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On the Existence for an Integral System Including m Equations

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Abstract. In this paper, we study an integral system

$$\begin{cases} u_i(x) = K_i(x)(|x|^{\alpha-n} * u_{i+1}^{p_{i+1}})(x), & i = 1,2,\dots,m-1, \\ u_m(x) = K_m(x)(|x|^{\alpha-n} * u_1^{p_1})(x). \end{cases}$$

When $\alpha \in (0,n)$, $p_i > 0$ $(i = 1,2,\dots,m)$, the Serrin-type condition is critical for existence of positive solutions for some double bounded functions $K_i(x)$ $(i = 1,2,\dots,m)$. When $\alpha \in (0,n)$, $p_i < 0$ $(i = 1,2,\dots,m)$, the system has no positive solution for any double bounded $K_i(x)$ $(i = 1,2,\dots,m)$. When $\alpha > n$, $p_i < 0$ $(i = 1,2,\dots,m)$, and $\max_i\{-p_i\} > \alpha/(\alpha-n)$, then the system exists positive solutions increasing with the rate $\alpha - n$.

1. Introduction

In this paper, we are concerned with an integral system including m equations. First, we observe a simple fact: the system

$$\begin{cases} u_i(x) = \displaystyle\int_{\mathbb{R}^n} \frac{u_{i+1}^{p_{i+1}}(y)}{|x-y|^{n-\alpha}}\,dy, & i = 1,2,\dots,m-1, \\ u_m(x) = \displaystyle\int_{\mathbb{R}^n} \frac{u_1^{p_1}(y)}{|x-y|^{n-\alpha}}\,dy, & m \geq 1,\ n \geq 1, \end{cases} \tag{1.1}$$

where $u_i(x) > 0$ in $\mathbb{R}^n$, $i = 1,2,\dots,m$, $\alpha \neq n$, satisfies the following result.

Theorem 1.1. *Equation* (1.1) *with $p_i \neq -1$ and the norms $\|u_i\|_{p_i+1}$ $(i = 1,2,\dots,m)$ are invariant under the rescaling transformation*

$$\overline{u}_i(x) = \mu^{\theta_i} u(\mu x), \quad \mu \neq 0,\ \theta_i \neq 0 \tag{1.2}$$

if and only if

$$p_i = \frac{n+\alpha}{n-\alpha} \quad (i = 1,2,\dots,m),\ \textit{when } m \textit{ is odd}; \tag{1.3}$$

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and

$$\begin{cases} p_1 = p_3 = \cdots = p_{m-1}, \quad p_2 = p_4 = \cdots = p_m, \\ \frac{1}{p_1+1} + \frac{1}{p_m+1} = \frac{n-\alpha}{n} \qquad \textit{when } m \textit{ is even.} \end{cases} \tag{1.4}$$

Now, $\theta_i = n/(p_i + 1)$.

When $m \in \{1, 2\}$, (1.1) is the Euler-Lagrange equation of the extremal function of the Hardy-Littlewood-Sobolev (HLS) inequality and the reversed HLS inequality (cf. [8, 15]). Papers [7] and [14] classified the regular solutions of (1.1) with $m = 1$ and the Sobolev exponent (1.3) as the radial form

$$u(x) = c\left(\frac{t}{t^2 + |x - x_0|^2}\right)^{(n-\alpha)/2}, \quad c, t > 0, \ x_0 \in \mathbb{R}^n.$$

When $m = 2$, many papers studied (1.1) with the Sobolev-type critical condition (1.4) including existence result (cf. [3, 21]), radial symmetry result (cf. [6]), regularity result (cf. [9]), integrability result (cf. [10, 11]), and asymptotic behavior (cf. [11, 22]).

When $m = 3$, Liu and Qiao (cf. [17]) gave a sufficient condition of which makes positive solutions of (1.1) be radially symmetric. Such a condition is different from (1.3).

For general m, if the right hand side of (1.1) is homogeneous about u_i, then (1.3) still ensures the method of moving planes works and the positive solutions are radially symmetric (cf. [4]). Afterwards, paper [13] generalized it to the system involving the Wolff potentials. Recently, Lü and Zhou studied an integral system with more general right hand side terms

$$\begin{cases} u_i(x) = \displaystyle\int_{\mathbb{R}^n} \frac{f_i(u(y))}{|x - y|^{n-\alpha}} \, dy, & i = 1, 2, \ldots, m, \\ u(x) = (u_1(x), u_2(x), \ldots, u_m(x)), & x \in \mathbb{R}^n. \end{cases}$$

They proved a radial symmetry result (cf. [19]).

When α is positive even number, (1.1) is also related to the study of PDEs system

$$\begin{cases} (-\Delta)^{\alpha/2} u_i(x) = u_{i+1}^{p_{i+1}}(x), \quad i = 1, 2, \ldots, m - 1, \\ (-\Delta)^{\alpha/2} u_m(x) = u_1^{p_1}(x). \end{cases} \tag{1.5}$$

Under some mild conditions, (1.1) is equivalent to the PDEs system above (cf. [5]). This PDEs system is helpful to understand the higher-order conformal PDEs (cf. [2, 16, 18, 24]).

