TAIWANESE JOURNAL OF MATHEMATICS
Vol. 23, No. 6, pp. 1435–1460, December 2019
DOI: 10.11650/tjm/181205

Traveling Waves for a Spatial SIRI Epidemic Model

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Abstract. The aim of this paper is to study the traveling waves in a spatial SIRI epidemic model arising from herpes viral infection. We obtain the complete information about the existence and non-existence of traveling waves in the model. Namely, we prove that when the basic reproduction number $\mathcal{R}_0 > 1$, there exists a critical wave speed $c^* > 0$ such that for each $c > c^*$, the model admits positive traveling waves; and for $c < c^*$, the model has no non-negative and bounded traveling wave. We also give some numerical simulations to illustrate our analytic results.

1. Introduction

In [19], Tudor proposed an $S \to I \to R \to I$ epidemic model for the spread of a herpes-type infection in either human or animal populations as follows

$$
\begin{aligned}
&\frac{dS(t)}{dt} = \mu - \mu S(t) - \beta S(t)I(t), \\
&\frac{dI(t)}{dt} = \beta S(t)I(t) + \delta R(t) - (\mu+\gamma)I(t), \\
&\frac{dR(t)}{dt} = \gamma I(t) - (\mu+\delta)R(t), \\
&S(0) + I(0) + R(0) = 1, \quad S(0) > 0,\ I(0) > 0,\ R(0) \geq 0,
\end{aligned} \tag{1.1}
$$

where S, I and R, respectively, are the fractions of susceptible, infectious and recovered subpopulations. Assumptions made in the system (1.1) are homogeneous mixing, the birth and death rates are assumed to be the same value μ, δ is the coefficient of the rate at which recovered individuals lose their immunity ($\delta = 0$ corresponds to permanent immunity), and γ is the coefficient of the rate at which infectious individuals change to removed individuals. β stands for the transmission coefficient from susceptible individuals to infectious individuals. The parameters μ, β and γ are positive, and δ is nonnegative.

In model (1.1), it is assumed that the susceptibles become infectious, then are removed with temporary immunity, and then become infectious again. And also, it was assumed

Received June 23, 2018; Accepted December 10, 2018.
Communicated by Cheng-Hsiung Hsu.
2010 *Mathematics Subject Classification.* 92D30, 35K57, 35C07.
Key words and phrases. traveling wave solutions, SIRI epidemic model, basic reproduction number, critical wave speed.
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that the rate at which susceptibles become infectives is the usual bilinear incidence βSI. As reported in [4, 13], the model (1.1) is also appropriate for the diseases such as human and bovine tuberculosis, recovered population may revert back to the infective class due to reactivation of the latent infection or incomplete treatment.

By using the characteristic equation technique, Tudor [19] showed that the basic reproduction number is a threshold parameter for the local stability of system (1.1). Furthermore, applying an elementary analysis of Liénard's equation [33, 36] and the classical Lyapunov theorem [36], Moreira and Wang [14] gave the sufficient conditions on the global stability of the disease-free and endemic equilibria in a general model with substituting the nonlinear incidence rate $I\varphi(S)$ by $\beta S(t)I(t)$. A more general SIRI model under the assumption that incidence of infection is given in an abstract, possibly bi-nonlinear form has been proposed and analyzed in Georgescu and Zhang [9], and the sufficient conditions for the global stability of equilibria are obtained by means of Lyapunov's second method.

Clearly, system (1.1) is one of ODE type, which could only reflect the epidemiological and demographic process as the time changes. Since the disease populations usually disperse spatially as well as involving in time, it is reasonable to consider the spatial structures in the model. Therefore, it gives us the motivation to investigate the PDE version of system (1.1). And also, to account for behavioral change and infection mechanism, we consider the saturated incidence rate [3] defined by $g(I) = I/(1 + \alpha I)$ in system (1.1). Here we propose the following spatial disease model

$$\begin{aligned}
&\frac{\partial S(t,x)}{\partial t} = d_1 \Delta S(t,x) + \mu - \mu S(t,x) - \beta S(t,x) g(I(t,x)),\\
&\frac{\partial I(t,x)}{\partial t} = d_2 \Delta I(t,x) + \beta S(t,x) g(I(t,x)) + \delta R(t,x) - (\mu+\gamma) I(t,x),\\
&\frac{\partial R(t,x)}{\partial t} = d_3 \Delta R(t,x) + \gamma I(t,x) - (\mu+\delta) R(t,x),
\end{aligned} \tag{1.2}$$

in which $S(t,x)$, $I(t,x)$ and $R(t,x)$ are the population sizes of susceptible, infected and recovered individuals at location $x \in \mathbb{R}^n$ and time $t \geq 0$, respectively, $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. The positive constants d_i, $i = 1, 2, 3$, are the corresponding diffusion coefficients. The parameters $\mu, \beta, \gamma > 0$ and $\delta \geq 0$ are constants as in system (1.1), and $\alpha > 0$ determines the saturation level when the infectious population is large.

It is noted that disease propagation in space is relevant to the so-called traveling wave solutions which are used to study the spread of infectious diseases. The traveling wave solution describes that the transition process of the disease population runs into the susceptible population from an initial disease-free equilibrium to the endemic equilibrium. Results on this topic may help people to predict how fast a disease invades geographically, and accordingly, take necessary measures in advance to prevent the disease, or at least, decrease possible negative consequences [15]. Recently, many researchers have studied the existence and non-existence of traveling wave solutions of the epidemic models of two equations, see, for example [2, 5, 6, 8, 11, 12, 16, 22–24, 26, 27, 29, 30] and the references cited therein. However, to the best of our knowledge, there are few literature dealing with the existence and non-existence of traveling waves for the epidemic models of three equations, except four types of simple epidemic disease-transmission models [1, 21, 28, 31, 35] and a class of reaction-diffusion systems of three equations [34]. But the results in [34] can not be applied directly to establish the existence of traveling wave solutions for system (1.2) since the system (1.2) does not satisfy the conditions (A5)(II) and (III) in [34]. This is the motivation for the current study.

The purpose of the current paper is to study the existence and non-existence of traveling wave solutions connecting the disease-free equilibrium and endemic equilibrium of system (1.2). We employ the Schauder fixed point theorem and construct the upper-lower solutions to establish the existence theorem (see Theorem 3.4 below). Namely, we will show that when the basic reproduction number $\mathcal{R}_0 > 1$, there exists a constant $c^* > 0$ such that (1.2) has a positive traveling wave solution if $c > c^*$. One important feature of our method, which is different from the ones [2, 8, 21–24], is that we need to construct the vector type of upper-lower solutions [7, 20, 25, 28] for system (2.1) (see Section 2.2) since system (2.1) consists of three equations. Further, we shall construct the appropriate Lyapunov function to show that the traveling wave converges to the endemic equilibrium as $t \to \infty$. Here we would like to comment that construction of the Lyapunov function is nontrivial and difficult because the corresponding wave profile system (2.1) is a second order differential system of three equations. Moreover, by the two-sided Laplace transform, we conclude the non-existence of traveling wave solutions for model (1.2) when $\mathcal{R}_0 > 1$ and $c \in (0, c^*)$.

This paper is organized as follows. In the next section, we give some preliminaries, that is, we study the eigenvalue problems for wave profile equation (2.1) and construct the vector type of upper-lower solutions, and then verify the conditions of the Schauder fixed point theorem. In Section 3, we establish the existence and non-existence of traveling waves in model (1.2). In Section 4, we carry out some numerical simulations to confirm our theoretical results and give a brief summary.

2. Preliminaries

2.1. The eigenvalue problems

In this subsection, we study the eigenvalue problems for the wave profile equation. First, we define the basic reproduction number of system (1.2) as

$$\mathcal{R}_0 := \frac{\beta(\mu + \delta)}{\mu(\mu + \delta + \gamma)}.$$

By a direct computation, we get the following conclusion.

