# Homoclinic classes for sectional-hyperbolic sets

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**Abstract** We prove that every sectional-hyperbolic Lyapunov stable set contains a nontrivial homoclinic class.

## 1. Introduction

A well-known problem in dynamics is to determine when a given system has periodic or nontrivial homoclinic classes. This has been completely solved for *hyperbolic sets*; namely, every nontrivial isolated hyperbolic set contains a nontrivial homoclinic class (and hence infinitely many periodic orbits). It is very natural to extend this solution beyond hyperbolicity. For instance, we can consider the *singular-hyperbolic sets*, introduced in [24], to put together both hyperbolic systems and certain robustly transitive sets with singularities like the *geometric Lorenz attractor* (see [1], [16]). It is tempting to say that every nontrivial isolated singular-hyperbolic set contains a nontrivial homoclinic class, but this is false in general (see [22]). However, Bautista and the third author [9] proved that if a three-dimensional singular-hyperbolic set is *attracting*, then it must contain a periodic orbit. This was proved in parallel with the claim by Arroyo and Pujals [8] that every three-dimensional singular-hyperbolic *transitive* attracting set is a nontrivial homoclinic class (see also [3]). Afterward, Nakai [25] extended [9] from attracting to *Lyapunov stable* sets while Reis [27] gave generic conditions under which a three-dimensional singular-hyperbolic attracting set exhibits infinitely many periodic orbits. In 2013, Pacifico and Reis [26] reported that every singular-hyperbolic attracting set of a three-dimensional flow contains a nontrivial homoclinic class. In 2005, Metzger and the third author [20], [21] introduced the notion of a *sectional-hyperbolic set* extending singular hyperbolicity to higher

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dimensions. More recently, the second author [19] was able to extend the existence of periodic orbits in [9] to all sectional-hyperbolic attracting sets.

In this article we will extend all these results by proving that every sectional-hyperbolic Lyapunov stable set has a nontrivial homoclinic class. In particular, they contain infinitely many periodic orbits too. Let us state it in a precise way.

By abuse of language, we call *flow* any $C^1$ vector field $X$ with induced flow $X_t$ of a compact connected manifold $M$ endowed with a Riemannian structure $\|\cdot\|$. We say that $\Lambda \subset M$ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. An invariant set $\Lambda$ is *Lyapunov stable* if for every neighbourhood $U$ of $\Lambda$ there is a neighborhood $V \subset U$ of $\Lambda$ such that $X_t(V) \subset U$ for all $t \geq 0$. A similar definition holds for maps. The set of singularities (i.e., zeroes of $X$) is denoted by $\mathrm{Sing}(X)$. We say that $\sigma \in \mathrm{Sing}(X)$ is hyperbolic if the derivative $DX(\sigma)$ has no purely imaginary eigenvalues. We say that a point $x$ is *periodic* if there is a minimal $t = t_x > 0$ such that $X_t(x) = x$. We say that a periodic point $x$ is *hyperbolic* if the eigenvalues of the derivative $DX_{t_x}(x)$ not corresponding to the flow direction are all different from 1 in modulus. In case there are eigenvalues of modulus less than and greater than 1 we say that the hyperbolic periodic point is a *saddle*.

As is well known (see [17]), through any periodic saddle $x$ there passes a pair of invariant manifolds, the so-called *strong stable and unstable manifolds* $W^{ss}(x)$ and $W^{uu}(x)$, tangent at $x$ to the eigenspaces corresponding to the eigenvalues of modulus less than and greater than 1, respectively. Saturating them with the flow, we obtain the stable and unstable manifolds $W^s(x)$ and $W^u(x)$, respectively.

Denote by $\mathrm{Cl}(\cdot)$ the closure operation. We say that $H \subset M$ is a *homoclinic class* if there is a periodic saddle $x$ such that

$$H = \mathrm{Cl}\big(\{q \in W^s(x) \cap W^u(x) : \dim\big(T_q W^s(x) \cap T_q W^u(x)\big) = 1\}\big).$$

A homoclinic class is *nontrivial* if it does not reduce to a single periodic orbit.