Next, we consider an integral system with double bounded coefficients

$$\begin{cases} u_i(x) = K_i(x) \displaystyle\int_{\mathbb{R}^n} \frac{u_{i+1}^{p_{i+1}}(y)}{|x - y|^{n-\alpha}} \, dy, & i = 1, 2, \ldots, m - 1, \\ u_m(x) = K_m(x) \displaystyle\int_{\mathbb{R}^n} \frac{u_1^{p_1}(y)}{|x - y|^{n-\alpha}} \, dy, & m \geq 1, \ n \geq 1, \end{cases} \tag{1.6}$$

where $u_i(x) > 0$, $p_i \neq -1$, $\alpha \neq n$, and $K_i(x)$ are double bounded $(i = 1, 2, \dots, m)$, i.e., we can find $c_1 \geq c_2 > 0$ such that

$$c_2 \leq K_i(x) \leq c_1, \quad i = 1, 2, \dots, m, \ x \in \mathbb{R}^n.$$

Clearly, (1.1) is a special case of (1.6) (i.e., $K_i(x) \equiv 1$ $(i = 1, 2, \dots, m)$).

This paper is concerned with the positive entire solutions of (1.6) in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Thus, there always holds

$$\alpha > 0.$$

In fact, if $\alpha \leq 0$, for any fixed $x_0 \in \mathbb{R}^n$, the following contradiction appears

$$u_m(x_0) \geq c \int_{B_{|x_0|/2}(x_0)} \frac{u_1^{p_1}(y)\,dy}{|x_0 - y|^{n-\alpha}} \geq c \int_{B_{|x_0|/2}(x_0)} \frac{dy}{|x_0 - y|^{n-\alpha}} = \infty.$$

First, we point out that the Serrin-type condition is critical for existence when the exponents p_i $(i = 1, 2, \dots, m)$ are positive.

Theorem 1.2. *Let $\alpha \in (0, n)$ and $p_i > 0$ $(i = 1, 2, \dots, m)$. Then* (1.6) *has positive solutions for some double bounded functions $K_i(x)$ $(i = 1, 2, \dots, m)$ if and only if*

$$\begin{cases} \prod_{i=1}^m p_i > 1 \quad \text{and} \\ \frac{\alpha}{\prod_{i=1}^m p_i - 1} \max\big\{(1 + p_1 + p_1p_2 + \cdots + p_1 \cdots p_{m-1}), \\ \qquad (1 + p_2 + p_2p_3 + \cdots + p_2 \cdots p_{m-1}p_m), \dots, \\ \qquad (1 + p_m + p_mp_1 + \cdots + p_mp_1 \cdots p_{m-2})\big\} < n - \alpha. \end{cases} \tag{1.7}$$

We call (1.7) the Serrin-type condition.

When $m = 1$, (1.7) is reduced to $p > n/(n-\alpha)$, where $n/(n-\alpha)$ is the Serrin exponent, which is critical for existence of positive solution (cf. [12]). When $m = 2$, (1.7) becomes $p_1p_2 > 1$ and $\max\big\{\frac{\alpha(1+p_1)}{p_1p_2-1}, \frac{\alpha(1+p_2)}{p_1p_2-1}\big\} < n - \alpha$. It is also the critical condition for existence of super-solutions of (1.1) with $m = 2$ (cf. [1]). In particular, when $m = 1$, $\alpha = 2$, by the properties of the Newton potential, the $C^2(\mathbb{R}^n)$-solution of (1.6) satisfies

$$-\Delta u(x) = K(x)u^p(x), \quad x \in \mathbb{R}^n.$$

It is related to the argument of the Kazdan-Warner condition appearing in the Nirenberg problem. In addition, it is associated with the argument of 'quasi-solutions' which were studied by Taliaferro (cf. [23]). When $m = \alpha = 2$, Serrin-type condition is also the critical condition for existence of super-solutions of the Lane-Emden system (cf. [20])

$$\begin{cases} -\Delta u_1(x) = u_2^{p_2}(x), \\ -\Delta u_2(x) = u_1^{p_1}(x). \end{cases}$$

Next, we consider the negative exponents case (i.e., $p_i < 0$ $(i = 1, 2, \ldots, m)$). When $m \in \{1, 2\}$, (1.1) and (1.6) are associated with the higher order equations in the lower dimensions space which appear in the study of conformal geometry (cf. [14, 25]), and the reversed HLS inequality (cf. [8, 21]).

The following two theorems are the generalization of the results in [26] and [11].

Theorem 1.3. *Let $\alpha \in (0, n)$ and $p_i < 0$ $(i = 1, 2, \ldots, m)$. Then* (1.6) *has no positive solution as long as either $K_i(x)$ and $K_{i+1}(x)$ for some $i \in \{1, 2, \ldots, m-1\}$, or $K_m(x)$ and $K_1(x)$ are larger than some positive constant.*

Theorem 1.4. *Let $\alpha > n$ and $p_i < 0$ $(i = 1, 2, \ldots, m)$. If $\min_i\{-p_i\} > \alpha/(\alpha - n)$, then* (1.6) *has a radial solution $u_i(x) = (1 + |x|^2)^{(\alpha-n)/2}$ $(i = 1, 2, \ldots, m)$, for some double bounded $K_i(x)$ $(i = 1, 2, \ldots, m)$.*

A natural problem is whether all positive solutions are increasing with rate $\alpha - n$.

Theorem 1.5. *Let u_i $(i = 1, 2, \ldots, m)$ be the positive solutions of* (1.6) *in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Assume $\alpha > n$ and $p_i < 0$ $(i = 1, 2, \ldots, m)$. If $\max_i\{-p_i\} > \alpha/(\alpha - n)$, then $\min_i\{-p_i\} > \alpha/(\alpha - n)$. In addition, $\exists\, c_1 \geq c_2 > 0$ such that for all i,*

$$c_2 \leq u_i(x)|x|^{n-\alpha} \leq c_1 \quad \text{when } |x| \to \infty. \tag{1.8}$$

Theorem 1.5 implies the integrability and the asymptotic behavior.