Lemma 2.1. (1) *If $\mathcal{R}_0 < 1$, then system (1.2) has only a disease-free equilibrium $E_0 = (1, 0, 0)$.*

(2) *If $\mathcal{R}_0 > 1$, then system (1.2) admits a positive constant endemic equilibrium $E^* = (S^*, I^*, R^*)$, where*

$$S^* = \frac{1}{\mathcal{R}_0}(1 + \alpha I^*), \quad I^* = \frac{\mu}{\beta + \alpha\mu}(\mathcal{R}_0 - 1), \quad R^* = \frac{\gamma}{\mu + \delta} I^*.$$

Furthermore,

$$0 < S^* < 1, \quad 0 < I^* < \frac{\mathcal{R}_0}{\alpha}, \quad 0 < R^* < \frac{\gamma \mathcal{R}_0}{\alpha(\mu + \delta)}.$$

A traveling wave solution of system (1.2) is a special solution $(S(t,x), I(t,x), R(t,x))$ taking the form

$$(S(t,x), I(t,x), R(t,x)) = (\widetilde{S}(\xi), \widetilde{I}(\xi), \widetilde{R}(\xi)), \quad \xi := \nu \cdot x + ct,$$

where $c > 0$ is the wave speed, $\nu \in \mathbb{R}^n$ is a unit vector denoting the direction of wave propagation, $\nu \cdot x$ is the usual inner product in $\mathbb{R}^n$, and $(S(\xi), I(\xi), R(\xi))$ (for convenience, we use $(S(\xi), I(\xi), R(\xi))$ instead of $(\widetilde{S}(\xi), \widetilde{I}(\xi), \widetilde{R}(\xi))$) satisfies the following wave profile equation

$$\begin{aligned} cS'(\xi) &= d_1 S''(\xi) + \mu - \mu S(\xi) - \beta S(\xi) g(I(\xi)), \\ cI'(\xi) &= d_2 I''(\xi) + \beta S(\xi) g(I(\xi)) + \delta R(\xi) - (\mu + \gamma) I(\xi), \\ cR'(\xi) &= d_3 R''(\xi) + \gamma I(\xi) - (\mu + \delta) R(\xi), \end{aligned} \tag{2.1}$$

and the boundary conditions

$$\lim_{\xi \to -\infty} (S(\xi), I(\xi), R(\xi)) = (1, 0, 0), \quad \lim_{\xi \to +\infty} (S(\xi), I(\xi), R(\xi)) = (S^*, I^*, R^*). \tag{2.2}$$

Linearizing the equations of I and R of (2.1) at $E_0 = (1, 0, 0)$, we get

$$\begin{aligned} d_2 I''(\xi) - cI'(\xi) + (\beta - \mu - \gamma) I(\xi) + \delta R(\xi) &= 0, \\ d_3 R''(\xi) - cR'(\xi) - (\mu + \delta) R(\xi) + \gamma I(\xi) &= 0. \end{aligned} \tag{2.3}$$

Plugging $I(\xi) = \eta_1 e^{\lambda\xi}$ and $R(\xi) = \eta_2 e^{\lambda\xi}$ into (2.3), we get the following eigenvalue problem

$$H(\lambda) := \det A(\lambda) = 0, \quad A(\lambda) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0, \quad \text{where} \quad A(\lambda) = \begin{pmatrix} h_2(\lambda) & \delta \\ \gamma & h_3(\lambda) \end{pmatrix}$$

with $h_2(\lambda) = d_2\lambda^2 - c\lambda + \beta - \mu - \gamma$ and $h_3(\lambda) = d_3\lambda^2 - c\lambda - \mu - \delta$.

Using the ideas in [11, 34], we give the following lemma.

Lemma 2.2. *Assume that $\mathcal{R}_0 > 1$ holds. Then there exists a positive constant c^* such that*

(1) *for any $c > c^*$, then the characteristic equation $H(\lambda) = 0$ has three positive roots $0 < \lambda_1 < \lambda_2 < \lambda_3$ and a negative root $\lambda_4 < 0$ with*

$$H(\lambda_1 + \epsilon) > 0, \quad h_2(\lambda_1) < 0, \quad h_3(\lambda_1) < 0$$

for $\epsilon \in (0, \lambda_2 - \lambda_1)$;

(2) *for any $0 < c < c^*$, there exists no positive constant λ^* such that*

$$H(\lambda^*) = 0, \quad h_2(\lambda^*) < 0, \quad h_3(\lambda^*) < 0; \tag{2.4}$$

(3) *for $0 < c < c^*$, the characteristic equation $H(\lambda) = 0$ has no roots with zero real parts.*

Proof. We divide the following three possible cases to show this lemma. (C1) $\beta > \mu + \gamma$, (C2) $\beta = \mu + \gamma$, (C3) $\beta < \mu + \gamma$. We only give the proof of the case (C1), since the proofs of the cases (C2) and (C3) are similar.

(1) For convenience, set

$$\lambda_2^{\pm} := \frac{c \pm \sqrt{c^2 - 4d_2(\beta - \mu - \gamma)}}{2d_2}, \quad \lambda_3^{\pm} := \frac{c \pm \sqrt{c^2 + 4d_3(\mu + \delta)}}{2d_3}.$$

Note that $\beta > \mu + \gamma$ and $\mu + \delta > 0$. Then $c > c_0 := 2\sqrt{d_2(\beta - \mu - \gamma)}$ implies that $\lambda_2^{\pm}$ and $\lambda_3^{\pm}$ are real, and $\lambda_3^- < 0 < \lambda_3^+$ and $0 < \lambda_2^- < \lambda_2^+$ hold. Therefore, there are the following three cases. (a) $\lambda_3^- < 0 < \lambda_2^- < \lambda_2^+ \le \lambda_3^+$, (b) $\lambda_3^- < 0 < \lambda_2^- \le \lambda_3^+ < \lambda_2^+$, (c) $\lambda_3^- < 0 < \lambda_3^+ \le \lambda_2^- < \lambda_2^+$.

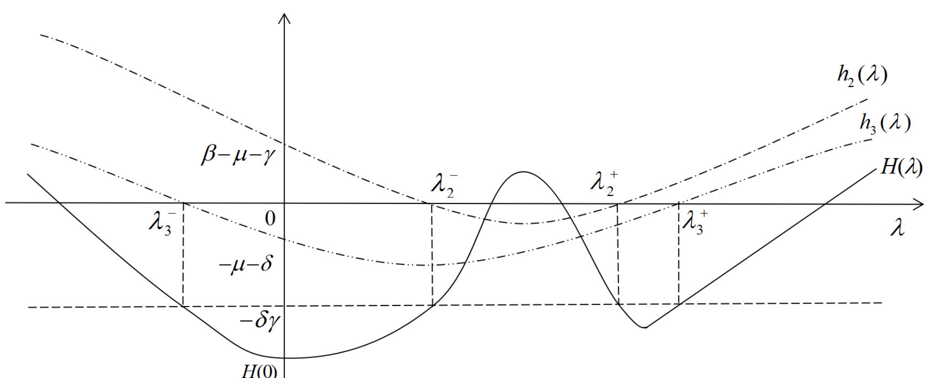

Figure 2.1: The graphs of $h_2(\lambda)$, $h_3(\lambda)$ and $H(\lambda)$.

Since the proofs of cases (a), (b) and (c) are similar, we only give the proof of the case (a). By the expression of $H(\lambda)$, we see that $H(\lambda)$ is decreasing in $\lambda \in (-\infty, \lambda_3^-)$, and $H(\lambda) < 0$ for $\lambda \in (\lambda_3^-, \lambda_2^-) \cup (\lambda_2^+, \lambda_3^+)$ and $H(\lambda)$ is increasing in $\lambda \in (\lambda_3^+, +\infty)$ (see Figure 2.1).

Observing the fact $H(\lambda_3^-) = H(\lambda_3^+) = -\delta\gamma < 0$, consequently, $H(\lambda) = 0$ has exactly two real roots with one in the interval $(-\infty, \lambda_3^-)$ and the other in the interval $(\lambda_3^+, +\infty)$. The simple calculation yields

$$\frac{dH}{dc} = -\lambda(h_2(\lambda) + h_3(\lambda)) > 0, \quad \forall\, \lambda \in (\lambda_2^-, \lambda_2^+).$$

Thus, we see that $H(\lambda)$ is increasing in c for any fixed $\lambda \in (\lambda_2^-, \lambda_2^+)$. Note that

$$\lim_{c\to+\infty} H\left(\frac{1}{\sqrt{c}}\right) = +\infty, \quad \lim_{c\to+\infty} h_2\left(\frac{1}{\sqrt{c}}\right) = -\infty, \quad \lim_{c\to+\infty} h_3\left(\frac{1}{\sqrt{c}}\right) = -\infty,$$

which imply $1/\sqrt{c} \in (\lambda_2^-, \lambda_2^+)$ for c large enough. Combing the monotonicity of $H(\lambda)$ in c with any fixed $\lambda \in (\lambda_2^-, \lambda_2^+)$, there exists a positive constant $c^* > c_0$ such that $H(\lambda) = 0$ has two positive roots in $(\lambda_2^-, \lambda_2^+)$ when $c > c^*$ and has no positive roots in $(\lambda_2^-, \lambda_2^+)$ when $c_0 < c < c^*$. Therefore, we have shown that $H(\lambda) = 0$ has three positive roots $0 < \lambda_1 < \lambda_2 < \lambda_3$ and a negative $\lambda_4 < 0$, and also, for $\varepsilon > 0$ small enough, it holds that

$$H(\lambda_1 + \varepsilon) > 0, \quad h_2(\lambda_1) < 0, \quad h_3(\lambda_1) < 0.$$

(2) It follows from the proof of (1) that $H(\lambda) = 0$ has no positive roots in $(\lambda_2^-, \lambda_2^+)$ when $c_0 < c < c^*$. Thus, for $c_0 < c < c^*$, there does not exist $\lambda^* > 0$ satisfying (2.4). And, for $c \le c_0$, it is clear that there is no $\lambda^* > 0$ satisfying $h_2(\lambda^*) < 0$. Hence, in this case $c < c^*$, there does not exist $\lambda^* > 0$ such that (2.4) holds.

