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Nonresonance and Resonance Problems for Nonlocal Elliptic Equations with Respect to the Fučik Spectrum

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Abstract. In this article, we consider the following problem

$$\begin{cases} (-\Delta)^s u = \alpha u^+ - \beta u^- + f(u) + h & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $n > 2s$, $0 < s < 1$, $(\alpha, \beta) \in \mathbb{R}^2$, $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function and $h \in L^2(\Omega)$. We prove the existence results in two cases: first, the nonresonance case where (α, β) is not an element of the Fučik spectrum. Second, the resonance case where (α, β) is an element of the Fučik spectrum. Our existence results follows as an application of the saddle point theorem. It extends some results, well known for Laplace operator, to the nonlocal operator.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $n > 2s$ and $s \in (0,1)$. Then consider the following problem

$$\begin{cases} (-\Delta)^s u = \alpha u^+ - \beta u^- + f(u) + h & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $(\alpha, \beta) \in \mathbb{R}^2$, $u^{\pm} = \max\{\pm u, 0\}$, $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function, $h \in L^2(\Omega)$ and $(-\Delta)^s$ is the fractional Laplacian operator defined as

$$(-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy \quad \text{for all } x \in \mathbb{R}^n.$$

In this article, we study the corresponding problem driven by the nonlocal operator $\mathcal{L}_K$ given by

$$\begin{cases} -\mathcal{L}_K u = \alpha u^+ - \beta u^- + f(u) + h & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

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The nonlocal operator $\mathcal{L}_K$ is defined as

$$\mathcal{L}_K u(x) := -\frac{1}{2}\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\, dy \quad \text{for all } x \in \mathbb{R}^n,$$

and the function $K\colon \mathbb{R}^n \setminus \{0\} \to (0,\infty)$ satisfies the following assumptions:

(K1) $mK \in L^1(\mathbb{R}^n)$, where $m(x) = \min\{|x|^2, 1\}$,

(K2) There exist $\lambda > 0$ and $s \in (0,1)$ such that $K(x) \geq \lambda |x|^{-(n+2s)}$,

(K3) $K(x) = K(-x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

In case $K(x) = |x|^{-(n+2s)}$, $\mathcal{L}_K$ is the fractional Laplace operator $(-\Delta)^s$. When $s = 1$, the fractional Laplacian operator becomes the usual Laplace operator. There has been done a lot of works related to the solvability of resonance problem with respect to the Fučik spectrum for the classical Laplace equation, see [7, 8, 13, 14] and references therein. The Fučik spectrum in the case of Laplacian and p-Laplacian equations with Dirichlet boundary condition has been studied by many authors [4, 5, 10].

Recently a lot of attention has been paid to the study of fractional and nonlocal equations of elliptic type due to concrete real world applications in finance, thin obstacle problem, optimization, quasi-geostrophic flow etc., see [1, 2, 18]. Dirichlet boundary value problem in case of fractional Laplacian with polynomial type nonlinearity using variational methods is studied in [15,16]. Fiscella, Servadei and Valdinoci in [11] studied the resonance problem with respect to the spectrum for a non local equation.

The Fučik spectrum of the nonlocal operator $\mathcal{L}_K$ is defined as the set

$$\Sigma_K := \{(\alpha,\beta) \in \mathbb{R}^2 \mid -\mathcal{L}_K u = \alpha u^+ - \beta u^- \text{ in } \Omega, u = 0 \text{ on } \mathbb{R}^n \setminus \Omega\}, \tag{1.2}$$

where u is a nontrivial solution. For $\alpha = \beta = \lambda$, the Fučik spectrum of (1.2) becomes the usual spectrum of $\mathcal{L}_K$. In this case, u satisfies

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.3}$$

Let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ denote the sequence of eigenvalues of (1.3) and $\{\phi_k\}_k$ denote the sequence of eigenfunctions corresponding to λ_k. Then it is proved in [16] that the first eigenvalue λ_1 of (1.3) is simple, isolated and can be characterized as follows:

$$\lambda_1 = \inf_{u \in X_0} \left\{ \int_Q (u(x) - u(y))^2 K(x-y)\, dxdy : \int_\Omega u^2 = 1 \right\}.$$

The authors also proved that the eigenfunctions corresponding to λ_1 is non-negative. Also such eigenfunction is strictly positive in Ω, thanks to [17]. Moreover, one can observe that Σ_K clearly contains (λ_k, λ_k) for each $k \in \mathbb{N}$ and Σ_K is symmetric with respect to the diagonal. In [12], it is shown that the two lines $\mathbb{R} \times \lambda_1$ and $\lambda_1 \times \mathbb{R}$ belongs to Σ_K and are isolated in Σ_K. Also a variational characterization of the second eigenvalue λ_2 of $-\mathcal{L}_K$ is studied in [12]. But here we will characterize a portion of Σ_K using the variational methods. That is, the eigenvalue pair will be obtained as minima or minimax values of an appropriate functional.

In the homogeneous case where $\alpha = \beta = \lambda$ and $f \equiv 0$, the solvability of (1.1) can be completely described by the Fredholm Alternative, which says that if λ is not an eigenvalue of $-\mathcal{L}_K$, then the problem has a unique solution for any h, and if λ is an eigenvalue of $-\mathcal{L}_K$, then the problem (1.1) has a solution if and only if h is orthogonal to the corresponding eigenspace.

For the nonhomogeneous case where $\alpha = \beta = \lambda$ and $f \neq 0$, Fiscella, Servadei and Valdinoci in [11], studied the existence results for the following problem

$$\begin{cases} -\mathcal{L}_K u + q(x)u = \lambda u + f(u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where f, q and h are sufficiently smooth functions. They showed that if λ is not an eigenvalue (nonresonance), then it has a solution with no further restriction on f and h, and if λ is an eigenvalue (resonance), then they need some extra conditions on f and h. Precisely, denoting by

$$f_l = \lim_{t \to -\infty} f(t) \quad \text{and} \quad f_r = \lim_{t \to \infty} f(t),$$

they assume that f_l and f_r exist, are finite and such that $f_l > f_r$ and

$$f_r \int_\Omega \phi^-(x)\,dx - f_l \int_\Omega \phi^+(x)\,dx < \int_\Omega h(x)\phi(x)\,dx < f_l \int_\Omega \phi^-(x)\,dx - f_r \int_\Omega \phi^+(x)\,dx$$

for any nontrivial ϕ in the eigenspace associated with λ. We would remark that these extra conditions on f and h are exactly the same required in the resonant setting, when dealing with the classical Laplace operator. Moreover, in the resonant case for fractional Laplacian, they are able to treat this case only if λ satisfies the following condition:

$$\begin{array}{c} \lambda \text{ is an eigenvalue of } -\mathcal{L}_K + q \text{ such that all the eigenfunctions} \\ \text{corresponding to } \lambda \text{ have nodal set with zero Lebesgue measure.} \end{array} \tag{1.4}$$

As usual, the nodal set of a function g in Ω is the level set $\{x \in \Omega : g(x) = 0\}$. The condition (1.4) has been established for the fractional Laplacian in [9]. For instant, in case of fractional Laplacian (1.4) is true when λ is its first eigenvalue however condition (1.4) is satisfied by every eigenvalue of the classical Laplace operator. At this point, we note that there is a difference in result between Nonlocal operator and Classical Laplcian operator. Therefore one can not directly extend the results for Laplace equation to nonlocal equation due to the nonlocal behaviour of the operator and the bounded support of the test function is not preserved.

In this paper, we studied the problem (1.1) with respect to the Fučik spectrum for nonlocal equation. Here we use the variational argument which was developed by Castro and Chang in [4] for the Laplace operator. To the best of our knowledge, no work has been done related to the solvability of resonance and nonresonance problems with respect to the Fučik spectrum for the nonlocal equation in (1.1). The results obtained here are somehow expected but new for the nonlocal operator.

Now for the nonresonance case, we assume that α lies strictly between consecutive eigenvalues of $(-\Delta)^s$, call them as $\lambda_k < \lambda_{k+1}$, and we also assume that $\alpha \leq \beta < \beta(\alpha)$, where $\beta(\alpha)$ is defined as

$$\beta(\alpha) := \sup\left\{\beta \geq \alpha \mid \widetilde{J}_{\alpha,b}(y) = J_{\alpha,b}(y + M_{\alpha,b}(y)) > 0 \text{ for all } b \in (\alpha,\beta), y \in X_2 \setminus \{0\}\right\},$$

where $J_{\alpha,b}$ is a function defined in (2.2) and $\{(\alpha,\beta) : \alpha \leq \beta < \beta(\alpha)\}$ contains no points in Σ_K, according to the Castro-Chang characterization (see Theorem 2 in [4]) which can be easily generalized in case of fractional Laplace operator and it is stated as follows:

Proposition 1.1. *If $\alpha \in (\lambda_k, \lambda_{k+1})$, $N(l) < \infty$ (multiplicity of eigenvalue λ_l) for $l \geq k+1$, and $\beta(\alpha)$ is defined as above. Then the following hold.*

(1) $(\alpha, \beta(\alpha))$ *is in the Fučik spectrum when* $\beta(\alpha) < +\infty$.

(2) *If* $\beta \in [\alpha, \beta(\alpha))$ *then* (α,β) *is not in the Fučik spectrum.*

(3) *For $\beta > \alpha$, (α,β) is in the Fučik spectrum if and only if the restriction of $\widetilde{J}_{\alpha,\beta}$ to $\{y \in X_2 \mid \|y\| = 1\}$ has a critical point on $\{y \in X_2 \mid \|y\| = 1, \widetilde{J}_{\alpha,\beta} = 0\}$.*

(4) *The function $\beta(\alpha)\colon (\lambda_j, \lambda_{j+1}) \to [0,+\infty]$, $\alpha \to \beta(\alpha)$ is non-increasing and continuous.*

Note that if $X_2\setminus\{0\}$ contains a non-negative function then $\beta(\alpha) < \infty$ for all $\alpha \in (\lambda_k, \lambda_{k+1})$.

In the prove of above theorem, one use the following global reduction principle (see [3]).