Corollary 1.6. *Under the same assumptions of Theorem* 1.5, *then $u_i^{-1}(x) \in L^s(\mathbb{R}^n)$ for all $s > n/(\alpha - n)$, where u_i $(i = 1, 2, \ldots, m)$ are positive solutions of* (1.6).

Corollary 1.7. *Let u_i $(i = 1, 2, \ldots, m)$ be positive solutions of* (1.1). *If the assumptions of Theorem* 1.5 *are true, then*

$$\lim_{|x|\to\infty} u_m(x)|x|^{n-\alpha} = \|u_1^{-1}\|_{-p_1}^{-p_1}, \quad \lim_{|x|\to\infty} u_i(x)|x|^{n-\alpha} = \|u_{i+1}^{-1}\|_{-p_{i+1}}^{-p_{i+1}}, \quad i = 1, 2, \ldots, m-1.$$

Remark 1.8. According to [5], (1.1) is equivalent to (1.5) when $p_i > 0$ $(i = 1, 2, \ldots, m)$. So Theorems 1.1 and 1.2 can be applied to 'some solutions' of the system of PDEs (1.5). Here, 'some solution' is the classical solution when $\alpha \in (0, n)$ is even. For other real values of α, 'some solution' $(u_1, u_2, \ldots, u_m)$ is the $H^{\alpha/2}(\mathbb{R}^n)$-weak solutions. Namely, for any nonnegative function $\phi \in C_0^\infty(\mathbb{R}^n)$, u_i satisfies

$$\int_{\mathbb{R}^n} |\xi|^\alpha \widehat{u}_i(\xi)\overline{\widehat{\phi}(\xi)}\, d\xi = \int_{\mathbb{R}^n} u_{i+1}^{p_{i+1}}(x)\phi(x)\, dx, \quad i = 1, 2, \ldots, m-1,$$
$$\int_{\mathbb{R}^n} |\xi|^\alpha \widehat{u}_m(\xi)\overline{\widehat{\phi}(\xi)}\, d\xi = \int_{\mathbb{R}^n} u_1^{p_1}(x)\phi(x)\, dx.$$

2. Proof of Theorem 1.1

By (1.1) and (1.2), we have

$$\overline{u}_i(x) = \mu^{\theta_i} \int_{\mathbb{R}^n} \frac{u_{i+1}^{p_{i+1}}(y)}{|\mu x - y|^{n-\alpha}}\, dy = \mu^{\theta_i+\alpha-\theta_{i+1}p_{i+1}} \int_{\mathbb{R}^n} \frac{\overline{u}_{i+1}^{p_{i+1}}(z)}{|x-z|^{n-\alpha}}\, dz.$$

Similarly,

$$\overline{u}_m(x) = \mu^{\theta_m+\alpha-\theta_1 p_1} \int_{\mathbb{R}^n} \frac{\overline{u}_1^{p_1}(z)}{|x-z|^{n-\alpha}}\, dz.$$

Thus, (1.1) is invariant under the rescaling transformation (1.2) if and only if

$$\begin{cases} \theta_i + \alpha = \theta_{i+1}p_{i+1}, \quad i = 1,2,\dots,m-1, \\ \theta_m + \alpha = \theta_1 p_1. \end{cases} \tag{2.1}$$

In addition,

$$\int_{\mathbb{R}^n} \overline{u}_i^{p_i+1}(x) = \mu^{\theta_i(1+p_i)-n} \int_{\mathbb{R}^n} u_i^{p_i+1}(z)\, dz, \quad i = 1,2,\dots,m.$$

Thus, the norm $\|u_i\|_{p_i+1}$ $(i = 1,2,\dots,m)$ are invariant under the transformation (1.2) if and only if

$$\theta_i(p_i+1) = n, \quad (i = 1,2,\dots,m). \tag{2.2}$$

If (2.1) and (2.2) hold, inserting $\theta_i p_i = n - \theta_i$ (which is implied by (2.2)) into (2.1) yields

$$\theta_i + \theta_{i+1} = \theta_m + \theta_1 = n - \alpha \quad (i = 1,2,\dots,m).$$

Combining this with $\theta_i = n/(p_i+1)$ (which is implied by (2.2)), we can see (1.3) and (1.4). On the contrary, if (1.3) and (1.4) are true, we also see that (2.1) and (2.2) hold. Theorem 1.1 is proved.

3. Case of $p_i > 0$

In this section, we prove Theorem 1.2.

3.1. Necessity

Theorem 3.1. *If* (1.7) *is not true, then* (1.6) *has no positive solution as long as* $K_i(x)$ *(*$i = 1,2,\dots,m$*) are larger than some positive constant.*

Remark 3.2. Theorem 3.1 shows that (1.1) has no positive super-solution as long as the Serrin-type condition (1.7) does not hold.

Proof of Theorem 3.1. Write

$$\begin{aligned}
H_1 &= 1 + p_2 + p_2p_3 + \cdots + p_2p_3\cdots p_m,\\
H_2 &= 1 + p_3 + p_3p_4 + \cdots + p_3p_4\cdots p_mp_1,\\
&\ \vdots\\
H_{m-1} &= 1 + p_m + p_mp_1 + \cdots + p_mp_1\cdots p_{m-2},\\
H_m &= 1 + p_1 + p_1p_2 + \cdots + p_1p_2\cdots p_{m-1}.
\end{aligned}$$

Without loss of generality, we assume $H_1 = \max_i H_i$.

Suppose (1.6) has positive solutions $u_i(x)$ $(i = 1, 2, \ldots, m)$, we will deduce a contradiction.