(3) Note that $H(\lambda) = 0$ can be rewritten as

$$d_2 d_3 \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where

$$a_3 = -c(d_2 + d_3) < 0, \quad a_2 = c^2 - (\mu + \gamma - \beta)d_3 - (\mu + \delta)d_2,$$

and

$$a_1 = c(\mu + \gamma - \beta + \mu + \delta), \quad a_0 = (\mu + \gamma - \beta)(\mu + \delta) - \delta\gamma.$$

Obviously, $\mathcal{R}_0 > 1$ follows $a_0 < 0$, which implies that $\lambda = 0$ is not the root of $H(\lambda) = 0$. If $\lambda = i\omega$ $(\omega > 0)$ is the root of $H(\lambda) = 0$, then, substituting $\lambda = i\omega$ into $H(\lambda) = 0$ and separating real and imaginary parts, we get

$$d_2 d_3 \omega^4 - a_2\omega^2 + a_0 = 0, \quad a_1 = a_3\omega^2. \tag{2.5}$$

It follows from the fact $a_3 < 0$ that $a_1 < 0$. And eliminating w in (2.5), we get

$$d_2 d_3 a_1^2 - a_1 a_2 a_3 + a_0 a_3^2 = 0.$$

An simple computation yields

$$\begin{aligned} & d_2 d_3 a_1^2 - a_1 a_2 a_3 + a_0 a_3^2 \\ = & -c^2\left[((\mu+\delta)d_2 - (\mu+\gamma-\beta)d_3)^2 + \delta\gamma(d_2+d_3)^2 - a_1 c(d_2+d_3)\right] < 0, \end{aligned}$$

a contradiction. This completes the proof. □

2.2. The upper and lower solutions

In the following, without mentioning, we always assume that $\mathcal{R}_0 > 1$ and $c > c^*$ hold. Let λ_1 be the eigenvalue defined as in Lemma 2.2(1) and $(\eta_1, \eta_2) \gg 0$ its associating eigenvector with

$$h_2(\lambda_1)\eta_1 + \delta\eta_2 = 0, \quad \gamma\eta_1 + h_3(\lambda_1)\eta_2 = 0. \tag{2.6}$$

Also, by Lemma 2.2(1), for a sufficient small $\epsilon \in (0, \lambda_2 - \lambda_1)$, we get

$$h_2(\lambda_1+\epsilon)h_3(\lambda_1+\epsilon) - \delta\gamma > 0, \quad h_2(\lambda_1+\epsilon) < 0, \quad h_3(\lambda_1+\epsilon) < 0.$$

Then we can choose a constant $h > 0$ such that

$$-\frac{\gamma}{h_3(\lambda_1+\epsilon)} < h < -\frac{h_2(\lambda_1+\epsilon)}{\delta}. \tag{2.7}$$

Motivated by the ideas [7, 20, 25, 28], we construct the vector type of upper-lower solutions for (2.1). For $\xi \in \mathbb{R}$, we define six continuous functions as follows:

$$\begin{aligned} &\overline{S}(\xi) = 1, && \underline{S}(\xi) = \max\left\{1 - \frac{1}{\sigma}e^{\sigma\xi}, \frac{\alpha\mu}{\alpha\mu+\beta}\right\}, \\ &\overline{I}(\xi) = \min\left\{\eta_1 e^{\lambda_1\xi}, \frac{\mathcal{R}_0}{\alpha}\right\}, && \underline{I}(\xi) = \max\left\{0, \eta_1 e^{\lambda_1\xi} - M e^{(\lambda_1+\epsilon)\xi}\right\}, \\ &\overline{R}(\xi) = \min\left\{\eta_2 e^{\lambda_1\xi}, \frac{\gamma\mathcal{R}_0}{\alpha(\mu+\delta)}\right\}, && \underline{R}(\xi) = \max\left\{0, \eta_2 e^{\lambda_1\xi} - M h e^{(\lambda_1+\epsilon)\xi}\right\}, \end{aligned}$$

where σ, ϵ, M are positive constants determined in the following lemmas.

Lemma 2.3. *The following inequalities hold.*

$$d_1\overline{S}''(\xi) - c\overline{S}'(\xi) + \mu - \mu\overline{S}(\xi) - \beta\overline{S}(\xi)g(\underline{I}(\xi)) \le 0, \quad \forall \xi \in \mathbb{R}, \tag{2.8}$$

$$d_2\overline{I}''(\xi) - c\overline{I}'(\xi) + \beta\overline{S}(\xi)g(\overline{I}(\xi)) + \delta\overline{R}(\xi) - (\mu+\gamma)\overline{I}(\xi) \le 0, \quad \forall \xi \ne \xi_1, \tag{2.9}$$

$$d_3\overline{R}''(\xi) - c\overline{R}'(\xi) + \gamma\overline{I}(\xi) - (\mu+\delta)\overline{R}(\xi) \le 0, \quad \forall \xi \ne \xi_2. \tag{2.10}$$

Here, $\xi_1 := \frac{1}{\lambda_1}\ln\frac{\mathcal{R}_0}{\alpha\eta_1}$, $\xi_2 := \frac{1}{\lambda_1}\ln\frac{\gamma\mathcal{R}_0}{\alpha(\mu+\delta)\eta_2}$, *and the function* $(\overline{S}(\xi), \overline{I}(\xi), \overline{R}(\xi))$ *is called an upper solution of* (2.1).

Proof. Note that the function $\overline{S}(\xi) = 1$ and $\underline{I}(\xi)$ is non-negative for all $\xi \in \mathbb{R}$. Then (2.8) holds. Next, we show the inequality (2.9) holds. Indeed, when $\xi < \xi_1$, $\overline{I}(\xi) = \eta_1 e^{\lambda_1 \xi}$, and note that the facts $\overline{S}(\xi) = 1$, $\overline{R}(\xi) \le \eta_2 e^{\lambda_1 \xi}$ for all $\xi \in \mathbb{R}$ and $0 < g(x) \le 1$ for all $x \ge 0$, then, by (2.6),

$$
\begin{aligned}
&d_2 \overline{I}''(\xi) - c\overline{I}'(\xi) + \beta \overline{S}(\xi) g(\overline{I}(\xi)) + \delta \overline{R}(\xi) - (\mu+\gamma)\overline{I}(\xi) \\
&\le d_2 \overline{I}''(\xi) - c\overline{I}'(\xi) + \beta \overline{I}(\xi) + \delta \overline{R}(\xi) - (\mu+\gamma)\overline{I}(\xi) \\
&\le (h_2(\lambda_1)\eta_1 + \delta\eta_2) e^{\lambda_1 \xi} = 0.
\end{aligned}
$$

When $\xi > \xi_1$, $\overline{I}(\xi) = \mathcal{R}_0/\alpha$. It follows from the facts $\overline{R}(\xi) \le \frac{\gamma \mathcal{R}_0}{\alpha(\mu+\delta)}$ for $\xi \in \mathbb{R}$ and $g(x) \le 1/\alpha$ for $x \ge 0$ that

$$
\begin{aligned}
&d_2 \overline{I}''(\xi) - c\overline{I}'(\xi) + \beta \overline{S}(\xi) g(\overline{I}(\xi)) + \delta \overline{R}(\xi) - (\mu+\gamma)\overline{I}(\xi) \\
&\le \frac{\beta}{\alpha} + \frac{\delta\gamma\mathcal{R}_0}{\alpha(\mu+\delta)} - \frac{\mu+\gamma}{\alpha}\mathcal{R}_0 = 0.
\end{aligned}
$$