Proposition 1.2. *Let H be a separable real Hilbert space. Let X_1, X_2 be closed subspaces such that $H = X_1 \oplus X_2$, and $J\colon H \to \mathbb{R}$ a functional of class C^1. If there exists $m > 0$ such that*

$$\langle \nabla J(x_1 + x_2) - \nabla J(y_1 + x_2), x_1 - y_1\rangle \leq -m\|x_1 - y_1\|^2$$

for all $x_1, y_1 \in X_1$, $x_2 \in X_2$, then there exists a continuous function $M\colon X_2 \to X_1$ such that

(1) $J(y + M(y)) = \max\{J(y + x) \mid x \in X_1\}$.

(2) $\widetilde{J}\colon Y \to \mathbb{R}$ *defined by* $\widetilde{J}(y) = J(y + M(y))$ *is of class* C^1.

(3) $x + y$ *is a critical point of* J *if and only if* $x = M(y)$ *and* y *is a critical point of* $\widetilde{J}$.

Now we state the main results.

Theorem 1.3 (Nonresonance case). *Assume* $\lambda_k < \alpha < \lambda_{k+1}$, $\alpha \le \beta < \beta(\alpha)$, $f\colon \mathbb{R} \to \mathbb{R}$ *is a bounded and continuous function, and* $h \in L^2(\Omega)$, *then the problem* (1.1) *has at least one weak solution.*

In the resonance case, we still assume that $\lambda_k < \alpha < \lambda_{k+1}$, but now assume that $\beta = \beta(\alpha)$, as above, where $(\alpha, \beta(\alpha)) \in \Sigma_K$. The solvability condition that we impose is the following:

Let $F(u) := \int_0^u f(t)\,dt$. If $\{u_k\} \in X_0$ (see (2.1)), is such that $\|u_k\|_{L^2} \to \infty$ and $\frac{u_k}{\|u_k\|_{L^2}}$ converges in $L^2(\Omega)$ to some v, a nontrivial Fučik eigenfunction associated with (α, β), then $\lim_{k\to\infty} \int_\Omega (F(u_k) + hu_k)\,dx = -\infty$. Moreover, this

$$\lim_{k\to\infty} \int_\Omega (F(u_k) + hu_k)\,dx = -\infty \tag{1.5}$$

is known as the generalization of Landesman-Lazer condition. The following theorem is the boarder line case comparing to Theorem 1.3.

Theorem 1.4 (Resonance case). *Assume* $\lambda_k < \alpha < \lambda_{k+1}$, $\beta = \beta(\alpha)$, $f\colon \mathbb{R} \to \mathbb{R}$ *is bounded and continuous satisfying* $\lim_{k\to\infty} \int_\Omega (F(u_k) + hu_k)\,dx = -\infty$, *where* F *is primitive of* f *and* $h \in L^2(\Omega)$. *Then the problem* (1.1) *has at least one weak solution.*

2. Preliminaries

In this section we will recall function spaces which was introduced by Servadei and Valdinoci in [15,16] and some standard results from Functional analysis and critical point Theory.

Now due to nonlocalness of the fractional Laplacian, we define the function spaces introduced by Servadei and Valdinoci in [15] as

$$X = \left\{u \mid u\colon \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^2, (u(x) - u(y))\sqrt{K(x-y)} \in L^2(Q)\right\},$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. The space X is endowed with the norm $\|u\|_X = \|u\|_{L^2(\Omega)} + \left(\int_Q |u(x) - u(y)|^2 K(x-y)\,dxdy\right)^{1/2}$. Then we define

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\} \tag{2.1}$$

equipped with the norm

$$\|u\| = \left(\int_Q |u(x) - u(y)|^2 K(x-y)\, dxdy \right)^{1/2},$$

is a Hilbert spaces. Note that the norm $\|\cdot\|$ on the space X_0 involves the interaction between Ω and $\mathbb{R}^n \setminus \Omega$. For more details on these function spaces and the embedding theorems, we refer to [6, 15].

We also recall the space $L^2(\Omega) := \{u\colon \Omega \to \mathbb{R} : u \text{ is measurable}, \int_\Omega u^2\, dx < \infty\}$ endowed with the norm $\|u\|_{L^2} = \left(\int_\Omega u^2\, dx \right)^{1/2}$ is a Hilbert space.

Definition 2.1. A function $u \in X_0$ is a weak solution of (1.1), if for every $v \in X_0$, u satisfies

$$\int_Q (u(x) - u(y))(v(x) - v(y))K(x-y)\, dxdy = \int_\Omega (\alpha u^+ - \beta u^- + f(u) + h)v\, dx.$$

Now we denote $X_1 := \operatorname{span}[\phi_1, \phi_2, \ldots, \phi_k]$. That is, the linear span of the first k eigenfunctions, and $X_2 := X_1^\perp = [\phi_{k+1}, \phi_{k+2}, \ldots]$. The sequence $\{\phi_k\}_{k\in\mathbb{N}}$ of eigenfunctions is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of X_0. By definition, the subspaces X_1 and X_2 are orthogonal and $X_0 = X_1 \oplus X_2$. The Fourier expansion of a function $u \in X_0$ is $u = \sum_{j=1}^\infty c_j \phi_j$. Then note that

$$\int_Q |u(x) - u(y)|^2 K(x-y)\, dxdy = \sum_{j=1}^\infty \lambda_j c_j^2 \quad \text{and} \quad \int_\Omega u^2\, dx = \sum_{j=1}^\infty c_j^2.$$

This has helpful consequence such as

$$\int_Q |u(x) - u(y)|^2 K(x-y)\, dxdy \le \lambda_k \int_\Omega u^2\, dx, \quad \forall\, u \in X_1,$$
$$\int_Q |v(x) - v(y)|^2 K(x-y)\, dxdy \ge \lambda_{k+1} \int_\Omega v^2\, dx, \quad \forall\, v \in X_2.$$

To analyze problem (1.2), we consider the functional

$$J_{\alpha,\beta}(u) = \frac{1}{2} \left(\int_Q |u(x) - u(y)|^2 K(x-y)\, dxdy - \alpha \int_\Omega (u^+)^2\, dx - \beta \int_\Omega (u^-)^2\, dx \right), \tag{2.2}$$

which is C^1 functional on X_0 with

$$\langle J'_{\alpha,\beta}(u), v \rangle = \int_Q (u(x) - u(y))(v(x) - v(y))K(x-y)\, dxdy - \alpha \int_\Omega u^+ v\, dx + \beta \int_\Omega u^- v\, dx.$$

One can easily see that the critical points of $J_{\alpha,\beta}$ are weak solutions of (1.2). It will be very useful to think of $J_{\alpha,\beta}$ as a C^1 functional on $\mathbb{R}^2 \times X_0$. That is,

$$J\colon \mathbb{R}^2 \times X_0 \to \mathbb{R}, \quad J(\alpha, \beta, u) := J_{\alpha,\beta}(u)$$

with the derivative given by

$$\begin{aligned}\langle J'(\alpha,\beta,u),(s,t,v)\rangle &= \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\,dxdy \\ &\quad -\alpha\int_\Omega u^+v+\beta\int_\Omega u^-v - s\int_\Omega (u^+)^2 - t\int_\Omega (u^-)^2.\end{aligned}$$

It is clear that $\|DJ\|_{(\mathbb{R}^2\times X_0)^*}$ is bounded on bounded subsets of $\mathbb{R}^2\times X_0$, and so J is uniformly Lipschitz continuous on any bounded subset of $\mathbb{R}^2\times X_0$.

To examine problem (1.1), we consider the functional

$$E_{\alpha,\beta}(u) = \frac{1}{2}\left(\int_Q |u(x)-u(y)|^2K(x-y)\,dxdy - \alpha\int_\Omega (u^+)^2 - \beta\int_\Omega (u^-)^2\right) - \int_\Omega (F(u)+hu),$$

where $F(u) := \int_0^u f(t)\,dt$. $E_{\alpha,\beta}$ is also a C^1 functional on X_0 with

$$\langle E'_{\alpha,\beta}(u),v\rangle = \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\,dxdy - \int_\Omega (\alpha u^+ - \beta u^- + f(u)+h)v\,dx.$$

It is straightforward to see that critical points of $E_{\alpha,\beta}$ are weak solutions of (1.1).

We will use the saddle point theorem to prove the existence of critical points. Before stating the theorem, we first define the Palais-Smale condition (PS).

Definition 2.2. Let $J\colon H\to\mathbb{R}$ be a C^1 functional on a Banach space H. Then we say that J satisfies (PS) if for any sequence $\{u_k\}\subset H$ such that $J(u_k)$ is bounded and $J'(u_k)\to 0$ in H^*, there is a converging subsequence of $\{u_k\}$.

Theorem 2.3 (Saddle Point Theorem)**.** *Let $J\colon X_0\to\mathbb{R}$ be a C^1 functional which satisfies Palais-Smale (PS) condition. Assume that there are sets $\mathcal{X}_1,\mathcal{X}_2\subset X_0$ such that*

(i) $\mathcal{X}_1 = \widetilde{\gamma}(S^{k-1})$, *where $\widetilde{\gamma}\colon S^{k-1}\to X_0$ is continuous.*

(ii) $\mathcal{X}_2$ *links with* $\mathcal{X}_1$, *i.e., if B is the unit ball in $\mathbb{R}^k$ and $\gamma\colon B\to X_0$ is a continuous function such that $\gamma\equiv\widetilde{\gamma}$ on S^{k-1}, then $\gamma(B)\cap\mathcal{X}_2\neq\emptyset$.*

(iii) $\sup_{x\in\mathcal{X}_1} J(x) < \inf_{y\in\mathcal{X}_2} J(y)$.

Then $c := \inf_{\gamma\in\Gamma}\sup_{x\in B} J(\gamma(x))$ is a critical point of J, where $\Gamma = \{\gamma\colon B\to X_0 : \gamma$ is continuous and $\gamma\equiv\widetilde{\gamma}$ on $S^{k-1}\}$.

3. The variational characterization of Fučik spectrum

In all that follows we assume that $\lambda_k<\alpha<\lambda_{k+1}$ and the points of Σ_K that we characterize will all lie in this vertical strip in the (α,β) plane. We assume that $\alpha\le\beta$, and note that opposite case can be treated via symmetric arguments. Our approach to find the critical points of $J_{\alpha,\beta}$ will take advantage of concavity to maximize in the X_1 direction, and then to use weak lower semicontinuity to minimize in the X_2 direction.