In fact, for $|x| \gg 1$,

$$u_1(x) \geq c\int_{B_{1/2}(0)} \frac{u_2^{p_2}(y)}{|x-y|^{n-\alpha}}\,dy \geq \frac{c}{|x|^{n-\alpha}} := \frac{c}{|x|^{h_{1,0}}}.$$

Thus,

$$\begin{aligned}
u_m(x) &\geq c\int_{B_{|x|/2}(x)} \frac{u_1^{p_1}(y)}{|x-y|^{n-\alpha}}\,dy \geq \frac{c}{|x|^{p_1h_{1,0}}}\int_{B_{|x|/2}(x)} \frac{dy}{|x-y|^{n-\alpha}}\\
&= \frac{c}{|x|^{p_1h_{1,0}-\alpha}} := \frac{c}{|x|^{h_{m,0}}},\\
u_{m-1}(x) &\geq c\int_{B_{|x|/2}(x)} \frac{u_m^{p_m}(y)}{|x-y|^{n-\alpha}}\,dy \geq \frac{c}{|x|^{p_mh_{m,0}-\alpha}} := \frac{c}{|x|^{h_{m-1,0}}},\\
&\ \vdots\\
u_2(x) &\geq c\int_{B_{|x|/2}(x)} \frac{u_3^{p_3}(y)}{|x-y|^{n-\alpha}}\,dy \geq \frac{c}{|x|^{p_3h_{3,0}-\alpha}} := \frac{c}{|x|^{h_{2,0}}},\\
u_1(x) &\geq c\int_{B_{|x|/2}(x)} \frac{u_2^{p_2}(y)}{|x-y|^{n-\alpha}}\,dy \geq \frac{c}{|x|^{p_2h_{2,0}-\alpha}} := \frac{c}{|x|^{h_{1,1}}}.
\end{aligned}$$

By induction, we can obtain

$$u_i(x) \geq \frac{c}{|x|^{h_{i,j}}} \quad \text{for } |x| \gg 1,\ i = 1, 2, \ldots, m,\ j = 0, 1, 2, \ldots, \tag{3.1}$$

where

$$\begin{aligned}
h_{1,0} &= n - \alpha, \quad h_{1,j} = p_2h_{2,j-1} - \alpha, \quad j = 1, 2, \ldots,\\
h_{i,j} &= p_{i+1}h_{i+1,j} - \alpha, \quad i = 2, 3, \ldots, m-1, \quad j = 0, 1, 2, \ldots,\\
h_{m,j} &= p_1h_{1,j} - \alpha, \quad j = 0, 1, 2, \ldots.
\end{aligned}$$

Therefore, for $j = 1, 2, \dots,$

$$h_{1,j} = \left(\prod_{i=1}^m p_i\right) h_{1,j-1} - \alpha H_1 = \cdots = \left(\prod_{i=1}^m p_i\right)^j h_{1,0} - \left[\sum_{k=0}^{j-1} \left(\prod_{i=1}^m p_i\right)^k\right] \alpha H_1.$$

When $\prod_{i=1}^m p_i = 1$, there holds $h_{1,j_0} = h_{1,0} - j_0 \alpha H_1 < 0$ for some large j_0. When $\prod_{i=1}^m p_i \in (0,1)$, letting $j \to \infty$, we get

$$h_{1,j} = \left(\prod_{i=1}^m p_i\right)^j h_{1,0} - \frac{1 - \left(\prod_{i=1}^m p_i\right)^j}{1 - \left(\prod_{i=1}^m p_i\right)} \alpha H_1 \to -\frac{\alpha H_1}{1 - \left(\prod_{i=1}^m p_i\right)} < 0.$$

We can find some j_0 such that $h_{1,j_0} < 0$.

When $\prod_{i=1}^m p_i > 1$, and $\frac{\alpha H_1}{(\prod_{i=1}^m p_i) - 1} > n - \alpha$, there exists suitably large j_0 such that

$$h_{1,j_0} = \left(\prod_{i=1}^m p_i\right)^{j_0} \left[h_{1,0} - \frac{\alpha H_1}{\left(\prod_{i=1}^m p_i\right) - 1}\right] + \frac{\alpha H_1}{\left(\prod_{i=1}^m p_i\right) - 1} < 0.$$

The existence of j_0 in (3.1) shows that at $u_1(x) = \infty$. It is impossible.

When

$$\prod_{i=1}^m p_i > 1 \quad \text{and} \quad \frac{\alpha H_1}{\left(\prod_{i=1}^m p_i\right) - 1} = n - \alpha, \tag{3.2}$$

we will also deduce a contradiction.

In fact, for $i = 1, 2, \dots, m-1$,

$$\begin{aligned} u_i(x) &\geq \int_{B_R(0)} \frac{u_{i+1}^{p_{i+1}}(y)}{|x-y|^{n-\alpha}} \, dy \geq \frac{c}{(|x|+R)^{n-\alpha}} \int_{B_R(0)} u_{i+1}^{p_{i+1}}(y) \, dy, \\ u_m(x) &\geq \frac{c}{(|x|+R)^{n-\alpha}} \int_{B_R(0)} u_1^{p_1}(y) \, dy. \end{aligned} \tag{3.3}$$