Finally, we prove (2.10) holds. In fact, note that $\overline{R}(\xi) = \eta_2 e^{\lambda_1 \xi}$ for $\xi < \xi_2$, and $\overline{I}(\xi) \le \eta_1 e^{\lambda_1 \xi}$ for all $\xi \in \mathbb{R}$, it follows, by (2.6), for $\xi < \xi_2$,

$$
d_3 \overline{R}''(\xi) - c\overline{R}'(\xi) + \gamma \overline{I}(\xi) - (\mu+\delta)\overline{R}(\xi) \le e^{\lambda_1 \xi}(\gamma\eta_1 + h_3(\lambda_1)\eta_2) = 0.
$$

When $\xi > \xi_2$, $\overline{R}(\xi) = \frac{\gamma\mathcal{R}_0}{\alpha(\mu+\delta)}$, and in view of $\overline{I}(\xi) \le \frac{1}{\alpha}\mathcal{R}_0$ for all $\xi \in \mathbb{R}$, we get

$$
d_3 \overline{R}''(\xi) - c\overline{R}'(\xi) + \gamma \overline{I}(\xi) - (\mu+\delta)\overline{R}(\xi) \le \frac{\gamma}{\alpha}\mathcal{R}_0 - \frac{(\mu+\delta)\gamma}{\alpha(\mu+\delta)}\mathcal{R}_0 = 0, \quad \forall \xi > \xi_2.
$$

The proof is completed. □

Lemma 2.4. *Let*

$$
0 < \sigma < \min\left\{1, \frac{\lambda_1}{2}, \frac{c+\mu}{d_1 + \beta\eta_1}\right\}, \quad 0 < \epsilon < \min\{\sigma, \lambda_2 - \lambda_1\}
$$

and

$$
M > \max\left\{\eta_1, \frac{\eta_2}{h}, -\frac{\beta\eta_1(1+\alpha\sigma\eta_1)}{\sigma(h_2(\lambda_1+\epsilon)+\delta h)}\right\}
$$

hold. Then the following inequalities hold.

$$
d_1 \underline{S}''(\xi) - c\underline{S}'(\xi) + \mu - \mu\underline{S}(\xi) - \beta\underline{S}(\xi) g(\overline{I}(\xi)) \ge 0, \quad \forall \xi \ne \xi_3, \tag{2.11}
$$

$$
d_2 \underline{I}''(\xi) - c\underline{I}'(\xi) + \beta\underline{S}(\xi) g(\underline{I}(\xi)) + \delta\underline{R}(\xi) - (\mu+\gamma)\underline{I}(\xi) \ge 0, \quad \forall \xi \ne \xi_4, \tag{2.12}
$$

$$
d_3 \underline{R}''(\xi) - c\underline{R}'(\xi) + \gamma\underline{I}(\xi) - (\mu+\delta)\underline{R}(\xi) \ge 0, \quad \forall \xi \ne \xi_5, \tag{2.13}
$$

where

$$
\xi_3 := \frac{1}{\sigma}\ln\frac{\sigma\beta}{\alpha\mu+\beta}, \quad \xi_4 := \frac{1}{\eta}\ln\frac{\eta_1}{M}, \quad \xi_5 := \frac{1}{\eta}\ln\frac{\eta_2}{Mh}.
$$

Here, the function $(\underline{S}(\xi), \underline{I}(\xi), \underline{R}(\xi))$ is called a lower solution of (2.1).

Proof. When $\xi > \xi_3$, $\underline{S}(\xi) = \frac{\alpha\mu}{\alpha\mu+\beta}$, then

$$\begin{aligned}&d_1\underline{S}''(\xi) - c\underline{S}'(\xi) + \mu - \mu\underline{S}(\xi) - \beta\underline{S}(\xi)g(\overline{I}(\xi))\\ &\geq d_1\underline{S}''(\xi) - c\underline{S}'(\xi) + \mu - \mu\underline{S}(\xi) - \frac{\beta}{\alpha}\underline{S}(\xi) = 0.\end{aligned}$$

Note that $\overline{I}(\xi) \leq \eta_1 e^{\lambda_1\xi}$ for all $\xi \in \mathbb{R}$, and

$$e^{(\lambda_1-\sigma)\xi} < e^{\frac{\lambda_1-\sigma}{\sigma}\ln\frac{\sigma\beta}{\alpha\mu+\beta}} = \left(\frac{\sigma\beta}{\alpha\mu+\beta}\right)^{(\lambda_1-\sigma)/\sigma} < \sigma^{(\lambda_1-\sigma)/\sigma} < \sigma, \quad \forall\,\xi < \xi_3.$$

Hence, it follows from the fact $\underline{S}(\xi) = 1 - \frac{1}{\sigma}e^{\sigma\xi}$ for $\xi < \xi_3$ that

$$\begin{aligned}&d_1\underline{S}''(\xi) - c\underline{S}'(\xi) + \mu - \mu\underline{S}(\xi) - \beta\underline{S}(\xi)g(\overline{I}(\xi))\\ &\geq d_1\underline{S}''(\xi) - c\underline{S}'(\xi) + \mu - \mu\underline{S}(\xi) - \beta\overline{I}(\xi)\\ &\geq -d_1\sigma e^{\sigma\xi} + ce^{\sigma\xi} + \frac{\mu}{\sigma}e^{\sigma\xi} - \beta\eta_1 e^{\lambda_1\xi}\\ &> \big(-d_1\sigma + c + \mu - \beta\eta_1 e^{(\lambda_1-\sigma)\xi}\big)e^{\sigma\xi}\\ &> (-d_1\sigma + c + \mu - \beta\eta_1\sigma)e^{\sigma\xi}.\end{aligned}$$

Here, we just need to choose the constant $\sigma < \frac{c+\mu}{d_1+\beta\eta_1}$, then (2.11) holds for all $\xi \neq \xi_3$.

Now, we show that (2.12) holds. In fact, for $\xi > \xi_4$, inequality (2.12) holds since $\underline{I}(\xi) = 0$ on $[\xi_4, \infty)$ and $\underline{R}(\xi)$ is non-negative for all $\xi \in \mathbb{R}$. For $\xi < \xi_4$, $\underline{I}(\xi) = \eta_1 e^{\lambda_1\xi} - Me^{(\lambda_1+\epsilon)\xi}$, and $\underline{R}(\xi) \geq \eta_2 e^{\lambda_1\xi} - Mhe^{(\lambda_1+\epsilon)\xi}$ for all $\xi \in \mathbb{R}$, and by the facts that $g(x) \geq x(1-\alpha x)$ for all $x \geq 0$, and

$$1 - \frac{1}{\sigma}e^{\sigma\xi} \leq \underline{S}(\xi) \leq 1, \quad \eta_1 e^{\lambda_1\xi} - Me^{(\lambda_1+\epsilon)\xi} \leq \underline{I}(\xi) \leq \eta_1 e^{\lambda_1\xi}, \quad \forall\,\xi \in \mathbb{R},$$

we get, for $\xi < \xi_4$,

$$\begin{aligned}&d_2\underline{I}''(\xi) - c\underline{I}'(\xi) + \beta\underline{S}(\xi)g(\underline{I}(\xi)) + \delta\underline{R}(\xi) - (\mu+\gamma)\underline{I}(\xi)\\ &\geq d_2\underline{I}''(\xi) - c\underline{I}'(\xi) + \beta\left(1 - \frac{1}{\sigma}e^{\sigma\xi}\right)\underline{I}(\xi)(1 - \alpha\underline{I}(\xi)) + \delta\underline{R}(\xi) - (\mu+\gamma)\underline{I}(\xi)\\ &\geq d_2\underline{I}''(\xi) - c\underline{I}'(\xi) + (\beta-\mu-\gamma)\underline{I}(\xi) + \delta\underline{R}(\xi) - \alpha\beta\underline{I}^2(\xi) - \frac{\beta}{\alpha}e^{\sigma\xi}\underline{I}(\xi)\\ &\geq e^{\lambda_1\xi}(h_2(\lambda_1)\eta_1 + \delta\eta_2) - e^{(\lambda_1+\epsilon)\xi}\left(M(h_2(\lambda_1+\epsilon) + \delta h) + \frac{\beta\eta_1}{\sigma}e^{(\sigma-\epsilon)\xi} + \alpha\beta\eta_1^2 e^{(\lambda_1-\epsilon)\xi}\right)\\ &> -e^{(\lambda_1+\epsilon)\xi}\left(M(h_2(\lambda_1+\eta) + \delta h) + \beta\eta_1\left(\frac{1}{\sigma} + \alpha\eta_1\right)\right),\end{aligned}$$

since $e^{(\lambda_1-\epsilon)\xi} < 1$ and $e^{(\sigma-\epsilon)\xi} < 1$ for $\xi < \xi_4 < 0$. By (2.7), we see $h_2(\lambda_1+\eta) + \delta h < 0$ and then only need to choose

$$M > -\frac{\beta\eta_1(1+\alpha\sigma\eta_1)}{\sigma(h_2(\lambda_1+\epsilon) + \delta h)}.$$

Hence, (2.12) holds.