We say that a compact invariant set $\Lambda$ *has a dominated splitting with respect to the tangent flow* if there is a continuous splitting $T_\Lambda M = E \oplus F$ into $DX_t$-invariant subbundles $E, F$ such that $DX_t|_E$ dominates $DX_t|_F$; namely, there are positive constants $K, \lambda$ satisfying

$$\big\|DX_t(p)\big|_{E_p}\big\| \cdot \big\|DX_{-t}\big(X_t(p)\big)\big|_{F_{X_t(p)}}\big\| \leq Ke^{-\lambda t}, \quad \forall p \in \Lambda, t \geq 0.$$

The splitting $T_\Lambda M = E \oplus F$ is called *sectional-hyperbolic* if $E$ is *contracting*, that is,

$$\big\|DX_t(p)\big|_{E_p}\big\| \leq Ke^{-\lambda t}, \quad \forall p \in \Lambda, t \geq 0,$$

and $F$ is *sectional expanding*, that is, $\dim(F) \geq 2$ and

$$\big|\det DX_t(p)\big|_L\big| \geq Ke^{\lambda t},$$

for every $p \in \Lambda$, $t \geq 0$, and every two-dimensional subspace $L \subset F_p$.

A compact invariant set is *sectional-hyperbolic* if its singularities are all hyperbolic and if it exhibits a sectional-hyperbolic splitting. We emphasize that this definition does require that all the singularities of a sectional-hyperbolic set be hyperbolic. We shall use this hypothesis in the proof of our main result below.

Of course, one can try to handle a general situation in which this requirement is dropped.

Sectional hyperbolicity is closely related to the notion of singular hyperbolicity defined elsewhere (see [24]). Indeed, sectional hyperbolicity implies singular hyperbolicity and they are equivalent in dimension three only. With these definitions we can state our main result.

THEOREM 1.1

*Every sectional-hyperbolic Lyapunov stable set contains a nontrivial homoclinic class.*

The above theorem can be added to a number of important results which have been appearing related to sectional hyperbolicity. Among these we can mention the structure of the strong stable manifolds (see [23]), existence of Sinai–Ruelle–Bowen (SRB) measures (see [4], [14], [28]), connecting lemmas (see [10]), decay of correlations (see [2], [5]), essential hyperbolicity (see [11]) and sensitivity to initial conditions (see [7], [11]), abundance of sectional-hyperbolic Lyapunov stable sets (see [6], [29]), and finally the solution of a conjecture by Palis (see [15]).

Our proof uses some recent results concerning SRB-like measures for continuous maps (see [12], [13]) and a version of Crovisier's [14, Proposition 1.4] for (locally) star flows stated in [29]. This allows us to prove the existence of nontrivial homoclinic classes not only for these Lorenz-like attractors but for any sectional-hyperbolic Lyapunov stable set.

## 2. Proof

We start with some terminology from [13]. As is well known, the space of probability measures of $M$ endowed with the weak* topology is metrizable; we denote by $d_*$ the corresponding metric. We say that a measure $\mu$ is *supported on* $H \subset M$ if its support $\mathrm{supp}(\mu)$ is contained in $H$. We denote by $\delta_y$ the Dirac measure supported on $y$.

If $f : M \to M$ is a continuous map, then we say that a Borel probability measure $\mu$ is an *invariant measure* if $\mu(f^{-1}(A)) = \mu(A)$ for every Borelian $A$. For any point $x \in M$ we denote by $p\omega(x)$ the set of all the Borel probability measures that are the accumulation points of the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}.$$

An invariant measure $\mu$ is *SRB-like for* $f$ if the set of points $x \in M$ for which there is $\nu \in p\omega(x)$ satisfying $d_*(\nu,\mu) < \epsilon$ has positive Lebesgue measure for all $\epsilon > 0$.

Applying [13, Theorem 1.3] we obtain the following existence result.