Therefore,

$$\begin{aligned} \int_{B_R(0)} u_i^{p_i}(x) \, dx &\geq \frac{c}{R^{p_i(n-\alpha)-n}} \left(\int_{B_R(0)} u_{i+1}^{p_{i+1}}(y) \, dy\right)^{p_i}, \quad i = 1, 2, \dots, m-1, \\ \int_{B_R(0)} u_m^{p_m}(x) \, dx &\geq \frac{c}{R^{p_m(n-\alpha)-n}} \left(\int_{B_R(0)} u_1^{p_1}(y) \, dy\right)^{p_m}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{B_R(0)} u_2^{p_2}(x) \, dx &\geq \frac{c}{R^{p_2(n-\alpha)-n}} \left(\int_{B_R(0)} u_3^{p_3}(y) \, dy\right)^{p_2} \\ &\geq \frac{c}{R^{p_2(n-\alpha)-n}} \left(\frac{c}{R^{p_3(n-\alpha)-n}}\right)^{p_2} \left(\int_{B_R(0)} u_4^{p_4}(y) \, dy\right)^{p_2 p_3} \\ &\geq \cdots \geq \frac{c}{R^H} \left(\int_{B_R(0)} u_2^{p_2}(y) \, dy\right)^{\prod_{i=1}^m p_i}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} H &:= (p_2 + p_2p_3 + \cdots + p_2p_3\cdots p_mp_1)(n-\alpha) - n(1 + p_2 + p_2p_3 + \cdots + p_2p_3\cdots p_m) \\ &= \left(\prod_{i=1}^m p_i - 1\right)n - (p_2 + p_2p_3 + \cdots + p_2p_3\cdots p_mp_1)\alpha \\ &= \left(\prod_{i=1}^m p_i - 1\right)n - \left(H_1 + \prod_{i=1}^m p_i - 1\right)\alpha. \end{aligned}$$

By (3.2), $H = 0$. Letting $R \to \infty$ in (3.4), we have $u_2 \in L^{p_2}(\mathbb{R}^n)$. By an analogous argument of (3.4), from (3.3) we also get

$$\int_{B_R(0)\setminus B_{R/2}(0)} u_2^{p_2}(x)\,dx \geq c\left(\int_{B_R(0)} u_2^{p_2}(y)\,dy\right)^{\prod_{i=1}^m p_i}.$$

Letting $R \to \infty$ and noting $u_2 \in L^{p_2}(\mathbb{R}^n)$, we have

$$0 = \int_{\mathbb{R}^n} u_2^{p_2}(y)\,dy.$$

This leads to $u_2(x) \equiv 0$. It is impossible. Theorem 3.1 is complete. □

3.2. Sufficiency

For $i = 1, 2, \ldots, m$, set

$$U_i(x) = \frac{1}{(1+|x|^2)^{\theta_i}}, \tag{3.5}$$

where

$$\theta_i = \frac{\alpha H_i}{2\left(\prod_{i=1}^m p_i - 1\right)}. \tag{3.6}$$

Noting $p_iH_i > \prod_{i=1}^m p_i - 1$ and

$$p_1H_1 - \left(\prod_{i=1}^m p_i - 1\right) = H_m, \quad p_iH_i - \left(\prod_{i=1}^m p_i - 1\right) = H_{i-1}, \quad i = 2, 3, \ldots, m, \tag{3.7}$$

from (1.7) we deduce that

$$\alpha < 2\theta_ip_i < n, \quad i = 1, 2, \ldots, m. \tag{3.8}$$

When $|x|$ is bounded, it is not difficult to see that L_0 is also bounded, where

$$L_0 := \int_{\mathbb{R}^n} \frac{U_1^{p_1}(y)\,dy}{|x-y|^{n-\alpha}}.$$

Therefore, there exists a double bounded function $K_1(x)$ such that $U_m(x) = K_1(x)L_0$. Thus, we only consider the case of $|x| \gg 1$.

Clearly, for $R > 0$,

$$L_0 = \sum_{l=1}^{4} L_l,$$

where

$$L_1 := \int_{B_R(0)} \frac{U_1^{p_1}(y)\,dy}{|x-y|^{n-\alpha}}, \qquad L_2 := \int_{B_{|x|/2}(x)} \frac{U_1^{p_1}(y)\,dy}{|x-y|^{n-\alpha}},$$
$$L_3 := \int_{B_{2|x|}(0)\setminus(B_R(0)\cup B_{|x|/2}(x))} \frac{U_1^{p_1}(y)\,dy}{|x-y|^{n-\alpha}}, \qquad L_4 := \int_{\mathbb{R}^n\setminus B_{2|x|}(0)} \frac{U_1^{p_1}(y)\,dy}{|x-y|^{n-\alpha}}.$$

When $|x| \gg 1$, there exists a double bounded function $K(x)$ such that

$$L_1 = K(x)(1+|x|^2)^{(\alpha-n)/2}.$$

When $|y-x| \le |x|/2$, $|x|/2 \le |y| \le 3|x|/2$. By (3.5), there exists a double bounded function $K(x)$ such that

$$L_2 = \frac{K(x)}{(1+|x|^2)^{\theta_1 p_1}} \int_{B_{|x|/2}(x)} \frac{dy}{|x-y|^{n-\alpha}} = \frac{K(x)}{(1+|x|^2)^{\theta_1 p_1-\alpha/2}}.$$

In view of $B_{2|x|}(0) \setminus B_{3|x|/2}(0) \subset B_{2|x|}(0) \setminus (B_R(0) \cup B_{|x|/2}(x)) \subset B_{2|x|}(0) \setminus B_R(0)$, by (3.5), there exist constants $C > c > 0$ such that

$$\frac{c}{|x|^{n-\alpha}} \int_{B_{2|x|}(0)\setminus B_{3|x|/2}(0)} \frac{dy}{|y|^{2\theta_1 p_1}} \le L_3 \le \frac{C}{|x|^{n-\alpha}} \int_{B_{2|x|}(0)\setminus B_R(0)} \frac{dy}{|y|^{2\theta_1 p_1}}.$$