Next, we verify that (2.13) holds. Clearly, (2.13) holds since $\underline{R}(\xi) = 0$ for $\xi > \xi_5$. When $\xi < \xi_5$, $\underline{R}(\xi) = \eta_2 e^{\lambda_1 \xi} - Mhe^{(\lambda_1+\epsilon)\xi}$, and note that $\underline{I}(\xi) \geq \eta_1 e^{\lambda_1 \xi} - Me^{(\lambda_1+\epsilon)\xi}$ for all $\xi \in \mathbb{R}$, we get

$$\begin{aligned}
&d_3\underline{R}''(\xi) - c\underline{R}'(\xi) + \gamma\underline{I}(\xi) - (\mu+\delta)\underline{R}(\xi) \\
&\geq e^{\lambda_1\xi}(h_2(\lambda_1)\eta_2 + \gamma\eta_1) - Me^{(\lambda_1+\epsilon)\xi}(hh_2(\lambda_1+\epsilon)+\gamma) \\
&= -Me^{(\lambda_1+\epsilon)\xi}(hh_3(\lambda_1+\epsilon)+\gamma).
\end{aligned}$$

By (2.7), we have $hh_3(\lambda_1+\epsilon)+\gamma < 0$, which follows that (2.13) holds. The proof is completed. $\square$

2.3. The solutions for (2.1)

In this subsection, we will use the upper-lower solutions $(\overline{S}, \overline{I}, \overline{R})$ and $(\underline{S}, \underline{I}, \underline{R})$ to verify that the conditions of the Schauder fixed point theorem hold.

Letting $r > \max\{\mu + \beta/\alpha, \mu+\gamma, \mu+\delta\}$ such that

$$H_1(S, I, R)(\xi) := rS(\xi) + \mu - \mu S(\xi) - \beta S(\xi) g(I(\xi))$$

is monotone increasing in $S \in [0,1]$, and monotone decreasing in $I \in [0, \mathcal{R}_0/\alpha]$ for all $\xi \in \mathbb{R}$, and

$$H_2(S, I, R)(\xi) := rI(\xi) + \beta S(\xi) g(I(\xi)) + \delta R(\xi) - (\mu+\gamma) I(\xi)$$

is monotone increasing in $S \in [0,1]$ and $I \in [0, \mathcal{R}_0/\alpha]$, $R \in \left[0, \frac{\gamma\mathcal{R}_0}{\alpha(\mu+\delta)}\right]$ for all $\xi \in \mathbb{R}$, and

$$H_3(S, I, R)(\xi) := rR(\xi) + \gamma I(\xi) - (\mu+\delta) R(\xi)$$

is monotone increasing in both $I \in [0, \mathcal{R}_0/\alpha]$ and $R \in \left[0, \frac{\gamma\mathcal{R}_0}{\alpha(\mu+\delta)}\right]$ for all $\xi \in \mathbb{R}$. Then, (2.1) can be written as

$$\begin{aligned}
d_1 S''(\xi) - cS'(\xi) - rS(\xi) + H_1(S, I, R)(\xi) &= 0, \\
d_2 I''(\xi) - cI'(\xi) - rI(\xi) + H_2(S, I, R)(\xi) &= 0, \\
d_3 R''(\xi) - cR'(\xi) - rR(\xi) + H_3(S, I, R)(\xi) &= 0.
\end{aligned} \tag{2.14}$$

Define the set

$$\Gamma = \{(S, I, R) \in C(\mathbb{R}, \mathbb{R}^3) : (\underline{S}, \underline{I}, \underline{R})(\xi) \leq (S, I, R)(\xi) \leq (\overline{S}, \overline{I}, \overline{R})(\xi), \forall \xi \in \mathbb{R}\}.$$

Obviously, Γ is nonempty, closed and convex in $C(\mathbb{R}, \mathbb{R}^3)$. Furthermore, we define an operator: $F = (F_1, F_2, F_3) \colon \Gamma \to C(\mathbb{R}, \mathbb{R}^3)$ by

$$F_i(S, I, R)(\xi) = \frac{1}{\Lambda_i}\left(\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi - x)} + \int_{\xi}^{+\infty} e^{\lambda_{i2}(\xi - x)}\right) H_i(S, I, R)(x)\, dx,$$

where $\Lambda_i = d_i(\lambda_{i2} - \lambda_{i1})$, and $\lambda_{i1} < 0 < \lambda_{i2}$ are the roots of $d_i\lambda^2 - c\lambda - r = 0$, $i = 1, 2, 3$.

One can easily see that any fixed point of F is a solution of (2.14), also is the solution of (2.1). Hence, the existence of the solution of (2.14) is reduced to verify that the operator F satisfies the conditions of the Schauder fixed point theorem. Next we divide the proof into the following three lemmas.

Lemma 2.5. *The operator F maps Γ into Γ.*

Proof. Given $(S, I, R) \in \Gamma$. Obviously, we only need to show that, for all $\xi \in \mathbb{R}$,

$$\underline{S}(\xi) \le F_1(S, I, R)(\xi) \le 1, \ \ \underline{I}(\xi) \le F_2(S, I, R)(\xi) \le \overline{I}(\xi), \ \ \underline{R}(\xi) \le F_3(S, I, R)(\xi) \le \overline{R}(\xi).$$

Based on the monotonicity of H_i, $i = 1, 2, 3$, we need to prove that

$$\begin{aligned}
\underline{S}(\xi) &\le F_1(\underline{S}, \overline{I}, \underline{R})(\xi) \le F_1(S, I, R)(\xi) \le F_1(\overline{S}, \underline{I}, \overline{R})(\xi) \le 1,\\
\underline{I}(\xi) &\le F_2(\underline{S}, \underline{I}, \underline{R})(\xi) \le F_2(S, I, R)(\xi) \le F_2(\overline{S}, \overline{I}, \overline{R})(\xi) \le \overline{I}(\xi),\\
\underline{R}(\xi) &\le F_3(\underline{S}, \underline{I}, \underline{R})(\xi) \le F_3(S, I, R)(\xi) \le F_3(\overline{S}, \overline{I}, \overline{R})(\xi) \le \overline{R}(\xi).
\end{aligned}$$

First, we show the inequality $\underline{S}(\xi) \le F_1(\underline{S}, \overline{I}, \underline{R})(\xi)$ holds for all $\xi \in \mathbb{R}$. Indeed, for $\xi \ne \xi_3$, by (2.11), we get

$$\begin{aligned}
F_1(\underline{S}, \overline{I}, \underline{R})(\xi) &= \frac{1}{\Lambda_1}\left(\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)}\right) H_1(\underline{S}, \overline{I}, \underline{R})(x)\,dx\\
&\ge \frac{1}{\Lambda_1}\left(\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)}\right)(-d_1\underline{S}''(x) + c\underline{S}'(x) + r\underline{S}(x))\,dx.
\end{aligned}$$

When $\xi > \xi_3$, it follows that

$$\begin{aligned}
&F_1(\underline{S}, \overline{I}, \underline{R})(\xi)\\
&\ge \frac{1}{\Lambda_1}\left(\int_{-\infty}^{\xi_3} + \int_{\xi_3}^{\xi}\right) e^{\lambda_{11}(\xi-x)}(-d_1\underline{S}''(x) + c\underline{S}'(x) + r\underline{S}(x))\,dx\\
&\quad + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)}(-d_1\underline{S}''(x) + c\underline{S}'(x) + r\underline{S}(x))\,dx\\
&= \frac{1}{\Lambda_1}\left(\int_{-\infty}^{\xi_3} e^{\lambda_{11}(\xi-x)} d(-d_1\underline{S}'(x) + c\underline{S}(x)) + r\int_{-\infty}^{\xi_3} e^{\lambda_{11}(\xi-x)}\underline{S}(x)\,dx\right.\\
&\qquad \left. + \int_{\xi_3}^{\xi} e^{\lambda_{11}(\xi-x)} d(-d_1\underline{S}'(x) + c\underline{S}(x)) + r\int_{\xi_3}^{\xi} e^{\lambda_{11}(\xi-x)}\underline{S}(x)\,dx\right)\\
&\quad + \frac{1}{\Lambda_1}\left(\int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)} d(-d_1\underline{S}'(x) + c\underline{S}(x)) + r\int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)}\underline{S}(x)\,dx\right)\\
&= \frac{1}{\Lambda_1}\Big(- (d_1\underline{S}'(\xi_3 - 0) - c\underline{S}(\xi_3) + d_1\lambda_{11}\underline{S}(\xi_3))e^{\lambda_{11}(\xi-\xi_3)}\\
&\qquad + (d_1\underline{S}'(\xi_3 + 0) - c\underline{S}(\xi_3) - d_1\lambda_{11}\underline{S}(\xi_3))e^{\lambda_{11}(\xi-\xi_3)}
\end{aligned}$$