3.1. Maximizing in the X_1 direction

In this subsection, we will show that the functional $J_{\alpha,\beta}$ attains a maximizer in the X_1 direction and the properties of the maximizer function. First, we prove the general inequality that is used to prove the concavity of the functional in X_1 direction.

Lemma 3.1. *Let $(\alpha_i, \beta_i) \in \mathbb{R}^2$ for $i = 1, 2$, be points satisfying $\alpha_i \leq \beta_i$, and let $s_i = \beta_i - \alpha_i$. Let $u_i \in X_1$ and $v_i \in X_2$ for $i = 1, 2$. Then there exist a $\delta = \alpha_2/\lambda_k - 1 > 0$ such that*

$$\begin{aligned} &\langle (J'_{\alpha_2,\beta_2}(u_2+v_2) - J'_{\alpha_1,\beta_1}(u_1+v_1)), (u_2-u_1)\rangle \\ &\leq -\delta\|u_2-u_1\|^2 + |\beta_2-\alpha_2|(\|u_2-u_1\|_{L^2} + \|v_2-v_1\|_{L^2})\|v_2-v_1\|_{L^2} \\ &\quad + |\alpha_2-\alpha_1|\|u_1\|_{L^2}\|u_2-u_1\|_{L^2} + |s_2-s_1|\|u_1+v_1\|_{L^2}\|u_2-u_1\|_{L^2}. \end{aligned} \tag{3.1}$$

Proof. Consider

$$\begin{aligned} &\langle J'_{\alpha_i,\beta_i}(u_i+v_i), (u_2-u_1)\rangle \\ &= \int_Q ((u_i+v_i)(x) - (u_i+v_i)(y))((u_2-u_1)(x) - (u_2-u_1)(y))K(x-y)\,dxdy \\ &\quad - \alpha_i \int_\Omega (u_i+v_i)^+(u_2-u_1) + \beta_i \int_\Omega (u_i+v_i)^-(u_2-u_1) \\ &= \int_Q ((u_i+v_i)(x) - (u_i+v_i)(y))((u_2-u_1)(x) - (u_2-u_1)(y))K(x-y)\,dxdy \\ &\quad - \alpha_i \int_\Omega (u_i+v_i)(u_2-u_1) + s_i \int_\Omega (u_i+v_i)^-(u_2-u_1). \end{aligned}$$

Then using the orthogonality of X_1 and X_2, we obtain

$$\begin{aligned} &\langle J'_{\alpha_i,\beta_i}(u_i+v_i), (u_2-u_1)\rangle \\ &= \int_Q (u_i(x) - u_i(y))((u_2-u_1)(x) - (u_2-u_1)(y))K(x-y)\,dxdy \\ &\quad - \alpha_i \int_\Omega u_i(u_2-u_1) + s_i \int_\Omega (u_i+v_i)^-(u_2-u_1). \end{aligned}$$

Subtracting the above expression for $i = 1, 2$ gives

$$\begin{aligned} &\langle (J'_{\alpha_2,\beta_2}(u_2+v_2) - J'_{\alpha_1,\beta_1}(u_1+v_1)), (u_2-u_1)\rangle \\ &= \int_Q |(u_2-u_1)(x) - (u_2-u_1)(y)|^2 K(x-y)\,dxdy - \int_\Omega (\alpha_2 u_2 - \alpha_1 u_1)(u_2-u_1) \\ &\quad + \int_\Omega (s_2(u_2+v_2)^- - s_1(u_1+v_1)^-)(u_2-u_1) \\ &= \|u_2-u_1\|^2 - \alpha_2 \int_\Omega |u_2-u_1|^2 + s_2 \int_\Omega ((u_2+v_2)^- - (u_1+v_1)^-)(u_2-u_1) \\ &\quad + (s_2-s_1)\int_\Omega (u_1+v_1)^-(u_2-u_1) - (\alpha_2-\alpha_1)\int_\Omega u_1(u_2-u_1). \end{aligned}$$

Now we analyze each term of the right hand side separately. First, it is clear from the definition of X_1 and the standard characterization of the eigenvalue of $(-\Delta)^s$ that

$$\|u_2 - u_1\|^2 - \alpha_2 \int_\Omega |u_2 - u_1|^2 \leq \left(1 - \frac{\alpha_2}{\lambda_k}\right) \|u_2 - u_1\|^2 = -\delta \|u_2 - u_1\|^2.$$

From the Hölder's inequality, we obtain

$$\int_\Omega u_1(u_2 - u_1) \leq \|u_1\|_{L^2} \|u_2 - u_1\|_{L^2}.$$

Using the relation $f = f^+ - f^-$, the monotonicity of $g(t) = t^-$, the fact that $|g(t_1) - g(t_2)| \leq |t_2 - t_1|$ and Hölder's inequality, we obtain

$$\begin{aligned}
&s_2 \int_\Omega ((u_2 + v_2)^- - (u_1 + v_1)^-)(u_2 - u_1) \\
&= s_2 \int_\Omega ((u_2 + v_2)^- - (u_1 + v_1)^-)((u_2 + v_2) - (u_1 + v_1)) \\
&\quad - s_2 \int_\Omega ((u_2 + v_2)^- - (u_1 + v_1)^-)(v_2 - v_1) \\
&= -s_2 \int_\Omega \left[((u_2 + v_2)^-)^2 + ((u_1 + v_1)^-)^2 + (u_2 + v_2)^+(u_1 + v_1)^- + (u_2 + v_2)^-(u_1 + v_1)^+\right] \\
&\quad - s_2 \int_\Omega ((u_2 + v_2)^- - (u_1 + v_1)^-)(v_2 - v_1) \\
&\leq s_2 \int_\Omega |(u_2 - u_1) + (v_2 - v_1)||v_2 - v_1| \\
&\leq s_2(\|u_2 - u_1\|_{L^2} + \|v_2 - v_1\|_{L^2})\|v_2 - v_1\|_{L^2}.
\end{aligned}$$

Combining the above inequalities together we obtain the desired result. □

First, we recall the definition of anticoercivity.

Definition 3.2. A functional $J\colon X_0 \to \mathbb{R}$ on a Banach space X_0 is called coercive if for every sequence $\{u_k\} \subset X_0$, $\|u_k\| \to +\infty$ implies $J(u_k) \to +\infty$. A functional J is called anticoercive if $-J$ is coercive.

Lemma 3.3. *For every $v \in X_2$, the functional $J_{\alpha,\beta}(\,\cdot\,, v)\colon X_1 \to \mathbb{R}$ is strictly concave and anticoercive.*

Proof. Taking $\alpha = \alpha_2 = \alpha_1$, $\beta = \beta_2 = \beta_1$ and $v_2 = v_1 = v$, in (3.1), we obtain

$$\langle (J'_{\alpha,\beta}(u_2 + v) - J'_{\alpha,\beta}(u_1 + v)), (u_2 - u_1) \rangle \leq -\delta \|u_2 - u_1\|^2,$$

which implies strict concavity. Then, the anticoercivity of $J_{\alpha,\beta}$ now follows from the strict concavity and the Fundamental Theorem of Calculus. □

Lemma 3.4. *For every $v \in X_2$, the functional $J_{\alpha,\beta}(\,\cdot\,, v)\colon X_1 \to \mathbb{R}$ achieves a unique maximum.*

Proof. Let $\{u_k + v\}$ be a maximizing sequence. Then anticoercivity of $J_{\alpha,\beta}$ implies that the sequence $\{u_k\}$ is bounded in X_1. Therefore the sequence $\{u_k\}$ has a weakly convergent subsequence. Also $J_{\alpha,\beta}$ is weakly upper semicontinuous, follows from concavity of $J_{\alpha,\beta}$. So, $J_{\alpha,\beta}$ achieves its maximum. Uniqueness follows easily from the strict concavity. □

Thus, the above result makes possible to define the definition.

Definition 3.5. The function $M_{\alpha,\beta}\colon X_2 \to X_1$ is defined as

$$M_{\alpha,\beta}(v) = \max_{u \in X_1} J_{\alpha,\beta}(u, v).$$

Now we investigate a few useful properties of $M_{\alpha,\beta}$. We start with homogeneity.

Lemma 3.6. *If $t \geq 0$ and $v \in X_2$, then $M_{\alpha,\beta}(tv) = tM_{\alpha,\beta}(v)$.*

Proof. *Case* 1*:* $t > 0$. Then by the maximizing property of $M_{\alpha,\beta}$, we have

$$J_{\alpha,\beta}(M_{\alpha,\beta}(tv) + tv) \geq J_{\alpha,\beta}(u + tv) \quad \text{for all } u \in X_1.$$

Using the homogeneity of $J_{\alpha,\beta}$, we obtain that

$$J_{\alpha,\beta}\left(\frac{M_{\alpha,\beta}(tv)}{t} + v\right) \geq J_{\alpha,\beta}\left(\frac{u}{t} + v\right) \quad \text{for all } u \in X_1.$$

Hence

$$J_{\alpha,\beta}\left(\frac{M_{\alpha,\beta}(tv)}{t} + v\right) \geq J_{\alpha,\beta}(u + v) \quad \text{for all } u \in X_1.$$

Thus for any $t > 0$, $M_{\alpha,\beta}(tv) = tM_{\alpha,\beta}(v)$.