Therefore, by (3.8), we can find a double bounded function $K(x)$ such that

$$L_3 = \frac{K(x)}{(1+|x|^2)^{\theta_1 p_1-\alpha/2}}.$$

When $|y| \ge 2|x|$, $|x-y| \ge |y|/2$. By (3.5) and (3.8), there exists a double bounded function $K(x)$ such that

$$L_4 = K(x) \int_{\mathbb{R}^n\setminus B_{2|x|}(0)} \frac{dy}{|y|^{n-\alpha+2\theta_1 p_1}} = \frac{K(x)}{(1+|x|^2)^{\theta_1 p_1-\alpha/2}}.$$

Combining the estimates of L_l $(l = 1, 2, 3, 4)$ and noting (3.8), we can find a double bounded function $K(x)$ such that

$$L_0 = \frac{K(x)}{(1+|x|^2)^{\theta_1 p_1-\alpha/2}}.$$

By (3.6) and (3.7), the result above implies

$$U_m(x) = K_1(x)L_0$$

for some double bounded function $K_1(x)$ when $|x| \gg 1$.

Similarly, we also obtain

$$U_i(x) = K_i(x)\int_{\mathbb{R}^n} \frac{U_{i+1}^{p_{i+1}}(y)\,dy}{|x-y|^{n-\alpha}}, \quad i = 1, 2, \dots, m-1$$

for some double bounded functions $K_i(x)$. So we find a class of solutions of forms as (3.5) with (3.6).

In addition, if a stronger condition (than (1.7))

$$\min\{p_i, i = 1, 2, \dots, m\} > \frac{n}{n-\alpha}$$

holds true, then there exists another class of solutions. In fact, we can choose $\theta_i = (n-\alpha)/2$ instead of (3.6), which implies

$$\min_i\{2\theta_i p_i\} > n. \tag{3.9}$$

Now, set $L_5 = L_0 - L_1 - L_2$. Thus,

$$\int_{\mathbb{R}^n \setminus B_{2|x|}(0)} \frac{U_1^{p_1}(y)\,dy}{|x-y|^{n-\alpha}} \le L_5 \le \int_{\mathbb{R}^n \setminus B_R(0)} \frac{U_1^{p_1}(y)\,dy}{|x-y|^{n-\alpha}}.$$

By (3.9), it follows that

$$\frac{c}{(1+|x|^2)^{\theta_1 p_1 - \alpha/2}} \le L_5 \le \frac{C}{(1+|x|^2)^{(n-\alpha)/2}}.$$

Combining with the estimates of L_1 and L_2 and noting (3.9), we can find a double bounded function $K(x)$ such that

$$L_0 = \frac{K(x)}{(1+|x|^2)^{(n-\alpha)/2}}.$$

In view of $\theta_i = (n-\alpha)/2$,

$$U_m(x) = K_1(x)L_0$$

for some double bounded function $K_1(x)$ when $|x| \gg 1$.

Similarly, we also obtain

$$U_i(x) = K_i(x)\int_{\mathbb{R}^n} \frac{U_{i+1}^{p_{i+1}}(y)\,dy}{|x-y|^{n-\alpha}}, \quad i = 1, 2, \dots, m-1$$

for some double bounded functions $K_i(x)$.

4. Case of $p_i < 0$

Proof of Theorem 1.3. The idea comes from [26]. Without loss of generality, we assume $K_m(x) \geq c$ and $K_1(x) \geq c$ for some constant $c > 0$.

By Lemma 3.11.3 in [27], for all $r > 0$,

$$\begin{aligned} \int_{B_r(0)} u_1(x)\,dx &= \int_{\mathbb{R}^n} \int_{B_r(0)} \frac{dx}{|x-y|^{n-\alpha}} u_2^{p_2}(y)\,dy \\ &\leq c \int_{\mathbb{R}^n} \frac{|B_r|}{|y|^{n-\alpha}} u_2^{p_2}(y)\,dy = c|B_r| u_1(0), \end{aligned} \tag{4.1}$$

where $c > 0$ is independent of r. Applying the Hölder inequality, we get

$$\begin{aligned} |B_r(0)| &= \int_{B_r(0)} u_1^{p_1/(p_1-1)}(x) u_1^{p_1/(1-p_1)}(x)\,dx \\ &\leq \left(\int_{B_r(0)} u_1(x)\,dx \right)^{p_1/(p_1-1)} \left(\int_{B_r(0)} u_1^{p_1}(x)\,dx \right)^{1/(1-p_1)}. \end{aligned}$$

Inserting (4.1) into this result, we obtain

$$|B_r(0)| \leq c \int_{B_r(0)} u_1^{p_1}(x)\,dx.$$

Multiplying by $r^{\alpha-n}$ yields

$$r^\alpha \leq c r^{\alpha-n} \int_{B_r(0)} u_1^{p_1}(x)\,dx \leq c \int_{B_r(0)} \frac{u_1^{p_1}(x)}{|x|^{n-\alpha}}\,dx \leq c u_m(0),$$

where $c > 0$ is independent of r. Letting $r \to \infty$, we can see $u_m(0) = \infty$. It is impossible. □

Proof of Theorem 1.4. For $\theta_i \neq 0$, set $U_i(x) = (1+|x|^2)^{\theta_i/2}$, $(i = 1, 2, \ldots, m)$. When $|x| \leq 2R$ for some large $R > 0$, $U_i(x)$ is proportional to $\int_{\mathbb{R}^n} \frac{U_{i+1}^{p_{i+1}}(y)}{|x-y|^{n-\alpha}}\,dy$ for $i = 1, 2, \ldots, m-1$, and $U_m(x)$ is proportional to $\int_{\mathbb{R}^n} \frac{U_1^{p_1}(y)}{|x-y|^{n-\alpha}}\,dy$. Thus we only consider the case of $|x| > 2R$.