LEMMA 2.1

*Every Lyapunov stable set of a continuous map $f$ supports an SRB-like measure.*

*Proof*

Let $\Lambda$ be a Lyapunov stable set of $f$, and fix any neighborhood $W$ of $\Lambda$. By Lyapunov stability we can arrange a neighborhood $U \subset W$ satisfying $X_t(V) \subset W$ for all $t \geq 0$. Defining $U = \bigcup_{t \geq 0} X_t(U)$ we obtain a neighborhood $U \subset W$ of $\Lambda$ satisfying $X_t(U) \subset U$ for all $t \geq 0$. From this we can construct a nested sequence $U_i$ of compact neighborhoods of $\Lambda$ such that $f(U_i) \subset U_i$ and $\bigcap_i U_i = \Lambda$. By the aforementioned result in [13] there is a sequence of SRB-like measures $\mu_i$ for $f|_{U_i}$, $\forall i \in \mathbb{N}$. By definition, such measures are also SRB-like measures for $f$. Again by [13], any accumulation measure of $\mu_i$ is SRB-like and supported on $\Lambda$. This ends the proof. $\square$

Next we recall some facts about Lyapunov exponents. Assume that $f$ is a diffeomorphism, and let $\mu$ be an invariant measure. By Oseledets's theorem, for every continuous invariant subbundle $F$ of $T_\Lambda M$ there exist a full measure set $R$ (called regular points) and, for all $x \in R$, a positive integer $k(x)$, real numbers $\chi_1(x) < \cdots < \chi_{k(x)}(x)$, and a splitting $F_x = E_x^1 \oplus \cdots \oplus E_x^{k(x)}$, depending measurably on $x \in R$, such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \left\| Df^n(x) v^i \right\| = \chi_i(x), \quad \forall v^i \in E_x^i \setminus \{0\}, 1 \leq i \leq k(x).$$

The numbers $\chi_i(x)$ (which depend measurably on $x \in R$) are the so-called *Lyapunov exponents* of $\mu$ along $F$.

The following is a corollary of the main result in [12].

LEMMA 2.2

*Let $\Lambda$ be a Lyapunov stable set of a flow $X$. If $\Lambda$ has a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow and $\mu$ is an SRB-like measure of the time-1 map $X_1$, then*

$$h_\mu(X_1) \geq \int \sum_{i=1}^{\dim(F)} \chi_i \, d\mu,$$

*where $\sum_{i=1}^{\dim(F)} \chi_i$ denotes the sum of the Lyapunov exponents along $F$.*

The next lemma proves the positivity of the integral of the sum of the Lyapunov exponents along the central subbundle of any sectional-hyperbolic set.

LEMMA 2.3

*Let $\Lambda$ be a compact invariant set of a flow $X$. If $\Lambda$ has a sectional-hyperbolic splitting $T_\Lambda M = E \oplus F$ and $\mu$ is an invariant measure of the time-1 map $X_1$*

*supported in* $\Lambda$, *then*

$$\int \sum_{i=1}^{\dim(F)} \chi_i \, d\mu > 0.$$

*Proof*
Since

$$\lim_{n\to\infty} \frac{1}{n} \log |\det DX_n|_F| = \sum_{i=1}^{\dim(F)} \chi_i,$$

the result follows easily from the sectional expansivity of $F$. □

From this we obtain the following corollary proving positive topological entropy for every sectional-hyperbolic Lyapunov stable set.

COROLLARY 2.4
*If* $\Lambda$ *is a sectional-hyperbolic Lyapunov stable set, then the time-*1 *map restricted to* $\Lambda$ *has positive topological entropy.*

*Proof*
By Lemma 2.1 we can take an SRB-like measure $\mu$ supported on $\Lambda$ for the restricted time-1 map $f = X_1|_\Lambda$. Combining Lemmas 2.2 and 2.3 we obtain $h_\mu(X_1) > 0$. Thus, the result follows by applying the variational principle to $X_1$. □

The last ingredient is the following lemma whose proof is contained in that of [29, Theorem 5.6]. Given a flow $X$ and a compact invariant set $\Lambda$, we say that $X$ *is a star flow on* $\Lambda$ if there exist a neighborhood $U$ of $\Lambda$ and $\mathcal{U}$ of $X$ in the $C^1$-topology such that every periodic orbit or singularity contained in $U$ of every flow $Y$ in $\mathcal{U}$ is hyperbolic.

LEMMA 2.5
*Let* $\Lambda$ *be a compact invariant set of a flow* $X$*. Suppose that* $X$ *is a star flow on* $\Lambda$*. Consider an ergodic measure* $\mu$ *of* $X$ *whose support* $\operatorname{supp}(\mu)$ *is neither a periodic orbit nor a singularity of* $X$*. If* $\operatorname{supp}(\mu) \subset \Lambda$*, then* $\operatorname{supp}(\mu)$ *intersects a nontrivial homoclinic class of* $X$*.*

Now we can prove our main result.