Case 2*:* $t = 0$. Then it only needs to argue that $M_{\alpha,\beta}(0) = 0$. It is immediate that $J_{\alpha,\beta}(0) = 0$. It suffices to show that $J_{\alpha,\beta}(u) < 0$ for $u \in X_1 \setminus \{0\}$. Recall that $\int_Q |u(x) - u(y)|^2 K(x-y)\, dxdy \leq \lambda_k \int_\Omega u^2\, dx$ for all $u \in X_1$ and that $\lambda_k < \alpha < \beta$. It follows that

$$\begin{aligned}
J_{\alpha,\beta}(u) &= \frac{1}{2}\left(\int_Q |u(x) - u(y)|^2 K(x-y) - \alpha \int_\Omega (u^+)^2\, dx - \beta \int_\Omega (u^-)^2\, dx\right) \\
&\leq \frac{1}{2}\left(\lambda_k \int_\Omega u^2\, dx - \alpha \int_\Omega (u^+)^2\, dx - \beta \int_\Omega (u^-)^2\, dx\right) \\
&\leq \frac{1}{2}\left(\lambda_k \int_\Omega u^2\, dx - \alpha \int_\Omega (u^+)^2\, dx - \alpha \int_\Omega (u^-)^2\, dx\right) \\
&= \frac{1}{2}(\lambda_k - \alpha) \int_\Omega |u|^2\, dx < 0.
\end{aligned}$$

This completes the proof. □

Lemma 3.7. *If $0 \neq v \in X_2$, then $M_{\alpha,\beta}(v) + v$ is sign-changing.*

Proof. Suppose not. Then we assume $w = M_{\alpha,\beta}(v) + v \gneqq 0$ in Ω. Let n represent the Fourier coefficient of w in the ϕ_1 direction. We note that $n > 0$ because $\int_\Omega w\phi_1 > 0$. Since we have maximized $J_{\alpha,\beta}$ with respect to X_1, we must have $\langle J'_{\alpha,\beta}(w), \phi_1\rangle = 0$. Thus

$$0 = \int_Q (w(x) - w(y))(\phi_1(x) - \phi_1(y))K(x-y)\,dxdy - \alpha \int_\Omega w^+\phi_1\,dx + \beta \int_\Omega w^-\phi_1\,dx.$$

But $w = w^+$ and $w^- \equiv 0$, so

$$0 = n \int_Q |\phi_1(x) - \phi_1(y)|^2 K(x-y)\,dxdy - n\alpha \int_\Omega \phi_1^2\,dx = n(\lambda_1 - \alpha)\int_\Omega \phi_1^2\,dx \neq 0,$$

a contradiction. Hence the result holds. □

In order to obtain the continuity property of $M_{\alpha,\beta}$, in the next lemma, we distinguish between the space X_2, which has the X_0 topology and Y_2, which is the set of points in X_2 endowed with the $L^2(\Omega)$ topology.

Lemma 3.8. *$M_{\alpha,\beta}$ is locally Lipschitz continuous as a function of $\mathbb{R}^2 \times Y_2$ into X_1.*

Proof. Putting $u_i = M_{\alpha_i,\beta_i}(v_i)$ for $i = 1, 2$ into (3.1), we get

$$\begin{aligned}
&\delta\|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\|^2 \\
&\leq |\beta_2 - \alpha_2|(\|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\|_{L^2} + \|v_2 - v_1\|_{L^2})\|v_2 - v_1\|_{L^2} \\
&\quad + |\alpha_2 - \alpha_1|\|M_{\alpha_1,\beta_1}(v_1)\|_{L^2}\|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\|_{L^2} \\
&\quad + |s_2 - s_1|\|M_{\alpha_1,\beta_1}(v_1) + v_1\|_{L^2}\|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\|_{L^2}.
\end{aligned}$$

By Poincare's inequality, we obtain

$$\begin{aligned}
&\delta\|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\|^2 \\
&\leq |\beta_2 - \alpha_2| \left(\frac{1}{\lambda_1}\|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\| + \|v_2 - v_1\|_{L^2}\right)\|v_2 - v_1\|_{L^2} \\
&\quad + |\alpha_2 - \alpha_1|\|M_{\alpha_1,\beta_1}(v_1)\|_{L^2}\frac{1}{\lambda_1}\|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\| \\
&\quad + |s_2 - s_1|\|M_{\alpha_1,\beta_1}(v_1) + v_1\|_{L^2}\frac{1}{\lambda_1}\|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\|.
\end{aligned}$$

Taking $v_2 = v$, $v_1 = 0$, $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$. Note that $M_{\alpha,\beta}(0) = 0$. Then the above inequality reduces to

$$\delta\|M_{\alpha,\beta}(v)\|^2 \leq |\beta - \alpha|\left(\frac{1}{\lambda_1}\|M_{\alpha,\beta}(v)\| + \|v\|_{L^2}\right)\|v\|_{L^2}.$$

From this inequality, one can show that $\|M_{\alpha,\beta}(v)\| \leq C\|v\|_{L^2}$ for an appropriate $C > 0$ depending on δ.

We now proceed to the main estimate. For a given v_1, we let $c_1 = \|M_{\alpha_1,\beta_1}(v_1)\|_{L^2}$, $c_2 = \|M_{\alpha_1,\beta_1}(v_1) + v_1\|_{L^2}$ and $w = \|M_{\alpha_2,\beta_2}(v_2) - M_{\alpha_1,\beta_1}(v_1)\|$. Then it follows that

$$\delta w^2 \leq (|\beta_2 - \alpha_2|\|v_2 - v_1\|_{L^2} + c_1|\alpha_2 - \alpha_1| + c_2|s_2 - s_1|)\frac{1}{\lambda_1}w + |\beta_2 - \alpha_2|\|v_2 - v_1\|_{L^2}^2.$$

Now, take $\gamma := (|\beta_2 - \alpha_2|\|v_2 - v_1\|_{L^2} + c_1|\alpha_2 - \alpha_1| + c_2|s_2 - s_1|)$. Then observe that $|\beta_2 - \alpha_2|\|v_2 - v_1\|_{L^2} \leq \gamma$, so

$$\delta w^2 \leq \frac{\gamma}{\lambda_1}w + \frac{\gamma^2}{|\beta_2 - \alpha_2|}.$$

Therefore, there exists a positive constant K such that $w \leq K\gamma$ and the result follows. □

Lemma 3.9. *For a given α and β, $M_{\alpha,\beta}\colon Y_2 \to X_1$ is Lipschitz continuous.*

Proof. Taking $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ in the proof of Lemma 3.8, one can easily see that $w \leq K_1\gamma$, where $\gamma = \|v_2 - v_1\|$, and K_1 has no dependence on c_1 and c_2. □

Lemma 3.10. *There is a $\rho > 0$ such that $\|M_{\alpha,\beta}(v)\| \leq \rho\|v\|_{L^2}$ for all $v \in X_2$.*

Proof. It follows from the Lipschitz continuity and the homogeneity properties of $M_{\alpha,\beta}$. □

Lemma 3.11. *Suppose that $\{v_k\}$ is bounded in X_2, and $\{\alpha_k\}$, $\{\beta_k\}$ are bounded sequences in $\mathbb{R}$ that satisfy our given restriction on (α, β). Then there exist subsequences, still denoted by $\{v_k\}$, $\{\alpha_k\}$ and $\{\beta_k\}$ such that $(\alpha_k, \beta_k) \to (\alpha, \beta)$ in $\mathbb{R}^2$, $v_k \rightharpoonup v$ in X_2, $v_k \to v$ in $L^2(\Omega)$ and $M_{\alpha,\beta}(v_k) \to M_{\alpha,\beta}(v)$ in X_1.*

Proof. The proof follows from the standard compactness arguments combined with the continuity established in Lemma 3.10. □

Lemma 3.12. *If $J_{\alpha,\beta}$ has a critical point at $w = u + v$, then $u = M_{\alpha,\beta}(v)$.*

Proof. $J'_{\alpha,\beta}(w) = 0$ since $J_{\alpha,\beta}$ has a critical point at $w = u + v$. Using Lemma 3.4, we have for each $v \in X_2$, $J_{\alpha,\beta}(\,\cdot\,, v)$ achieves a unique maximum. Thus $u = M_{\alpha,\beta}(v) = \max_{u_1 \in X_1} J_{\alpha,\beta}(u_1)$. □

Given the last lemma, it makes sense to restrict our search for critical points to the set $\mathcal{X}_2 := \{M_{\alpha,\beta}(v) + v : v \in X_2\}$. We define $\widetilde{J}_{\alpha,\beta}\colon X_2 \to \mathbb{R}$ as

$$\widetilde{J}_{\alpha,\beta}(v) = J_{\alpha,\beta}(M_{\alpha,\beta}(v) + v).$$

Lemma 3.13. *The functional $\widetilde{J}_{\alpha,\beta}$ is differentiable and its derivative is continuous, i.e., $\widetilde{J}_{\alpha,\beta} \in C^1(X_2, \mathbb{R})$.*

Proof. Using the maximum property and the continuity of $M_{\alpha,\beta}$, as well as the fact that $J_{\alpha,\beta}$ is C^1 on X_0, we have the following inequality

$$\begin{aligned}
\widetilde{J}_{\alpha,\beta}(v_2) - \widetilde{J}_{\alpha,\beta}(v_1) &= J_{\alpha,\beta}(M_{\alpha,\beta}(v_2) + v_2) - J_{\alpha,\beta}(M_{\alpha,\beta}(v_1) + v_1) \\
&\leq J_{\alpha,\beta}(M_{\alpha,\beta}(v_2) + v_2) - J_{\alpha,\beta}(M_{\alpha,\beta}(v_2) + v_1) \\
&= \langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v_2) + v_1), (v_2 - v_1)\rangle + o(\|v_2 - v_1\|) \\
&= \langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v_1) + v_1), (v_2 - v_1)\rangle + o(\|v_2 - v_1\|) \\
&\quad + \langle (J'_{\alpha,\beta}(M_{\alpha,\beta}(v_2) + v_1) - J'_{\alpha,\beta}(M_{\alpha,\beta}(v_1) + v_1)), (v_2 - v_1)\rangle \\
&= \langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v_1) + v_1), (v_2 - v_1)\rangle + o(\|v_2 - v_1\|).
\end{aligned}$$

Similarly we can show

$$\widetilde{J}_{\alpha,\beta}(v_2) - \widetilde{J}_{\alpha,\beta}(v_1) \geq \langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v_1) + v_1), (v_2 - v_1)\rangle + o(\|v_2 - v_1\|).$$

Hence the functional $\widetilde{J}_{\alpha,\beta}$ is continuously differentiable. □

From the above Lemma 3.13, we also note the following identity

$$\widetilde{J}'_{\alpha,\beta}(v) = J'_{\alpha,\beta}(M_{\alpha,\beta}(v) + v). \tag{3.2}$$

Lemma 3.14. *$v \in X_2$ is a critical point of $\widetilde{J}_{\alpha,\beta}$ if and only if $M_{\alpha,\beta}(v) + v$ is a critical point of $J_{\alpha,\beta}$.*

Proof. Assume that $M_{\alpha,\beta}(v)+v$ is a critical point of $J_{\alpha,\beta}$. Then $\langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v)+v), w\rangle = 0$ for all $w \in X_0$. In particular $\langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v)+v), w\rangle = 0$ for all $w \in X_2$. Now using equation (3.2), we have $\langle \widetilde{J}'_{\alpha,\beta}(v), w\rangle = 0$ for all $w \in X_2$, so v is a critical point of $\widetilde{J}_{\alpha,\beta}$.