$$
\begin{aligned}
&\qquad - (d_1\lambda_{11}^2 - c\lambda_{11} - r)\int_{-\infty}^{\xi_3} e^{\lambda_{11}(\xi-x)}\underline{S}(x)\,dx - (d_1\underline{S}'(\xi) - c\underline{S}(\xi) - d_1\lambda_{11}\underline{S}(\xi))\\
&\qquad - (d_1\lambda_{11}^2 - c\lambda_{11} - r)\int_{\xi_3}^{\xi} e^{\lambda_{11}(\xi-x)}\underline{S}(x)\Big)\\
&\quad + \frac{1}{\Lambda_1}\Big((d_1\underline{S}'(\xi) - c\underline{S}(\xi) + d_1\lambda_{22}\underline{S}(\xi)) - (d_1\lambda_{12}^2 - c\lambda_{12} - r)\int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)}\underline{S}(x)\,dx\Big)\\
&= \underline{S}(\xi) + \frac{d_1}{\Lambda_1}e^{\lambda_{11}(\xi-\xi_3)}(\underline{S}'(\xi_3+0) - \underline{S}'(\xi_3-0))\\
&\geq \underline{S}(\xi), \quad \text{since } \underline{S}'(\xi_3-) \leq 0 \text{ and } \underline{S}'(\xi_3+) = 0.
\end{aligned}
$$

Similarly, when $\xi < \xi_3$, we also show $F_1(\underline{S},\overline{I},\underline{R})(\xi) \geq \underline{S}(\xi)$ for all $\xi \in \mathbb{R}$. By the continuity of both $\underline{S}(\xi)$ and $F_1(\underline{S},\overline{I},\underline{R})(\xi)$, we obtain $F_1(\underline{S},\overline{I},\underline{R})(\xi) \geq \underline{S}(\xi)$ for all $\xi \in \mathbb{R}$.

On the other hand, for any $\xi \in \mathbb{R}$, it follows from (2.8) that

$$
\begin{aligned}
F_1(\overline{S},\underline{I},\overline{R})(\xi) &= \frac{1}{\Lambda_1}\left(\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)}\right) H_1(\overline{S},\underline{I},\overline{R})(x)\,dx\\
&\leq \frac{1}{\Lambda_1}\left(\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)}\right)(-d_1\overline{S}''(x) + c\overline{S}'(x) + r\overline{S}(x))\,dx\\
&= \frac{1}{\Lambda_1}\left(\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-x)}\right)(r\overline{S}(x))\,dx\\
&= 1.
\end{aligned}
$$

So we have shown $\underline{S}(\xi) \leq F_1(S,I,R)(\xi) \leq \overline{S}(\xi)$ for all $\xi \in \mathbb{R}$.

The proofs of $F_i(S,I,R)(\xi)$, $i = 2,3$, are similar to that of $F_1(S,I,R)(\xi)$ and are omitted. Hence, we complete the proof. □

For $0 < \rho < \min\{-\lambda_{11}, -\lambda_{21}, -\lambda_{31}\}$, define

$$B_\rho(\mathbb{R},\mathbb{R}^3) = \{\Phi \in C(\mathbb{R},\mathbb{R}^3) : \|\Phi\|_\rho < +\infty\}$$

with the norm

$$\|\Phi\|_\rho = \max\left\{\sup_{\xi\in\mathbb{R}} |S(\xi)|e^{-\rho|\xi|}, \sup_{\xi\in\mathbb{R}} |I(\xi)|e^{-\rho|\xi|}, \sup_{\xi\in\mathbb{R}} |R(\xi)|e^{-\rho|\xi|}\right\}.$$

Then $B_\rho(\mathbb{R},\mathbb{R}^3)$ is a Banach space with the decay norm $\|\cdot\|_\rho$.

Lemma 2.6. *The operator $F = (F_1,F_2,F_3)\colon \Gamma \to \Gamma$ is continuous with respect to the norm $\|\cdot\|_\rho$.*

Proof. Note that the function $G(S,I) := Sg(I)$ has bounded partial derivatives with respect to S and I. For example, we see that the partial derivative $G(S,I)$ with respect to

S is $g(I)$ is bounded by $1/\alpha$. Similarly, we can show that the partial derivative $G(S,I)$ with respect to I is also bounded by S. Hence, for any $\Phi_1 = (S_1, I_1, R_1), \Phi_2 = (S_2, I_2, R_2) \in \Gamma$, we get

$$|S_1(\xi)g(I_1(\xi)) - S_2(\xi)g(I_2(\xi))| \le \frac{1}{\alpha}|S_1(\xi) - S_2(\xi)| + |I_1(\xi) - I_2(\xi)|.$$

By the above equality, it easy to see that there is a constant $L > 0$ such that

$$\begin{aligned}
&|H_1(\Phi_1)(\xi) - H_1(\Phi_2)(\xi)| \le L\big(|S_1(\xi) - S_2(\xi)| + |I_1(\xi) - I_2(\xi)|\big),\\
&|H_2(\Phi_1)(\xi) - H_2(\Phi_2)(\xi)| \le L\big(|S_1(\xi) - S_2(\xi)| + |I_1(\xi) - I_2(\xi)| + |R_1(\xi) - R_2(\xi)|\big),\\
&|H_3(\Phi_1)(\xi) - H_3(\Phi_2)(\xi)| \le L\big(|I_1(\xi) - I_2(\xi)| + |R_1(\xi) - R_2(\xi)|\big).
\end{aligned}$$

Hence

$$|H_i(\Phi_1)(\xi) - H_i(\Phi_2)(\xi)|e^{-\rho|\xi|} \le L\|\Phi_1 - \Phi_2\|_\rho, \quad \forall\, \xi \in \mathbb{R},\ i = 1, 2, 3.$$

Consequently,

$$\begin{aligned}
&|F_1(\Phi_1)(\xi) - F_1(\Phi_2)(\xi)|e^{-\rho|\xi|}\\
&\le \frac{L}{\Lambda_1}\left(\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi - x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi - x)}\right) e^{\rho|x| - \rho|\xi|}\, dx\, \|\Phi_1 - \Phi_2\|_\rho\\
&\le \frac{L}{\Lambda_1}\left(\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi - x)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi - x)}\right) e^{\rho|x - \xi|}\, dx\, \|\Phi_1 - \Phi_2\|_\rho\\
&= \frac{L}{\Lambda_1}\left(\frac{1}{\lambda_{12} - \rho} - \frac{1}{\lambda_{11} + \rho}\right)\|\Phi_1 - \Phi_2\|_\rho,
\end{aligned}$$

which indicates that the operator F_1 is continuous with respect to the norm $\|\cdot\|_\rho$.

By the similar arguments as above, we can also show that the operators $F_i \colon \Gamma \to \Gamma$, $i = 2, 3$, are continuous with respect to the norm $\|\cdot\|_\rho$. The proof is completed. $\square$

Lemma 2.7. *The operator $F = (F_1, F_2, F_3) \colon \Gamma \to \Gamma$ is compact with respect to the norm $\|\cdot\|_\rho$.*

The proof of Lemma 2.7 is similar to that of [21, Lemma 6], see also [22, Lemma 3.5] or [23, Lemma 2.8], we omit the details.

3. Existence and non-existence of traveling waves

3.1. Existence of traveling waves

In this subsection, we will establish the existence of traveling waves for system (1.2). To this end, we first give the propositions of the solutions of (2.1).