*Proof of Theorem 1.1*
Let $\Lambda$ be a sectional-hyperbolic Lyapunov stable set of a flow $X$. It is well known (see [3]) that $X$ is a star flow on $\Lambda$.

Since the entropy is positive by Corollary 2.4, the variational principle produces an ergodic measure $\mu$ whose support $\operatorname{supp}(\mu)$ not only is contained in $\Lambda$ but also is neither a periodic orbit nor a singularity. Applying Lemma 2.5 we obtain that $\operatorname{supp}(\mu)$ intersects a nontrivial homoclinic class $H$. In particular, $H \cap \Lambda \neq \emptyset$. Since $\Lambda$ is Lyapunov stable, we conclude that $H \subset \Lambda$. For completeness, we include the proof of this last assertion.

Choose a compact neighborhood $U$ of $\Lambda$. By Lyapunov stability, there is a neighborhood $W$ of $\Lambda$ such that

$$X_r(W) \subset U, \quad \forall r \geq 0.$$

Take $y \in H$. Since $H \cap \Lambda \neq \emptyset$ and $W$ is a neighborhood of $\Lambda$, there is $x \in H \cap \operatorname{Int}(W)$, where $\operatorname{Int}(W)$ denotes the interior of $W$. But $H$ is the omega-limit set of some point $z \in H$ by Birkhoff–Smale's theorem [18], so there is $t_0 > 0$ such that $X_{t_0}(z)$ is nearby $x$. In particular, we can assume that $X_{t_0}(z) \in W$. Since $y \in H$ there is a sequence $s_n \to \infty$ such that $X_{s_n+t_0}(z) \to y$. As $X_{t_0}(z) \in W$ and $s_n > 0$ for all $n$, we obtain $X_{s_n+t_0}(z) \in U$ for all $n$ by taking $r = s_n + t_0$ above. As $U$ is compact and $X_{s_n+t_0}(z) \to y$, we conclude that $y \in U$. Consequently, $H \subset U$ for every neighborhood $U$ of $\Lambda$, proving that $H \subset \Lambda$. This completes the proof. $\square$

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## References

[1] V. S. Afraĭmovič, V. V. Bykov, and L. P. Sil'nikov, *The origin and structure of the Lorenz attractor*, Dokl. Akad. Nauk SSSR **234** (1977), no. 2, 336–339. MR 0462175.

[2] V. Araújo, S. Galatolo, and M. J. Pacifico, *Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors*, Math. Z. **276** (2014), 1001–1048. MR 3175169. DOI 10.1007/s00209-013-1231-0.

[3] V. Araújo and M. J. Pacifico, *Three-Dimensional Flows*, Ergeb. Math. Grenzgeb. (3) **53**, Springer, Heidelberg, 2010. MR 2662317. DOI 10.1007/978-3-642-11414-4.

[4] V. Araújo, M. J. Pacifico, E. R. Pujals, and M. Viana, *Singular-hyperbolic attractors are chaotic*, Trans. Amer. Math. Soc. **361** (2009), no. 5, 2431–2485. MR 2471925. DOI 10.1090/S0002-9947-08-04595-9.

[5] V. Araújo and P. Varandas, *Robust exponential decay of correlations for singular-flows*, Comm. Math. Phys. **311** (2012), 215–246. MR 2892469. DOI 10.1007/s00220-012-1445-8.

[6] A. Arbieto, C. A. Morales, and B. Santiago, *Lyapunov stability and sectional-hyperbolicity for higher-dimensional flows*, Math. Ann. **361** (2015), 67–75. MR 3302612. DOI 10.1007/s00208-014-1061-3.

[7] A. Arbieto, C. A. Morales, and L. Senos, *On the sensitivity of sectional-Anosov flows*, Math. Z. **270** (2012), 545–557. MR 2875848. DOI 10.1007/s00209-010-0811-5.

[8] A. Arroyo and E. R. Pujals, *Dynamical properties of singular-hyperbolic attractors*, Discrete Contin. Dyn. Syst. **19** (2007), 67–87. MR 2318274. DOI 10.3934/dcds.2007.19.67.