Clearly,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{U_i^{p_i}(y)}{|x-y|^{n-\alpha}}\,dy &= \int_{B_1(0)} \frac{U_i^{p_i}(y)}{|x-y|^{n-\alpha}}\,dy + \int_{B_{2|x|}(0)\setminus B_1(0)} \frac{U_i^{p_i}(y)}{|x-y|^{n-\alpha}}\,dy \\ &\quad + \int_{\mathbb{R}^n \setminus B_{2|x|}(0)} \frac{U_i^{p_i}(y)}{|x-y|^{n-\alpha}}\,dy \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{4.2}$$

First, there exists $C > 0$ such that

$$C^{-1}|x|^{\alpha-n} \leq I_1 \leq C|x|^{\alpha-n}. \tag{4.3}$$

When $y \in \mathbb{R}^n \setminus B_{2|x|}(0)$, $\frac{1}{2}|y| \leq |x-y| \leq \frac{3}{2}|y|$. Therefore,

$$C^{-1} \int_{2|x|}^{\infty} r^{\alpha+p_i\theta_i} \frac{dr}{r} \leq I_3 \leq C \int_{2|x|}^{\infty} r^{\alpha+p_i\theta_i} \frac{dr}{r}.$$

In order to ensure $I_3 < \infty$, θ_i should satisfy

$$\alpha + p_i\theta_i < 0. \tag{4.4}$$

Thus,

$$C^{-1}|x|^{\alpha+p_i\theta_i} \leq I_3 \leq C|x|^{\alpha+p_i\theta_i}, \quad |x| > 2R. \tag{4.5}$$

When $y \in B_{2|x|}(0)$, $|x-y| \leq 3|x|$. When $3|x|/2 \leq |y| \leq 2|x|$, $|x-y| \geq |x|/2$. Therefore,

$$C^{-1}|x|^{\alpha-n} \int_{3|x|/2}^{2|x|} r^{n+p_i\theta_i} \frac{dr}{r} \leq I_2 \leq C|x|^{\alpha-n} \int_{1}^{2|x|} r^{n+p_i\theta_i} \frac{dr}{r}.$$

Clearly, (4.4) implies $n + p_i\theta_i < 0$. Therefore,

$$C^{-1}|x|^{\alpha+p_i\theta_i} \leq I_2 \leq C|x|^{\alpha-n}.$$

Combining this with (4.3), (4.5) and (4.2), and noting (4.4), we get

$$C^{-1}|x|^{\alpha-n} \leq \int_{\mathbb{R}^n} \frac{U_i^{p_i}(y)}{|x-y|^{n-\alpha}}\,dy \leq C|x|^{\alpha-n}, \quad |x| > 2R$$

for $i = 1, 2, \ldots, m$. Take

$$\theta_i = \alpha - n.$$

There holds

$$\begin{aligned} &C^{-1} \int_{\mathbb{R}^n} \frac{U_i^{p_i}(y)}{|x-y|^{n-\alpha}}\,dy \leq U_{i-1}(x) \leq C \int_{\mathbb{R}^n} \frac{U_i^{p_i}(y)}{|x-y|^{n-\alpha}}\,dy, \quad (i = 2, 3, \ldots, m) \\ &C^{-1} \int_{\mathbb{R}^n} \frac{U_1^{p_1}(y)}{|x-y|^{n-\alpha}}\,dy \leq U_m(x) \leq C \int_{\mathbb{R}^n} \frac{U_1^{p_1}(y)}{|x-y|^{n-\alpha}}\,dy. \end{aligned} \tag{4.6}$$

Setting

$$\begin{aligned} K_i(x) &= U_i(x) \left[\int_{\mathbb{R}^n} \frac{U_{i+1}^{p_{i+1}}(y)}{|x-y|^{n-\alpha}}\,dy \right]^{-1}, \quad (i = 1, 2, \ldots, m-1) \\ K_m(x) &= U_m(x) \left[\int_{\mathbb{R}^n} \frac{U_1^{p_1}(y)}{|x-y|^{n-\alpha}}\,dy \right]^{-1}, \end{aligned}$$

we can see that $U_i(x) = (1+|x|^2)^{(\alpha-n)/2}$ $(i = 1, 2, \ldots, m)$ solve (1.6), and (4.6) implies $K_i(x)$ are double bounded. $\square$

Proof of Theorem 1.5. Without loss of generality, we assume $-p_1 = \max_i\{-p_i\}$. Thus the condition of Theorem 1.5 leads to

$$\alpha + p_1(\alpha - n) < 0. \tag{4.7}$$

Clearly, for $|x| \gg 1$,

$$u_i(x) \ge c \int_{B_1(0)} \frac{u_{i+1}^{p_{i+1}}(y)}{|x-y|^{n-\alpha}}\, dy \ge c|x|^{\alpha-n}, \quad i = 1, 2, \dots, m-1, \tag{4.8}$$

$$u_m(x) \ge c \int_{B_1(0)} \frac{u_1^{p_1}(y)}{|x-y|^{n-\alpha}}\, dy \ge c|x|^{\alpha-n}. \tag{4.9}$$