Proposition 3.1. *Assume that* $\mathcal{R}_0 > 1$ *holds. Then for any* $c > c^*$*,* (2.1) *admits a non-trivial and positive solution* $(S(\xi), I(\xi), R(\xi))$ *satisfying*

$$\lim_{\xi \to -\infty} (S(\xi), I(\xi), R(\xi)) = (1, 0, 0). \tag{3.1}$$

Moreover,

$$0 < S(\xi) \leq 1, \quad 0 < I(\xi) \leq \frac{\mathcal{R}_0}{\alpha}, \quad 0 < R(\xi) \leq \frac{\gamma \mathcal{R}_0}{\alpha(\mu + \delta)}, \quad \forall\, \xi \in \mathbb{R} \tag{3.2}$$

and

$$\lim_{\xi \to -\infty} e^{-\lambda_1 \xi} I(\xi) = \eta_1, \quad \lim_{\xi \to -\infty} e^{-\lambda_1 \xi} R(\xi) = \eta_2. \tag{3.3}$$

Proof. By Lemmas 2.5, 2.6 and 2.7, the Schauder fixed point theorem implies that there exists a pair of $(S, I, R) \in \Gamma$, which is a fixed point of the operator F. Consequently, $(S(\xi), I(\xi), R(\xi))$ is a solution of (2.1) satisfying

$$0 < S(\xi) \leq 1, \quad 0 \leq I(\xi) \leq \frac{\mathcal{R}_0}{\alpha}, \quad 0 \leq R(\xi) \leq \frac{\gamma \mathcal{R}_0}{\alpha(\mu + \delta)}, \quad \forall\, \xi \in \mathbb{R}.$$

And also, noting that $(S, I, R) \in \Gamma$, it is easy to see that (3.1) and (3.3) hold.

Next, we claim that $I(\xi) > 0$ and $R(\xi) > 0$ for all $\xi \in \mathbb{R}$. Indeed, if there exists $\xi_0 \in \mathbb{R}$ such that $I(\xi_0) = 0$, then there exist constants $a, b \in \mathbb{R}$ such that $a < \frac{1}{\eta} \ln \frac{\eta_1}{M} < b$ and $\xi_0 \in (a, b)$. This implies $I(\xi)$ attains its minimum in (a, b) for any $\xi \in [a, b]$. It follows from the second equation of (2.1) that

$$-d_2 I''(\xi) + c I'(\xi) + (\mu + \gamma) I(\xi) \geq 0, \quad \xi \in [a, b].$$

By the elliptic strong maximum principle (see, [32, Lemma 2.1.2]), it follows that $I(\xi) \equiv 0$ for $\xi \in [a, b]$. But, $I(\xi) \geq \underline{I}(\xi) > 0$ for $\xi \in \left[a, \frac{1}{\eta} \ln \frac{\eta_1}{M}\right)$, a contradiction. Similarly, we can show $R(\xi) > 0$ for any $\xi \in \mathbb{R}$. The proof is completed. □

Proposition 3.2. *Let* $(S(\xi), I(\xi), R(\xi))$ *be a positive solution of* (2.1) *satisfying* (3.2)*. Then there exist positive constants* L_i*,* $i = 1, 2, \ldots, 6$*, such that*

$$-L_1 S(\xi) < S'(\xi) < L_2 S(\xi), \quad -L_3 I(\xi) < I'(\xi) < L_4 I(\xi), \quad -L_5 R(\xi) < R'(\xi) < L_6 R(\xi)$$

for all $\xi \geq 0$*. Furthermore, there is a constant* $C_1 > 0$ *such that*

$$\left| \frac{S'(\xi)}{S(\xi)} \right| + \left| \frac{I'(\xi)}{I(\xi)} \right| + \left| \frac{R'(\xi)}{R(\xi)} \right| \leq C_1 \quad \textit{for } \xi \geq 0. \tag{3.4}$$

Proof. (1) We show that $-L_1S(\xi) < S'(\xi)$ for all $\xi \geq 0$ if L_1 is a positive constant sufficiently large such that $-L_1S(0) < S'(0)$ and $cL_1 \geq \beta/\alpha$.

Let $\Phi_1(\xi) := S'(\xi) + L_1S(\xi)$ for $\xi \geq 0$. It suffices to show that $\Phi_1(\xi) > 0$ for all $\xi \geq 0$. In fact, note that $\Phi_1(0) > 0$, for contradiction, we assume that there exists $\xi_1 > 0$ such that $\Phi_1(\xi_1) = 0$ and $\Phi_1'(\xi_1) \leq 0$ hold. Then there are two possibilities: either

$$\Phi_1(\xi) \leq 0, \quad \forall\, \xi \geq \xi_1, \tag{3.5}$$

or there is $\xi_2 \geq \xi_1$ such that

$$\Phi_1(\xi_2) = 0 \quad \text{and} \quad \Phi_1'(\xi_2) \geq 0. \tag{3.6}$$

For the first case, (3.5) follows that $S'(\xi) \leq -L_1S(\xi)$ for all $\xi \geq \xi_1$. Note that $g(I) < 1/\alpha$, hence we deduce from the first equation of (2.1) that

$$d_1S''(\xi) = cS'(\xi) - \mu + \mu S(\xi) + \beta S(\xi)g(I(\xi)) < \left(-cL_1 + \frac{\beta}{\alpha}\right) S(\xi) \leq 0, \quad \forall\, \xi \geq \xi_1,$$

which implies that $S'(\xi)$ is decreasing in $[\xi_1, \infty)$. Hence $S'(\xi) \leq S'(\xi_1) \leq -L_1S(\xi_1) < 0$ for all $\xi \geq \xi_1$, which contradicts the fact $0 < S(\xi) \leq 1$.

For the second case, (3.6) yields that

$$S'(\xi_2) = -L_1S(\xi_2) < 0, \quad S''(\xi_2) \geq -L_1S'(\xi_2) > 0.$$

It follows from the first equation of (2.1) that

$$0 = d_1S''(\xi_2) - cS'(\xi_2) + \mu - \mu S(\xi_2) - \beta S(\xi_2)g(I(\xi_2)) > cL_1S(\xi_2) - \frac{\beta}{\alpha}S(\xi_2) \geq 0,$$

a contradiction again.

(2) We show that $S'(\xi) < L_2S(\xi)$ for all $\xi \geq 0$ if L_2 is a positive constant sufficiently large such that $S'(0) < L_2S(0)$ and $d_1L_2^2 - cL_2 \geq \beta/\alpha$ hold.

Let $\Phi_2(\xi) := S'(\xi) - L_2S(\xi)$ for $\xi \geq 0$. We now show that $\Phi_2(\xi) < 0$ for all $\xi \geq 0$. For contradiction, noting that $\Phi_2(0) < 0$, we can assume that there exists $\xi_3 \geq 0$ such that $\Phi_2(\xi_3) = 0$ and $\Phi_2'(\xi_3) \geq 0$. Then

$$S'(\xi_3) = L_2S(\xi_3), \quad S''(\xi_3) \geq L_2S'(\xi_3) = L_2^2S(\xi_3).$$

Thus, we deduce from the first equation of (3.1) that

$$0 = d_1S''(\xi_3) - cS'(\xi_3) + \mu - \mu S(\xi_3) - \beta S(\xi_3)g(I(\xi_3)) > \left(d_1L_2^2 - cL_2 - \frac{\beta}{\alpha}\right) S(\xi_3) \geq 0,$$

which is a contradiction.

(3) Similar to proof of the equality $-L_1S(\xi) < S'(\xi)$ for all $\xi \geq 0$, we can show that $-L_3I(\xi) < I'(\xi)$ for all $\xi \geq 0$ if L_3 is a positive constant sufficiently large such that $-L_3I(0) < I'(0)$ and $cL_3 \geq \mu + \gamma$. As the same as the proof of the inequality $S'(\xi) < L_2S(\xi)$ for all $\xi \geq 0$, we also show that $I'(\xi) < L_4I(\xi)$ holds for any $\xi \geq 0$ if L_4 is a positive constant sufficiently large such that $I'(0) < L_4I(0)$ and $d_2L_4^2 - cL_4 \geq \mu + \gamma$. We omit the details.

(4) The proof of the inequality of $-L_5R(\xi) < R'(\xi) < L_6R(\xi)$ for all $\xi > 0$ is similar to that of $-L_3I(\xi) < I'(\xi) < L_4I(\xi)$ for all $\xi > 0$, we also omit the details. The proof is completed. □

Proposition 3.3. *Let* $(S(\xi), I(\xi), R(\xi))$ *be a positive solution of* (2.1) *satisfying* (3.2). *Then* $\lim_{\xi\to+\infty}(S(\xi), I(\xi), R(\xi)) = (S^*, I^*, R^*)$.