Conversely, suppose that v is a critical point of $\widetilde{J}_{\alpha,\beta}$. Then as above, $\langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v) + v), w\rangle = 0$ for all $w \in X_2$. Recall that $M_{\alpha,\beta}(v)$ maximizes $J_{\alpha,\beta}(u + v)$ for $u \in X_1$. Hence $\langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v) + v), u\rangle = 0$ for all $u \in X_1$. Thus $\langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v) + v), w\rangle = 0$ for all $w \in X_0$. □

Lemma 3.15. *$\widetilde{J}_{\alpha,\beta}(tv) = t^2\widetilde{J}_{\alpha,\beta}(v)$ for all $t \geq 0$ and for all $v \in X_2$.*

Proof. Using the homogeneity of $J_{\alpha,\beta}$ and $M_{\alpha,\beta}$, we have

$$\begin{aligned}
\widetilde{J}_{\alpha,\beta}(tv) &= J_{\alpha,\beta}(M_{\alpha,\beta}(tv) + tv) = J_{\alpha,\beta}(tM_{\alpha,\beta}(v) + tv) \\
&= t^2 J_{\alpha,\beta}(M_{\alpha,\beta}(v) + v) = t^2\widetilde{J}_{\alpha,\beta}(v),
\end{aligned}$$

which completes the proof. □

Now the homogeneity leads to the following lemma.

Lemma 3.16. *If $v \in X_2$ is a critical point of $\widetilde{J}_{\alpha,\beta}$ then $\widetilde{J}_{\alpha,\beta}(v) = 0$.*

Proof. Differentiating $\widetilde{J}_{\alpha,\beta}(tv) = t^2\widetilde{J}_{\alpha,\beta}(v)$ with respect to t to get $\langle \widetilde{J}'_{\alpha,\beta}(tv), v\rangle = 2t\widetilde{J}_{\alpha,\beta}(v)$. Then the result follows by taking $t = 1$. □

As with $J_{\alpha,\beta}$, it is useful to think of $\widetilde{J}_{\alpha,\beta}$ as a function on $\mathbb{R}^2 \times X_2$ as $\widetilde{J}_{\alpha,\beta}(v) := \widetilde{J}(\alpha, \beta, v)$. Then we establish the following

Lemma 3.17. *$\widetilde{J}(\alpha, \beta, \cdot\,) := \widetilde{J}_{\alpha,\beta}(\cdot)$ is strictly decreasing in α and β.*

Proof. Assume that $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$, where at least one of the inequality is strict. Then using the definition of $J_{\alpha,\beta}$, the fact that $M_{\alpha,\beta}(v) + v$ is sign changing and the maximizing property of $M_{\alpha,\beta}$, we obtain

$$\begin{aligned}\widetilde{J}(\alpha_2, \beta_2, v) &= J(\alpha_2, \beta_2, M(\alpha_2, \beta_2, v) + v)\\ &< J(\alpha_1, \beta_1, M(\alpha_2, \beta_2, v) + v)\\ &\le J(\alpha_1, \beta_1, M(\alpha_1, \beta_1, v) + v),\end{aligned}$$

which completes the proof. □

Lemma 3.18. *Given any positive number R, there is a positive number C such that*

$$|\widetilde{J}(\alpha_2, \beta_2, v) - \widetilde{J}(\alpha_1, \beta_1, v)| \le C(|\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|),$$

whenever $\max\{|\alpha_1|, |\alpha_2|, |\beta_1|, |\beta_2|, \|v\|\} \le R$.

Proof. Combining the Lipschitz continuity of $J_{\alpha,\beta}$ and $M_{\alpha,\beta}$ in Lemma 3.10, we obtain the desired result. We also notice that the bound is on $\|v\|$ rather than just $\|v\|_{L^2}$. This is because the Lipschitz constant on $J_{\alpha,\beta}$ depends on a bound in X_0. □

3.2. Minimizing in the X_2 direction

We note that to find the critical points of $J_{\alpha,\beta}$ on X_0 has been reduced to find the critical points of $\widetilde{J}_{\alpha,\beta}$ on X_2. We know that $\widetilde{J}_{\alpha,\beta}$ is homogeneous, so it suffices to look for critical points on $\mathcal{S}_{X_2} := \{v \in X_2 : \|v\|_{L^2} = 1\}$, a weakly closed set in X_0, i.e., if $\{v_k\} \subset \mathcal{S}_{X_2}$ and $v_k \rightharpoonup v$ weakly in X_0, then $v_k \to v$ strongly in L^2 so $\|v\|_{L^2} = 1$ and $v \in \mathcal{S}_{X_2}$.

Lemma 3.19. *$\widetilde{J}_{\alpha,\beta}$ achieves a global minimum on $\mathcal{S}_{X_2}$.*

Proof. It is easy to see that $\widetilde{J}_{\alpha,\beta}$ is bounded below on $\mathcal{S}_{X_2}$. Let $\{v_k\} \subset \mathcal{S}_{X_2}$ be a minimizing sequence for $\widetilde{J}_{\alpha,\beta}$ and let $m = \inf_{v\in\mathcal{S}_{X_2}} \widetilde{J}_{\alpha,\beta}(v)$. Then one can easily see that $\|v_k\|$ is bounded. So after passing to a subsequence, we have $v_k \rightharpoonup v_0$ weakly in X_0 and $v_k \to v_0$

strongly in $L^2(\Omega)$ with $\|v_0\|_{L^2} = 1$. By the continuity and compactness of $M_{\alpha,\beta}$, we have $M_{\alpha,\beta}(v_k) \to M_{\alpha,\beta}(v_0)$ in X_0. Using these observation as well as the weak lower semicontinuity of X_0 norm, we obtain $v_0 \in \mathcal{S}_{X_2}$ such that $\widetilde{J}_{\alpha,\beta}(v_0) = \inf_{v\in\mathcal{S}_{X_2}} \widetilde{J}_{\alpha,\beta}(v)$. □

If v_0 is a critical point of $\widetilde{J}_{\alpha,\beta}$ restricted to $\mathcal{S}_{X_2}$, then one can not conclude that it is a critical point of $\widetilde{J}_{\alpha,\beta}$ on X_2. For this, one must check the direction orthogonal to the surface $\mathcal{S}_{X_2}$.

Lemma 3.20. *$v_0 \in X_2$ is a nontrivial critical point of $\widetilde{J}_{\alpha,\beta}$ if and only if v_0 is a critical point of $\widetilde{J}_{\alpha,\beta}$ restricted to $\mathcal{S}_{X_2}$ and $\widetilde{J}_{\alpha,\beta}(v_0) = 0$.*

Proof. This is a standard fact for homogeneous operator, since every nontrivial element of X_2 can be written as tv for some $v \in S_{X_2}$ and for some $t > 0$. Computing derivatives separately with respect to t and v gives the result. For this, one can follow the proof of Lemma 3.16. □

Lemma 3.21. *If u is a nontrivial critical point of $J_{\alpha,\beta}$ if and only if $u = M_{\alpha,\beta}(v_0) + v_0$, where $\frac{v_0}{\|v_0\|_{L^2}}$ is a critical point of $\widetilde{J}_{\alpha,\beta}$ restricted to $\mathcal{S}_{X_2}$ and $\widetilde{J}_{\alpha,\beta}(v_0) = 0$.*

Proof. It is a direct consequence of Lemma 3.20. □

Definition 3.22. $m(\alpha,\beta) := \min_{v\in\mathcal{S}_{X_2}} \widetilde{J}_{\alpha,\beta}(v)$.

Lemma 3.23. *$m(\alpha,\beta)$ is Lipschitz continuous and is strictly decreasing as a function of both α and β. Moreover, $m(\alpha,\alpha) > 0$.*

Proof. Let (α_1,β_1) and (α_2,β_2) be two points in the plane. Let v_1 and v_2 be the corresponding global minimizers on $\mathcal{S}_{X_2}$, and let $w_{ij} = M_{\alpha_i,\beta_i}(v_j)+v_j$ for $i,j = 1,2$. Then using the minimizing property of v_i and then the maximizing property of M_{α_j,β_j}, we obtain

$$\begin{aligned} m(\alpha_i,\beta_i) &= J_{\alpha_i,\beta_i}(M_{\alpha_i,\beta_i}(v_i) + v_i) \le J_{\alpha_i,\beta_i}(M_{\alpha_i,\beta_i}(v_j) + v_j) \\ &= J_{\alpha_j,\beta_j}(M_{\alpha_i,\beta_i}(v_j) + v_j) + \frac{1}{2}(\alpha_j - \alpha_i)\int_\Omega (w_{ij}^+)^2 + \frac{1}{2}(\beta_j - \beta_i)\int_\Omega (w_{ij}^-)^2 \\ &\le J_{\alpha_j,\beta_j}(M_{\alpha_j,\beta_j}(v_j) + v_j) + \frac{1}{2}(\alpha_j - \alpha_i)\int_\Omega (w_{ij}^+)^2 + \frac{1}{2}(\beta_j - \beta_i)\int_\Omega (w_{ij}^-)^2 \\ &= m(\alpha_j,\beta_j) + \frac{1}{2}(\alpha_j - \alpha_i)\int_\Omega (w_{ij}^+)^2 + \frac{1}{2}(\beta_j - \beta_i)\int_\Omega (w_{ij}^-)^2. \end{aligned}$$

Since this inequality holds for $i = 1$, $j = 2$ and $i = 2$, $j = 1$, we have

$$|m(\alpha_2,\beta_2) - m(\alpha_1,\beta_1)| \le c(|\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|),$$

where $c = \max\{\|w_{12}\|_{L^2}, \|w_{21}\|_{L^2}\}$.

