MAXIMAL RANKS AND INTEGER POINTS
ON A FAMILY OF ELLIPTIC CURVES II

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ABSTRACT. We extend a result of Spearman which provides a sufficient condition for elliptic curves of the form
\( y^2 = x^3 - dx \), with \( d = p, 2p \) and \( p \) prime, to have Mordell-Weil rank 3.

As in Spearman's work, the condition given here involves the existence of integer points on these curves.

1. Introduction. In two recent papers [9, 10], Spearman provided criteria for elliptic curves of the form \( y^2 = x^3 - dx \), with \( d = p, 2p \) and \( p \) prime, to have maximal rank. Specifically, in the case \( d = p \), Spearman proved that if \( p = u^4 + v^4 \) for some integers \( u, v \), then the rank of \( y^2 = x^3 - px \) is 2, while in the case \( d = 2p \), he proved that if \( 2p = (u^2 + 2v^2)^4 + (u^2 - 2v^2)^4 \), for integers \( u, v \), then the rank of \( y^2 = x^3 - 2px \) is 3. In [11], it was shown that the above condition for the case \( d = p \) can be described in terms of the set of integer points lying on such curves. The condition given includes all curves \( y^2 = x^3 - px \) for which \( p = u^4 + v^4 \), but also includes a larger class of curves. We note however that although the result in [11] applies to more curves than those in [9], there is no closed form for those \( p \), such as the polynomial given by Spearman. The purpose of the present paper is to give an analogous sufficient condition for curves of the form \( y^2 = x^3 - 2px \) to have Mordell-Weil rank 3, where the condition is given in terms of rational points on these curves. This is more general than the approach taken in [11], wherein a similar condition was given, but stated in terms of integer points. We thank the anonymous referee for this suggestion. We note that, as in [11], there does not appear to be a closed form for these primes which satisfy the condition given in this paper.

As noted above, for curves of the form

\[
E_{-2p} : y^2 = x^3 - 2px,
\]

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