leq \exp\{-\sum_{j=0}^{n} k\lambda_j\}\|x_0-q\|^2 \to 0 \text{ as } n\to\infty,$$

which contradicts with the assumption that for all $n \geq 0$, $\|x_n-q\| > 1$. ∎

**Remark 2.** In Theorems 2.1 and 2.2, all assumptions are satisfied except the boundedness of $\{Tx_n\}$ and $R(I-T)$ which are replaced by the boundedness of $T$, then the conclusions of Theorems 2.1 and 2.2 hold true. See Xu and Roach [8], and authors [9].

The next result extends [5, Theorem 3] to the case of an Ishikawa iterative process. Namely, we consider the following Ishikawa process:

$$\text{(IS)} \begin{cases} x_{n+1} = x_n - \alpha_n A y_n - \alpha_n\beta_n A x_n, \\ y_n = x_n - \beta_n A x_n,\ n \geq 0. \end{cases}$$

**Theorem 2.3.** *Let $A : X \to X$ be a demiclosed quasi-accretive operator. Assume that there exists a strictly increasing function $\psi : R^+ \to R^+$, $\psi(0) = 0$, such that*

$$(2.9) \qquad < Ay_n, J(y_n - P_0y_n) > \geq \psi(\|y_n - P_0y_n\|)\|Ay_n\|,\ n \geq 0.$$

*Furthermore, assume that the following conditions are satisfied:*

($H_1$) $0 < \alpha_n < 1$, $0 \leq \beta_n < 1$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n\beta_n < \infty$;
($H_2$) $\sup\{\|Ax_n\|; n \geq 0\} < \infty$ *and* $\sup\{\|Ay_n\||n \geq 0\} < \infty$;
($H_3$) $\sum_{n=0}^{\infty}((J(x_n - P_0x_n) - J(y_n - P_0y_n)) < \infty$ *and* $\sum_{n=0}^{\infty} \rho_X(\alpha_n) < \infty$;
($H_4$) $\|P_0x_n - P_0y_n\| \to 0$ *as* $n \to \infty$.

*Then $\{x_n\}$, defined by* (IS), converges strongly to an element of $N(A)$.

*Proof.* Set $j(x_n) = J(x_n - P_0x_n)$, $j(y_n) = J(y_n - P_0y_n)$, $c_1 = \sup\{\|Ax_n\||n \geq 0\}$, and $c_2 = \sup\{\|Ay_n\||n \geq 0\}$.

Using Lemma 1.3 and (IS) we have

$$
\begin{aligned}
\|x_{n+1}-P_0x_{n+1}\|^2 &\le \|x_{n+1}-P_0x_n\|^2\\
&= \|x_n-\alpha_nAy_n-\alpha_n\beta_nAx_n-P_0x_n\|^2\\
&\le \|x_n-P_0x_n\|^2-2\alpha_n< Ay_n, J(x_n-P_0x_n)>\\
&\quad -2\alpha_n\beta_n< Ax_n, J(x_n-P_0x_n)>\\
&\quad +k\max\{\|x_n-P_0x_n\|+\alpha_n\|Ay_n\|+\alpha_n\beta_n\|Ax_n\|,\frac{c}{2}\}\\
&\quad \cdot\rho_X(\alpha_n\|Ay_n\|+\alpha_n\beta_n\|Ax_n\|)\\
&\le \|x_n-P_0x_n\|^2\\
&\quad -2\alpha_n< Ay_n, J(x_n-P_0x_n)-J(y_n-P_0y_n>\\
&\quad -2\alpha_n< Ay_n, J(y_n-P_0y_n)>\\
&\quad +k_1\max\{\|x_n-P_0x_n\|+\alpha_n(c_1+c_2),\frac{c}{2}\}\rho_X(\alpha_n)\\
&\le \|x_n-P_0x_n\|^2-2\alpha_nb_n-2\alpha_n\psi(\|y_n-P_0y_n\|)\|Ay_n\|\\
&\quad +k_1\max\{\|x_n-P_0x_n\|+\alpha_n(c_1+c_2),\frac{c}{2}\}\rho_X(\alpha_n),
\end{aligned}
\tag{2.10}
$$

where $k_1$ is some positive constant and

$$b_n =< Ay_n, J(x_n-P_0x_n)-J(y_n-P_0y_n)> .$$

Here we have used the fact that $\rho_X(\tau)$ is nondecreasing and there exists some constant $c_0>0$ such that $\frac{\rho_X(\eta)}{\eta^2}\le\frac{c_0\rho_X(\tau)}{\tau^2}$, for all $\eta\ge\tau>0$. Arguing as in the proof of Theorem 2.1, we can show that $\|x_n-P_0x_n\|$ is bounded and $\lim_{n\to\infty}\|x_n-P_0x_n\|$ exists. From (2.10) we see that

$$\sum_{n=0}^{\infty}\alpha_n\psi(\|y_n-P_0y_n\|)\|Ay_n\|<\infty.$$

Now, we consider the following two possible cases:

Case 1. $\lim_{n\to\infty}\inf\psi(\|y_n-P_0y_n\|)=0$.

In this case, by the properties of $\psi$, we see that $\lim_{n\to\infty}\inf\|y_n-P_0y_n\|=0$. Assumption ($H_1$) implies $\lim_{n\to\infty}\inf\beta_n=0$. Without any loss of generality, we assume that $\beta_n\to0$ as $n\to\infty$. Then $y_n-x_n=-\beta_nAx_n\to0$ as $n\to\infty$. By ($H_4$), we have $\lim_{n\to\infty}\inf\|x_n-P_0x_n\|=0$. Consequently, $\lim_{n\to\infty}\|x_n-P_0x_n\|=0$ since $\lim_{n\to\infty}\|x_n-P_0x_n\|$ exists. Arguing as in the proof of Theorem 2.1, we can prove that $x_n\to x$ as $n\to\infty$. Hence $x\in N(A)$.

Case 2. $\sum_{n=0}^{\infty}\alpha_n\|Ay_n\|<\infty$.

In this case, by ($H_1$) and (IS), we have

$$\sum_{n=0}^{\infty}\|x_{n+1}-x_n\|\le\sum_{n=0}^{\infty}\alpha_n\|Ay_n\|+c_1\sum_{n=0}^{\infty}\alpha_n\beta_n<\infty,$$

and hence $\{x_n\}$ must be Cauchy. Assume that $x_n \to z$ as $n \to \infty$ Then $y_n \to z$ as $n \to \infty$. On the other hand, $\sum_{n=0}^{\infty} \alpha_n \|Ay_n\| < \infty$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ implies $\lim_{n\to\infty} \inf \|Ay_n\| = 0$. Therefore, $z \in N(A)$ since $A$ is demiclosed. ∎

**Remark 3.** If we take $\beta_n \equiv 0$, then (IS) becomes $x_{n+1} = x_n - \alpha_n Ax_n$, $n \geq 0$. In this case, conditions $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$, $\sum_{n=0}^{\infty} (J(x_n - P_0 x_n) - J(y_n - P_0 y_n)) < \infty$ and $\|P_0 x_n - P_0 y_n\| \to 0$ as $n \to \infty$ are satisfied trivially.

**Remark 4.** It is easy to see that our Theorem 2.3 works for the case that $A$ is multi-valued.

**Remark 5.** We don't know whether the assumptions $\sum_{n=0}^{\infty} (J(x_n - P_0 x_n) - J(y_n - P_0 y_n)) < \infty$ and $(H_4)$ can be removed. It is also interesting to discuss the relations between Theorem 2.3 and Chidume [4, Theorem 2].

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