Moreover, if $\alpha_2 \geq \alpha_1$ and $\beta_2 \geq \beta_1$ where at least one of these inequalities is strict, then $m(\alpha_2, \beta_2) < m(\alpha_1, \beta_1)$. This last conclusion uses the fact that w_{ij} is sign-changing.

If $\alpha = \beta$, then for $w \in X_0$ we have

$$\begin{aligned} J_{\alpha,\beta}(w) = J_{\alpha,\alpha}(w) &= \frac{1}{2}\left(\int_Q |w(x) - w(y)|^2 K(x-y)\,dxdy - \alpha \int_\Omega w^2\,dx\right) \\ &= \frac{1}{2}\sum_{j=1}^{\infty}(\lambda_j - \alpha)c_j^2, \end{aligned}$$

where we are applying the Fourier decomposition of w. Write $w = u + v$ using the usual decomposition of X_0. Then the coefficient $(\lambda_i - \alpha)$ are strictly negative for $j \leq k$, so it follows that we can maximize in the X_1 direction by choosing $c_j = 0$ for $j = 1, \ldots, k$, i.e., $M_{\alpha,\beta}(v) \equiv 0$. Thus we have

$$\widetilde{J}_{\alpha,\beta}(v) = J_{\alpha,\beta}(v) = \frac{1}{2}\sum_{j=k+1}^{\infty}(\lambda_j - \alpha)c_j^2.$$

Also the coefficients $(\lambda_j - \alpha)$ are strictly positive and increasing for $j = k+1, k+2, \ldots$. Also $\sum_{j=k+1}^{\infty} c_j^2 = \|v\|_{L^2} = 1$. Using the Lagrange multipliers, one can show that the critical points of this sum occur when $c_j \equiv \pm 1$ for one j and $c_j = 0$ for all other j. The minimizing choice is when $c_{k+1} = 1$ and $c_j = 0$ for $j > k+1$. Hence the minimizer is $v = \pm\phi_{k+1}$ and $m(\alpha, \alpha) = \widetilde{J}_{\alpha,\beta}(v) = \frac{1}{2}(\lambda_{k+1} - \alpha) > 0$. □

Lemma 3.24. *$m(\alpha, \lambda_{k+1}) > 0$.*

Proof. Let $v \in \mathcal{S}_{X_2}$ and let $\beta = \lambda_{k+1}$. Then using $\alpha < \lambda_{k+1}$ and v^+ is nontrivial, we obtain

$$\begin{aligned} \widetilde{J}_{\alpha,\beta}(v) &= J_{\alpha,\beta}(M_{\alpha,\beta}(v) + (v)) \geq J_{\alpha,\beta}(v) \\ &> \frac{1}{2}\left(\int_Q |v(x) - v(y)|^2 K(x-y)\,dxdy - \lambda_{k+1}\int_\Omega v^2\,dx\right) \geq 0, \end{aligned}$$

which follows the result. □

All of the lemmas above have been leading to the following theorem.

Theorem 3.25. *Assume that $\lambda_k < \alpha < \lambda_{k+1}$. Then one of the following is true:*

(1) *$m(\alpha, \beta) > 0$ and $(\alpha, \beta) \notin \Sigma_K$ for all $\beta \geq \alpha$.*

(2) *There is a unique $\beta(\alpha) > \lambda_{k+1}$, such that $m(\alpha, \beta(\alpha)) = 0$. Moreover, $(\alpha, \beta(\alpha)) \in \Sigma_K$, but $(\alpha, \beta) \notin \Sigma_K$ if $\alpha \leq \beta < \beta(\alpha)$.*

Proof. Case 1. If $\alpha \le \beta \le \lambda_{k+1}$ then $0 < m(\alpha, \lambda_{k+1}) \le m(\alpha, \beta)$ by Lemmas 3.24 and 3.23. This implies $(\alpha, \beta) \notin \Sigma_K$ by Lemma 3.20.

Case 2. If $\beta > \lambda_{k+1}$ then there is a $\beta(\alpha)$ such that $m(\alpha, \beta(\alpha)) = 0$. Clearly, one can easily see that $\beta(\alpha)$ is unique. If not, there are two $\beta_1(\alpha)$ and $\beta_1(\alpha)$ such that $m(\alpha, \beta_i(\alpha)) = 0$ for $i = 1, 2$. Now $0 = m(\alpha, \beta_1(\alpha)) < m(\alpha, \beta_i(\alpha)) = 0$, which gives a contradiction. Since $m(\alpha, \beta(\alpha)) = 0$, we have $(\alpha, \beta(\alpha)) \in \Sigma_K$ by Lemma 3.19. Now, if $\alpha \le \beta < \beta(\alpha)$ then $0 = m(\alpha, \beta(\alpha)) < m(\alpha, \beta)$ which implies $(\alpha, \beta) \notin \Sigma_K$. $\square$

Lemma 3.26. *The curve $(\alpha, \beta(\alpha))$ is Lipschitz continuous, strictly decreasing, and contains the point $(\lambda_{k+1}, \lambda_{k+1})$.*

Proof. Consider two points (α_1, β_1) and (α_2, β_2) on Σ_K, characterized as above, with $\alpha_2 > \alpha_1$. Let v_i be a minimizer of $J_{\alpha_i,\beta_i}(M_{\alpha_i,\beta_i}(v)+v)$ such that $\|v_i\|_{L^2} = 1$. In particular, we know that $J_{\alpha_i,\beta_i}(M_{\alpha_i,\beta_i}(v_i) + v_i) = 0$ and that $J_{\alpha_i,\beta_i}(M_{\alpha_i,\beta_i}(v) + v) \ge 0$ for all $v \in X_2$. Let $w_i = M_{\alpha_i,\beta_i}(v_i) + v_i$, then we have

$$\begin{aligned}
0 &= 2J_{\alpha_1,\beta_1}(w_1)\\
&= \int_Q |w_1(x) - w_1(y)|^2 K(x-y)\,dxdy - \alpha_1 \int_\Omega (w_1^+)^2\,dx - \beta_1 \int_\Omega (w_1^-)^2\,dx\\
&> \int_Q |w_1(x) - w_1(y)|^2 K(x-y)\,dxdy - \alpha_2 \int_\Omega (w_1^+)^2\,dx - \beta_1 \int_\Omega (w_1^-)^2\,dx,
\end{aligned}$$

where we obtain strict inequality using the fact that $\alpha_2 > \alpha_1$ and that w_1 is sign changing so that w_1^+ is nontrivial. It follows that $m(\alpha_2, \beta_1) < 0$. Since $m(\alpha, \beta)$ is strictly decreasing in β and $m(\alpha_2, \beta_2) = 0$, it must be the case that $\beta_2 < \beta_1$, i.e., $\beta(\alpha)$ is strictly decreasing. Now consider

$$\begin{aligned}
2J_{\alpha_2,\beta_1}(w_2) &= \int_Q |w_2(x) - w_2(y)|^2 K(x-y)\,dxdy - \alpha_2 \int_\Omega (w_2^+)^2 - \beta_1 \int_\Omega (w_2^-)^2\\
&= (\beta_2 - \beta_1) \int_\Omega (w_2^-)^2.
\end{aligned}$$

It follows that $m(\alpha_2, \beta_1) \le \frac{1}{2}(\beta_2 - \beta_1) \int_\Omega (w_2^-)^2 < 0$. Thus

$$|\beta_2 - \beta_1| \le 2\frac{1}{\int_\Omega (w_2^-)^2} |m(\alpha_2, \beta_1)| = 2\frac{1}{\int_\Omega (w_2^-)^2} |m(\alpha_2, \beta_1) - m(\alpha_2, \beta_2)|.$$

The Lipschitz estimate for $\beta(\alpha)$ follows from the Lipschitz estimate for $m(\alpha, \beta)$. $\square$

4. Nonresonance and resonance case for problem (1.1)

4.1. The nonresonance case

In this section we assume that $(\alpha, \beta) \in \mathbb{R}^2$ such that $\lambda_k < \alpha < \lambda_{k+1}$ and $\alpha \le \beta < \beta(\alpha)$. By the characterization of the Fučik spectrum in Theorem 3.25 and Lemma 3.26, we know that $(\alpha, \beta) \notin \Sigma_K$. Then one should expect that (1.1) is solvable without further restrictions on either f or h, by analogy with the Fredholm Alternative for the linear case. This is indeed the case.

For notational convenience let $E = E_{\alpha,\beta}$ and $J = J_{\alpha,\beta}$. Notice that

$$E(u) = J(u) - \int_\Omega (F(u) + hu). \tag{4.1}$$

We will see that the geometry of J dominates the geometry of E, so that the saddle geometry is easily proved in this case.

Lemma 4.1. *There is a positive constant K such that $|\int_\Omega (F(u) + hu)| \leq K\|u\|_{L^2}$ for all $u \in X_0$.*

Proof. Since f is bounded, there is an $M > 0$ such that $|f(t)| \leq M$ for all $t \in \mathbb{R}$. It immediately follows that $|F(t)| \leq M|t|$ for all t. Thus

$$\begin{aligned} \left|\int_\Omega (F(u) + hu)\right| &\leq \int_\Omega |F(u) + hu| \leq \int_\Omega (M + |h|)|u| \\ &\leq \left(\int_\Omega (M + |h|)^2\right)^{1/2} \left(\int_\Omega u^2\right)^{1/2}. \end{aligned}$$

□

Lemma 4.2. *E is anticoercive when restricted to X_1.*

Proof. Let $u \in X_1$, then using $\alpha \leq \beta$, $\int_Q |u(x) - u(y)|^2 K(x - y)\, dxdy \leq \lambda_k \int_\Omega u^2\, dx$ for all $u \in X_1$ and Lemma 4.1, we have

$$\begin{aligned} E(u) &= J(u) - \int_\Omega (F(u) + hu) \\ &\leq \left(1 - \frac{\alpha}{\lambda_k}\right)\|u\|^2 + \left(\int_\Omega (M + |h|)^2\right)^{1/2} \left(\int_\Omega u^2\right)^{1/2} \\ &\leq \left(1 - \frac{\alpha}{\lambda_k}\right)\|u\|^2 + C\|u\| \to -\infty \end{aligned}$$

as $\|u\| \to \infty$, since $\lambda_k < \alpha$. □

Lemma 4.3. *The functional E is bounded below and coercive on $\mathcal{X}_2 := \{M_{\alpha,\beta}(v) + v : v \in X_2\}$.*