On the other hand,

$$\begin{aligned} u_m(x) = K_m(x) \int_{B_R(0)} \frac{u_1^{p_1}(y)}{|x-y|^{n-\alpha}}\, dy + K_m(x) \int_{B_{2|x|}(0)\setminus \int_{B_R(0)}} \frac{u_1^{p_1}(y)}{|x-y|^{n-\alpha}}\, dy \\ + K_m(x) \int_{\mathbb{R}^n \setminus B_{2|x|}(0)} \frac{u_1^{p_1}(y)}{|x-y|^{n-\alpha}}\, dy := J_1 + J_2 + J_3. \end{aligned}$$

For large $R > 0$,

$$J_1 \le C \int_{B_R(0)} \frac{u_1^{p_1}(y)}{|x-y|^{n-\alpha}}\, dy \le C|x|^{\alpha-n}, \quad |x| \gg 1.$$

When $y \in B_{2|x|}(0)$, $|x - y| \le 3|x|$. Therefore, by (4.8) and (4.7),

$$J_2 \le C|x|^{\alpha-n} \int_1^{2|x|} r^{n-p_1(n-\alpha)} \frac{dr}{r} \le C|x|^{\alpha-n}, \quad |x| \gg 1.$$

When $|y| > 2|x|$, $|x - y| \le \frac{3}{2}|y|$. Therefore, by (4.8) and (4.7),

$$J_3 \le C \int_{2|x|}^{\infty} r^{\alpha+p_1(\alpha-n)} \frac{dr}{r} \le C|x|^{\alpha+p_1(\alpha-n)} \le C, \quad |x| \gg 1.$$

Combining the estimates of J_1, J_2, J_3 with (4.9), we get

$$0 < C^{-1} \le u_m(x)|x|^{n-\alpha} \le C, \quad |x| \gg 1. \tag{4.10}$$

When $|y| > 2|x|$, $|x - y| \ge \frac{1}{2}|y|$. By (4.8) and (4.10),

$$\infty > u_{m-1}(x) \ge c \int_{\mathbb{R}^n \setminus B_{2|x|}(0)} \frac{u_m^{p_m}(y)}{|x-y|^{n-\alpha}}\, dy \ge c \int_{2|x|}^{\infty} r^{\alpha+p_m(\alpha-n)} \frac{dr}{r}.$$

This implies

$$-p_m > \frac{\alpha}{\alpha - n}. \tag{4.11}$$

Replacing (4.7) by (4.11) and by the same estimates of J_1, J_2, J_3, we also obtain from (4.10) that

$$0 < C^{-1} \le u_{m-1}(x)|x|^{n-\alpha} \le C, \quad |x| \gg 1.$$

By induction, we can see that for $i = 1, 2, \dots, m$,

$$-p_i > \frac{\alpha}{\alpha - n} \quad \text{and} \quad 0 < C^{-1} \le u_i(x)|x|^{n-\alpha} \le C.$$

Theorem 1.5 is proved. □

Proof of Corollary 1.6. Theorem 1.5 shows that there exists $R > 0$ such that $|u_i(x)| \le c|x|^{\alpha-n}$, $|x| > R$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} u_i^{-s}(x)\,dx &= \int_{B_R(0)} u_i^{-s}(x)\,dx + \int_{\mathbb{R}^n\setminus B_R(0)} u_i^{-s}(x)\,dx \\ &\le c + c\int_R^\infty r^{n-s(\alpha-n)}\,\frac{dr}{r}.\end{aligned}$$

Therefore, $u_i^{-1} \in L^s(\mathbb{R}^n)$ as long as $s > n/(\alpha - n)$. □

Proof of Corollary 1.7. Theorem 1.5 shows $\min_i\{-p_i\} > \alpha/(\alpha - n) > n/(\alpha - n)$. By Corollary 1.6,

$$u_i^{-1} \in L^{-p_i}(\mathbb{R}^n), \quad i = 1, 2, \dots, m. \tag{4.12}$$

When $y \in B_R(0)$ for $R > 0$, $\left|\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}} - 1\right| \le 2$ for large $|x|$. By (4.12), we can use the Lebesgue dominated convergence theorem to obtain

$$\lim_{R\to\infty}\lim_{|x|\to\infty}\int_{B_R(0)}\left|\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}} - 1\right| u_1^{p_1}(y)\,dy = 0.$$

When $|y| \le 2|x|$, $|x - y| \le 3|x|$. Therefore, by (4.12),

$$\lim_{R\to\infty}\lim_{|x|\to\infty}\int_{B_{2|x|}(0)\setminus B_R(0)}\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}u_1^{p_1}(y)\,dy = 0.$$

When $|y| > 2|x|$, $|x - y| \le \frac{3}{2}|y|$. Therefore, by (1.8),

$$\begin{aligned}\lim_{|x|\to\infty}\int_{\mathbb{R}^n\setminus B_{2|x|}(0)}\frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}}u_1^{p_1}(y)\,dy &\le c\lim_{|x|\to\infty}|x|^{n-\alpha}\int_{2|x|}^\infty r^{\alpha+p_1(\alpha-n)}\,\frac{dr}{r} \\ &= c\lim_{|x|\to\infty}|x|^{n+p_1(\alpha-n)} = 0.\end{aligned}$$

Combining these estimates, we have

$$\lim_{|x|\to\infty} u_m(x)|x|^{n-\alpha} = \|u_1^{-1}\|_{-p_1}^{-p_1}.$$

Similarly, we can also get

$$\lim_{|x|\to\infty} u_{i-1}(x)|x|^{n-\alpha} = \|u_i^{-1}\|_{-p_i}^{-p_i}, \quad i = 2, 3, \dots, m. \qquad □$$

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