[9] S. Bautista and C. A. Morales, *Existence of periodic orbits for singular-hyperbolic sets*, Mosc. Math. J. **6** (2006), 265–297, 406. MR 2270614.

[10] ———, *A sectional-Anosov connecting lemma*, Ergodic Theory Dynam. Systems **30** (2010), 339–359. MR 2599883. DOI 10.1017/S0143385709000157.

[11] ———, *On the essential hyperbolicity of sectional-Anosov flows*, Proc. Amer. Math. Soc. **144** (2016), 205–215. MR 3415589. DOI 10.1090/proc/12686.

[12] E. Catsigeras, M. Cerminara, and H. Enrich, *The Pesin entropy formula for $C^1$ diffeomorphisms with dominated splitting*, Ergodic Theory Dynam. Systems **35** (2015), 737–761. MR 3334902. DOI 10.1017/etds.2013.93.

[13] E. Catsigeras and H. Enrich, *SRB-like measures for $C^0$-dynamics*, Bull. Pol. Acad. Sci. Math. **2478** (2011), 151–164. MR 2852870. DOI 10.4064/ba59-2-5.

[14] S. Crovisier, *Partial hyperbolicity far from homoclinic bifurcations*, Adv. Math. **226** (2011), 673–726. MR 2735772. DOI 10.1016/j.aim.2010.07.013.

[15] S. Crovisier and D. Yang, *On the density of singular hyperbolic three-dimensional vector fields: a conjecture of Palis*, C. R. Math. Acad. Sci. Paris **353** (2015), 85–88. MR 3285153. DOI 10.1016/j.crma.2014.10.015.

[16] J. Guckenheimer and R. F. Williams, *Structural stability of Lorenz attractors*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 59–72. MR 0556582.

[17] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant Manifolds*, Lecture Notes in Math. **583**, Springer, Berlin, 1977. MR 0501173.

[18] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems* with a supplementary chapter by A. Katok and L. Mendoza, Encyclopaedia Math. Appl. **54**, Cambridge Univ. Press, Cambridge, 1995. MR 1326374. DOI 10.1017/CBO9780511809187.

[19] A. M. López, *Existence of periodic orbits for sectional Anosov flows*, preprint, arXiv:1407.3471v1 [math.DS].

[20] R. Metzger and C. A. Morales, *Sectional-hyperbolic systems*, preprint, http://preprint.impa.br/FullText/Metzger__Wed_Jun_29_14_55_48_BRT_2005/MM4.pdf.

[21] ———, *Sectional-hyperbolic systems*, Ergodic Theory Dynam. Systems **28** (2008), 1587–1597. MR 2449545. DOI 10.1017/S0143385707000995.

[22] C. A. Morales, *A note on periodic orbits for singular-hyperbolic flows*, Discrete Contin. Dyn. Syst. **11** (2004), 615–619. MR 2083434. DOI 10.3934/dcds.2004.11.615.

[23] ———, *Strong stable manifolds for sectional-hyperbolic sets*, Discrete Contin. Dyn. Syst. **17** (2007), 553–560. MR 2276427. DOI 10.3934/dcds.2007.17.553.

[24] C. A. Morales, M. J. Pacifico, and E. R. Pujals, *On $C^1$ robust singular transitive sets for three-dimensional flows*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), 81–86. MR 1649489. DOI 10.1016/S0764-4442(97)82717-6.

[25] N. Nakai, *Existence of periodic orbits for singular-hyperbolic Lyapunov stable sets*, preprint, arXiv:1501.04339v1 [math.DS].

[26] M. J. Pacifico and J. Reis, lecture at “8a Edição Workshop de Topología & Dinâmica 2013,” 2013.

[27] J. Reis, *Existence of infinitely many periodic orbits for sectional-Anosov flows*, Ph.D. thesis, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil, 2011.

[28] E. A. Sataev, *Invariant measures for singularly hyperbolic attractors*, Mat. Sb. **201** (2010), no. 3, 107–160; English translation in Sb. Math. **201** (2010), 419–470. MR 2656321. DOI 10.1070/SM2010v201n03ABEH004078.

[29] Y. Shi, S. Gan, and L. Wen, *On the singular-hyperbolicity of star flows*, J. Mod. Dyn. **8** (2014), 191–219. MR 3277201. DOI 10.3934/jmd.2014.8.191.



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