Proof. Since $\beta < \beta(\alpha)$, we know that $\inf_{\mathcal{S}_{X_2}} \widetilde{J}(v) \geq c$ for some c. It follows that for any $v \in \mathcal{X}_2$,

$$J(M(v) + v) = \widetilde{J}(v) = \|v\|_{L^2}^2 \widetilde{J}\left(\frac{v}{\|v\|_{L^2}}\right) \geq c\|v\|_{L^2}^2.$$

Now, for $u = M(v) + v$ we have

$$E(u) \geq c\|v\|_{L^2}^2 - \left(\int_\Omega (M + |h|)^2\right)^{1/2} \|u\|_{L^2}.$$

Recall that $\|M_{\alpha,\beta}(v)\| \le c\|v\|_{L^2}$ for all $v \in X_2$. Then it follows that $\|u\|_{L^2} \le k\|v\|_{L^2}$ for some $k > 0$ and all $v \in X_2$. Thus the inequality for E becomes

$$E(u) \ge c\|v\|_{L^2}^2 - k\left(\int_\Omega (M + |h|)^2\right)^{1/2} \|v\|_{L^2}.$$

Hence one can easily conclude that E is bounded below and coercive on $\mathcal{X}_2$. $\square$

As a result of this estimates above we can choose $R > 0$ such that

$$\sup_{u \in X_1, \|u\|=R} E(u) < \inf_{v \in \mathcal{X}_2} E(v).$$

In the next lemma we show that $\partial B_R(0) := \{x \in X_1 : \|x\| = R\}$ and $\mathcal{X}_2$ link. Note that $\partial B_R(0)$ is clearly embedding of S^{k-1} in X_0.

Lemma 4.4. *Let $\gamma\colon \overline{B_R(0)} \subset X_1 \to X_0$ be a continuous function and write $\gamma(x) = \gamma_{X_1}(x) + \gamma_{X_2}(x)$, where $\gamma_{X_1}(x) \in X_1$ and $\gamma_{X_2}(x) \in X_2$. We assume that γ fixes ∂B_R, so $\gamma_{X_1}(x) = x$ and $\gamma_{X_2}(x) = 0$ for all $x \in \partial\overline{B_R(0)}$, then $\gamma(\overline{B_R(0)}) \cap \mathcal{X}_2 \neq \emptyset$.*

Proof. We must show that there is an $x \in \overline{B_R(0)}$ such that $\gamma_{X_1}(x) = M(\gamma_{X_2}(x))$, so it is reasonable to study the solutions of the equation $G(x) = 0$ where $G\colon \overline{B_R(0)} \to X_1$: $G(x) = \gamma_{X_1}(x) - M(\gamma_{X_2}(x))$. It is clear that G is continuous. Also, if $x \in \partial\overline{B_R(0)}$, then $G(x) = x \neq 0$ and so the Brouwer degree $\deg(G, \overline{B_R(0)}, 0)$ is well defined. Consider the homotopy $h(t,x) = tG(x) + (1-t)x$, where $t \in [0,1]$ and $x \in \overline{B_R(0)}$. For $x \in \partial\overline{B_R(0)}$ we have $h(t,x) = tx + (1-t)x = x \neq 0$, so $\deg(G, \overline{B_R(0)}, 0) = \deg(I, \overline{B_R(0)}, 0) = 1$ where I represents the identity map. Hence $G(x) = 0$ has a solution in $\overline{B_R(0)}$. $\square$

Lemma 4.5. *Assume $K\colon \mathbb{R}^n \setminus \{0\} \to (0,\infty)$ satisfies assumptions* (K1)–(K3)*, $f\colon \mathbb{R} \to \mathbb{R}$ is bounded and continuous, and $h \in L^2(\Omega)$. Let $c \in \mathbb{R}$ and let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in X_0 such that*

$$E(u_k) \le c \tag{4.2}$$

and

$$\sup\{|\langle E'(u_k), \phi\rangle : \phi \in X_0, \|\phi\| = 1|\} \to 0 \tag{4.3}$$

as $k \to \infty$. Then, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X_0. Moreover, there exists $u_0 \in X_0$ such that, up to a subsequence

$$\|u_k - u_0\| \to 0 \quad \text{as } k \to \infty.$$

Proof. Let $\{u_k\} \subset X_0$ be such that (4.2) and (4.3) hold, i.e., $E(u_k)$ is bounded and $E'(u_k) \to 0$ in X_0^*. Then we show that $\{u_k\}$ is bounded in X_0. Suppose by contradiction that $\|u_k\|_{L^2}$ is unbounded. Then without loss of generality we may assume that $\|u_k\|_{L^2}$ is increasing to ∞. Consider $v_k := \frac{u_k}{\|u_k\|_{L^2}}$. Then,

$$\begin{aligned}\frac{E(u_k)}{\|u_k\|_{L^2}^2} &= \frac{1}{2}\int_Q |v_k(x)-v_k(y)|^2 K(x-y)\,dxdy - \frac{\alpha}{2}\int_\Omega (v_k^+)^2 - \frac{\beta}{2}\int_\Omega (v_k^-)^2 \\ &\quad - \frac{1}{\|u_k\|_{L^2}^2}\int_\Omega (F(u_k)+hu_k).\end{aligned}$$

Now equation (4.2) implies that $\frac{E(u_k)}{\|u_k\|_{L^2}^2} \to 0$. Also $\frac{\alpha}{2}\int_\Omega (v_k^+)^2 + \frac{\beta}{2}\int_\Omega (v_k^-)^2 + \frac{1}{\|u_k\|_{L^2}^2}\int_\Omega (F(u_k) + hu_k)$ is bounded. Then it follows that $\{v_k\}$ is bounded in X_0, a reflexive space (being a Hilbert space), so up to a subsequence, there exists $v_0 \in X_0$ such that $v_k \rightharpoonup v_0$ weakly in X_0, $v_k \to v_0$ strongly in $L^2(\Omega)$ and $\|v_0\|_{L^2} = 1$. Now for any $w \in X_0$, we consider

$$\begin{aligned}\left\langle \frac{E'(u_k)}{\|u_k\|_{L^2}}, w\right\rangle &= \int_Q (v_k(x)-v_k(y))(w(x)-w(y))K(x-y)\,dxdy - \alpha\int_\Omega (v_k^+)w \\ &\quad + \beta\int_\Omega (v_k^-)w - \frac{1}{\|u_k\|_{L^2}}\left(\int_\Omega (f(u_k)+h)w\right).\end{aligned}$$

Using the boundedness of f it is clear that $\frac{1}{\|u_k\|_{L^2}}\int_\Omega (f(u_k)+h)w \to 0$. Also using the L^2 convergence of v_k, it is clear that v_k^+ and v_k^- converges to v_0^+ and v_0^- respectively in L^2. So,

$$-\alpha\int_\Omega (v_k^+)w + \beta\int_\Omega (v_k^-)w \to -\alpha\int_\Omega (v_0^+)w + \beta\int_\Omega (v_0^-)w.$$

By the weak convergence of v_k in X_0, we have for every $\phi \in X_0$,

$$\begin{aligned}&\int_Q (v_k(x)-v_k(y))(\phi(x)-\phi(y))K(x-y)\,dxdy \\ &\to \int_Q (v_0(x)-v_0(y))(\phi(x)-\phi(y))K(x-y)\,dxdy\end{aligned}$$

as $k \to \infty$. Thus using the above discussion, we obtain $\langle \frac{E'(u_k)}{\|u_k\|_{L^2}}, w\rangle \to 0$. Hence

$$0 = \int_Q (v_0(x)-v_0(y))(w(x)-w(y))\,dxdy - \alpha\int_\Omega (v_0^+)w + \beta\int_\Omega (v_0^-)w \quad \forall\, w \in X_0.$$

Therefore v_0 is a nontrivial weak solution of (1.2). This contradicts the fact that $(\alpha,\beta) \notin \Sigma_K$. Hence $\{u_k\}$ is bounded in L^2. Now

$$E(u_k) = \frac{1}{2}\|u_k\|^2 - \frac{\alpha}{2}\int_\Omega (u_k^+)^2 - \frac{\beta}{2}\int_\Omega (u_k^-)^2 - \int_\Omega (F(u_k)+hu_k).$$

We see that $E(u_k)$, $\int_\Omega (u_k^+)^2$, $\int_\Omega (u_k^-)^2$ and $\int_\Omega (F(u_k)+hu_k)$ are all bounded, so $\|u_k\|$ must be bounded.

Since $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in X_0, there exists $u_0 \in X_0$ such that up to a subsequence u_k converges to u_0 weakly in X_0, i.e., for every $\phi \in X_0$,

$$\begin{aligned}&\int_Q (u_k(x)-u_k(y))(\phi(x)-\phi(y))K(x-y)\,dxdy\\ &\to \int_Q (u_0(x)-u_0(y))(\phi(x)-\phi(y))K(x-y)\,dxdy\end{aligned}$$

as $k\to\infty$. Moreover, $u_k \to u_0$ strongly in $L^\mu(\Omega)$ for any $\mu \in [1, 2_s^*)$ and $u_k(x) \to u_0(x)$ a.e. in $\mathbb{R}^n$ as $k\to\infty$. Now,

$$\begin{aligned}&\langle E'(u_k),(u_k-u_0)\rangle\\ &= \int_Q (u_k(x)-u_k(y))((u_k-u_0)(x)-(u_k-u_0)(y))K(x-y)\,dxdy\\ &\quad -\alpha\int_\Omega (u_k^+)(u_k-u_0)+\beta\int_\Omega (u_k^-)(u_k-u_0)-\int_\Omega (f(u_k)+h)(u_k-u_0).\end{aligned} \tag{4.4}$$

Also using the L^2 boundedness of u_k^+, u_k^-, and $f(u_k)+h$ and the fact that $u_k \to u_0$ strongly in L^2, we obtain

$$-\alpha\int_\Omega (u_k^+)(u_k-u_0)+\beta\int_\Omega (u_k^-)(u_k-u_0)-\int_\Omega (f(u_k)+h)(u_k-u_0)\to 0. \tag{4.5}$$

From (4.3), we have $\langle E'(u_k),(u_k-u_0)\rangle \to 0$. Thus using this, (4.4) and (4.5), we obtain

$$\int_Q (u_k(x)-u_k(y))((u_k-u_0)(x)-(u_k-u_0)(y))K(x-y)\,dxdy \to 0 \quad \text{as } k\to\infty.$$

Hence, using this and the weak convergence of u_k, we obtain

$$\int_Q |u_k(x)-u_k(y)|^2K(x-y)\,dxdy \to \int_Q |u_0(x)-u_0(y)|^2K(x-y)\,dxdy \quad \text{as } k\to\infty.$$

It follows that $u_k \to u_0$ strongly in X_0. □

Proof of Theorem 1.3. By the saddle point theorem we can now conclude the proof. □

4.2. The resonance case

In this section, we study the problem (1.1) in the presence of a resonance, namely when $(\alpha,\beta)\in\mathbb{R}^2$ is an element of Fučik spectrum. This kind of problem is harder to solve than the nonresonant one and we have to impose further conditions on the nonlinearities. We assume that $\beta=\beta(\alpha)$. Many of the argument from the previous section are still applicable. Two notable exceptions are establishing a lower bound for E on $\mathcal{X}_2$ and proving (PS). Since this case is analogous to the case $\lambda=\lambda_{k+1}$ in this Fredholm Alternative, we should expect that the solutions will only exist if a generalized orthogonality condition is satisfied. Such conditions were first studied in 70s and known as Landesman-Lazer conditions [13]. We will assume (1.5): $\lim_{k\to\infty}\int_\Omega (F(u_k)+hu_k)\,dx = -\infty$, a generalized Landesman-Lazer condition.

Lemma 4.6. *If* (1.5) *is satisfied, then E define in* (4.1) *is bounded below on $\mathcal{X}_2$.*

Proof. Suppose that $\{u_k\}\subset \mathcal{X}_2$ such that $E(u_k)\to-\infty$. We write $u_k = M_{\alpha,\beta}(v_k)+v_k$. Then arguments similar to those in the proof of Lemma 4.3, we see that no subsequence of $\{u_k\}$ lies in a set of the form $\{u\in\mathcal{X}_2 : u = M_{\alpha,\beta}(v)+v, \widetilde{J}_{\alpha,\beta}(v)\geq c\|v\|^2_{L^2}\}$, where $c>0$. Thus $\widetilde{J}_{\alpha,\beta}(\frac{v_k}{\|v_k\|_{L^2}})\to 0$ and $\frac{v_k}{\|v_k\|_{L^2}}$ must be a minimizing sequence of $\widetilde{J}_{\alpha,\beta}$. Similarly, using the same arguments in the proof of Lemma 3.19 one can note that $\frac{v_k}{\|v_k\|_{L^2}}\rightharpoonup v_0$ weakly in X_0 and $\frac{v_k}{\|v_k\|_{L^2}}\to v_0$ strongly in $L^2(\Omega)$. This implies that $\frac{u_k}{\|u_k\|_{L^2}}\rightharpoonup\phi$ weakly in X_0 and $\frac{u_k}{\|u_k\|_{L^2}}\to\phi$ strongly in $L^2(\Omega)$, where $\phi = M_{\alpha,\beta}(v)+v$ is a nontrivial eigenfunction associated with (α,β). By (1.5), we know that $\lim_{k\to\infty}\int_\Omega(F(u_k)+hu_k) = -\infty$ and it immediately follows that $E(u_k)\to\infty$, a contradiction. Hence E is bounded below on $\mathcal{X}_1$. $\square$

Lemma 4.7. *Assume $K\colon\mathbb{R}^n\setminus\{0\}\to(0,\infty)$ satisfies assumptions* (K1)–(K3)*, f is a bounded and continuous function and $h\in L^2(\Omega)$. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in X_0 such that* (4.2) *and* (4.3) *hold. Then, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X_0 if* (1.5) *is satisfied.*

Proof. The first part of the proof is identical the argument in the proof of Lemma 4.5. We start with the hypothetical sequence $\{u_k\}$ such that (4.2) and (4.3) hold. Suppose $\|u_k\|_{L^2}$ is unbounded. Then argue up to the point, where we have $v_k\rightharpoonup v_0$ weakly in X_0, $v_k\to v_0$ strongly in $L^2(\Omega)$, where $\|v_0\|_{L^2}=1$ and v_0 is an eigenfunction associated with (α,β). Of course, in the resonance case this is not yet a contradiction, so a further argument is needed.

Write $u_k = w_k+v_k = \widetilde{w}_k + M_{\alpha,\beta}(v_k)+v_k$. Now using the fact that $\langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v_k)+v_k),u\rangle = 0$ for all $u\in X_1$ and Lemma 3.1, we have

$$\begin{aligned}\langle E'(u_k),\widetilde{w}_k\rangle &= \langle J'_{\alpha,\beta}(u_k),\widetilde{w}_k\rangle - \int_\Omega (f(u_k)+h)\widetilde{w}_k\\ &= \langle J'_{\alpha,\beta}(\widetilde{w}_k+M_{\alpha,\beta}(v_k)+v_k),\widetilde{w}_k\rangle - \int_\Omega (f(u_k)+h)\widetilde{w}_k\\ &= \langle J'_{\alpha,\beta}(\widetilde{w}_k+M_{\alpha,\beta}(v_k)+v_k),\widetilde{w}_k\rangle - \langle J'_{\alpha,\beta}(M_{\alpha,\beta}(v_k)+v_k),\widetilde{w}_k\rangle\\ &\quad - \int_\Omega (f(u_k)+h)\widetilde{w}_k\\ &\leq -\delta\|\widetilde{w}_k\|^2_{L^2} - \int_\Omega (f(u_k)+h)\widetilde{w}_k.\end{aligned}$$

It follows that $\widetilde{w}_k$ is bounded. Note that $\langle J'_{\alpha,\beta}(u_k),\widetilde{w}_k\rangle$ must also be bounded.

Now consider

$$\begin{aligned} E(u_k) &= J_{\alpha,\beta}(u_k) - \int_\Omega (F(u_k) + hu_k) \\ &\geq J_{\alpha,\beta}(\widetilde{w}_k + M_{\alpha,\beta}(v_k) + v_k) - J_{\alpha,\beta}(M_{\alpha,\beta}(v_k) + v_k) - \int_\Omega (F(u_k) + hu_k), \end{aligned}$$

because $J_{\alpha,\beta}(M_{\alpha,\beta}(v_k) + v_k) \geq 0$. Let $g(t) = J_{\alpha,\beta}(M_{\alpha,\beta}(v_k) + v_k + t\widetilde{w}_k)$. It follows from the properties of $J_{\alpha,\beta}$ that $g'(0) = 0$ and $g'(t)$ is decreasing. By the Mean value Theorem $g(1) - g(0) = g'(c)$, for some $c \in (0,1)$. Hence $g(1) - g(0) \geq g'(1)$. It follows that

$$J_{\alpha,\beta}(\widetilde{w}_k + M_{\alpha,\beta}(v_k) + v_k) - J_{\alpha,\beta}(M_{\alpha,\beta}(v_k) + v_k) \geq \langle J'_{\alpha,\beta}(u_k), \widetilde{w}_k \rangle$$

and thus

$$E(u_k) \geq \langle J'_{\alpha,\beta}(u_k), \widetilde{w}_k \rangle - \int_\Omega (F(u_k) + hu_k).$$

But the first term on the right-hand side is bounded and the second goes to $-\infty$ by (1.5). This contradicts the assumption that $E(u_k)$ is bounded. Hence $\{u_k\}$ is bounded in $L^2(\Omega)$, the remaining proof follows exactly as in the proof of Lemma 4.5. □

Proof of Theorem 1.4. One can conclude the proof from Lemmas 4.6, 4.7 and saddle point theorem. □

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References

[1] C. Bucur and E. Valdinoci, *Nonlocal Diffusion and Applications*, Lecture Notes of the Unione Matematica Italiana **20**, Springer, Bologna, 2016.

[2] L. A. Caffarelli, S. Salsa and L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. **171** (2008), no. 2, 425–461.

[3] A. Castro, *Hammerstein integral equations with indefinite kernel*, Internat. J. Math. Math. Sci. **1** (1978), no. 2, 187–201.

[4] A. Castro and C. Chang, *A variational characterization of the Fucik spectrum and applications*, Rev. Colombiana Mat. **44** (2010), no. 1, 23–40.

[5] M. Cuesta, D. de Figueiredo and J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations **159** (1999), no. 1, 212–238.

[6] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[7] P. Drábek and S. B. Robinson, *On the Fredholm alternative for the Fučík spectrum*, Abstr. Appl. Anal. **2010** (2010), Art. ID 125464, 20 pp.

[8] ______, *On the solvability of resonance problems with respect to the Fučik spectrum*, J. Math. Anal. Appl. **418** (2014), no. 2, 884–905.

[9] M. M. Fall and V. Felli, *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*, Comm. Partial Differential Equations **39** (2014), no. 2, 354–397.

[10] D. de Figueiredo and J.-P. Gossez, *On the first curve of the Fučik spectrum of an elliptic operator*, Differential Integral Equations **7** (1994), no. 5-6, 1285–1302.

[11] A. Fiscella, R. Servadei and E. Valdinoci, *Asymptotically linear problems driven by fractional Laplacian operators*, Math. Methods Appli. Sci. **38** (2015), no. 16, 3551–3563.

[12] S. Goyal and K. Sreenadh, *On the Fučik spectrum of non-local elliptic operators*, NoDEA Nonlinear Differential equations Appl. **21** (2014), no. 4, 567–588.

[13] E. M. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. **19** (1969/1970), 609–623.

[14] E. Massa, *On a variational characterization of the Fučík spectrum of the Laplacian and a superlinear Sturm-Liouville equation*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 3, 557–577.

[15] R. Servadei and E. Valdinoci, *Mountain pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), no. 2, 887–898.

[16] ______, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), no. 5, 2105–2137.

[17] ______, *Weak and viscosity solutions of the fractional Laplace equation*, Publ. Mat. **58** (2014), no. 1, 133–154.

[18] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), no. 1, 67–112.

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