Proof. Define

$$\mathcal{D} = \left\{(S, I, R) \in C^1(\mathbb{R}, \mathbb{R}^3_+) : \left|\frac{S'(\xi)}{S(\xi)}\right| + \left|\frac{I'(\xi)}{I(\xi)}\right| + \left|\frac{R'(\xi)}{R(\xi)}\right| \leq C_1, \forall\, \xi \geq 0\right\},$$

where $C_1 > 0$ is defined by Proposition 3.2. Obviously, by (3.4), we know the set $\mathcal{D} \neq \emptyset$. Hence, for each $(S, I, R) \in \mathcal{D}$, consider the Lyapunov function $W(S, I, R)\colon \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$W(S, I, R)(\xi) = W_1(S, I, R)(\xi) + W_2(S, I, R)(\xi), \quad \forall\, \xi \geq 0,$$

where

$$W_1(S, I, R)(\xi) = cS^*\mathcal{L}\left(\frac{S(\xi)}{S^*}\right) + cI^*\mathcal{L}\left(\frac{I(\xi)}{I^*}\right) + \frac{c\delta}{\mu+\delta}R^*\mathcal{L}\left(\frac{R(\xi)}{R^*}\right),$$

in which the function $\mathcal{L}(x) = x - 1 - \ln x$ for $x > 0$, and

$$W_2(S, I, R)(\xi) = d_1S'(\xi)\left(\frac{S^*}{S(\xi)} - 1\right) + d_2I'(\xi)\left(\frac{I^*}{I(\xi)} - 1\right) + \frac{\delta d_3}{\mu+\delta}R'(\xi)\left(\frac{R^*}{R(\xi)} - 1\right).$$

We first claim that there is a constant $C_2 \in \mathbb{R}$ such that

$$C_2 \leq W(S, I, R)(\xi) < \infty, \quad \forall\, \xi > 0. \tag{3.7}$$

Indeed, by (3.2) and the definition of the function $\mathcal{L}$, we know

$$0 \leq W_1(S, I, R)(\xi) < \infty, \quad \forall\, \xi > 0. \tag{3.8}$$

Note that, by (3.2) and (3.4), we see that there exists a constant $C_3 > 0$ such that

$$|S'(\xi)| + |I'(\xi)| + |R'(\xi)| \leq C_3, \quad \forall\, \xi > 0.$$

By (3.4) and the last inequality, we get

$$|W_1(S,I,R)(\xi)| \le d_1 S^* \left|\frac{S'(\xi}{S(\xi)}\right| + d_2 I^* \left|\frac{I'(\xi)}{I(\xi)}\right| + \frac{\delta d_3 R^*}{\mu+\delta}\left|\frac{R'(\xi)}{R(\xi)}\right| \\ + d_1|S'(\xi)| + d_2|I'(\xi)| + \frac{\delta d_3}{\mu+\delta}|R'(\xi)| < \infty,$$

which combining with (3.8) implies that the inequality (3.7) holds.

Next, we show that $W(S,I,R)(\xi)$ is non-increasing in $\xi > 0$. In fact, a direct calculation leads to

$$\frac{dW}{d\xi} = W_3(S,I,R) - d_1 S^* \left(\frac{S'}{S}\right)^2 - d_2 I^* \left(\frac{I'}{I}\right)^2 - \frac{\delta d_3}{\mu+\delta} R^* \left(\frac{R'}{R}\right)^2,$$

where

$$W_3(S,I,R) = \left(1 - \frac{S^*}{S}\right)(\mu - \mu S - \beta S g(I)) + \left(1 - \frac{I^*}{I}\right)(\beta S g(I) + \delta R - (\mu+\gamma)I) \\ + \frac{\delta}{\mu+\delta}\left(1 - \frac{R^*}{R}\right)(\gamma I - (\mu+\delta)R).$$

Using the relation at the endemic equilibrium

$$\mu = \mu S^* + \beta S^* g(I^*), \quad \beta S^* g(I^*) + \delta R^* = (\mu+\gamma)I^*, \quad \delta R^* = \frac{\delta\gamma}{\mu+\delta} I^*,$$

we get

$$\begin{aligned}
&W_3(S,I,R)\\
&= \left(1 - \frac{S^*}{S}\right)(\mu(S^* - S) + \beta S^* g(I^*) - \beta S g(I))\\
&\quad + \left(1 - \frac{I^*}{I}\right)(\beta S g(I) - (\mu+\gamma)I) + \delta R\left(1 - \frac{I^*}{I}\right)\\
&\quad + \frac{\delta}{\mu+\delta}\left(\gamma I - (\mu+\delta)R - \gamma R^* \frac{I}{R} + (\mu+\delta)R^*\right)\\
&= \left(1 - \frac{S^*}{S}\right)(\mu(S^* - S) + \beta S^* g(I^*)) + \beta S g(I)\frac{S^*}{S} - \beta S^* g(I^*)\frac{I}{I^*}\\
&\quad - \beta S^* g(I^*) S g(I) \frac{1+\alpha I^*}{S^* I} + \beta S^* g(I^*) + \frac{\gamma\delta}{\mu+\delta} I^* \left(2 - \frac{I^* R}{R^* I} - \frac{R^* I}{I^* R}\right)\\
&= -\frac{\mu}{S}(S - S^*)^2 + \frac{\gamma\delta}{\mu+\delta} I^* \left(2 - \frac{I^* R}{R^* I} - \frac{R^* I}{I^* R}\right)\\
&\quad + \beta S^* g(I^*)\left(2 - \frac{S^*}{S} + \frac{I}{I^*}\frac{1+\alpha I^*}{1+\alpha I} - \frac{I}{I^*} - \frac{1+\alpha I^*}{S^* I} S g(I)\right)\\
&= -\frac{\mu}{S}(S - S^*)^2 + \frac{\gamma\delta}{\mu+\delta} I^* \left(2 - \frac{I^* R}{R^* I} - \frac{R^* I}{I^* R}\right)\\
&\quad - \beta S^* g(I^*)\left(\mathcal{L}\left(\frac{S^*}{S}\right) + \mathcal{L}\left(\frac{S(1+\alpha I^*)}{S^*(1+\alpha I)}\right) + \mathcal{L}\left(\frac{1+\alpha I}{1+\alpha I^*}\right) + \frac{\alpha(I - I^*)^2}{I^*(1+\alpha I)(1+\alpha I^*)}\right).
\end{aligned}$$

As we know, $-\frac{\mu}{S}(S-S^*)^2 \leq 0$, $2-\frac{I^*R}{R^*I}-\frac{R^*I}{I^*R} \leq 0$, and the function $\mathcal{L}(x)$ is always greater than or equal to zero for all $x>0$, and $\mathcal{L}(x)=0$ if and only if $x=1$. Consequently, $W(S,I,R)(\xi)$ is non-increasing. Note that $\frac{dW}{d\xi}=0$ if and only if

$$(S(\xi),I(\xi),R(\xi)) \equiv (S^*,I^*,R^*) \quad \text{and} \quad (S'(\xi),I'(\xi),R'(\xi)) \equiv 0, \quad \forall\, \xi \geq 0.$$

Thus, $\lim_{\xi\to+\infty}(S(\xi),I(\xi),R(\xi))=(S^*,I^*,R^*)$. This completes the proof. □

By Propositions 3.1 and 3.3, we obtain the following existence of traveling wave solutions for (1.2).

Theorem 3.4. *Assume that $\mathcal{R}_0>1$ holds. Then for every $c>c^*$, system* (2.1) *has a positive solution $(S(\xi),I(\xi),R(\xi))$ satisfying* (2.2), (3.2) *and* (3.3). *That is, system* (1.2) *admits a positive traveling wave solution with speed c connecting the disease-free equilibrium $E_0(1,0,0)$ and endemic equilibrium $E^*(S^*,I^*,R^*)$.*

3.2. Non-existence of traveling waves

In this subsection, by the two-sided Laplace transform, we will establish the non-existence of traveling wave solutions for system (1.2) when $\mathcal{R}_0>1$ and $c\in(0,c^*)$. To apply the two-sided Laplace transform, the prior estimate of exponential decay is need.

Lemma 3.5. *Assume that $\mathcal{R}_0>1$ holds. If $c<c^*$ and $(S(\xi),I(\xi),R(\xi))$ is a non-negative and bounded solution of* (2.1) *satisfying* (2.2), *then there exist two positive constants α and α_0 such that*

$$\sup_{\xi\in\mathbb{R}}\{I(\xi)e^{-\alpha\xi}\}<+\infty, \quad \sup_{\xi\in\mathbb{R}}\{|I'(\xi)|e^{-\alpha\xi}\}<+\infty, \quad \sup_{\xi\in\mathbb{R}}\{|I''(\xi)|e^{-\alpha\xi}\}<+\infty, \tag{3.9}$$

$$\sup_{\xi\in\mathbb{R}}\{R(\xi)e^{-\alpha\xi}\}<+\infty, \quad \sup_{\xi\in\mathbb{R}}\{|R'(\xi)|e^{-\alpha\xi}\}<+\infty, \quad \sup_{\xi\in\mathbb{R}}\{|R''(\xi)|e^{-\alpha\xi}\}<+\infty \tag{3.10}$$

and

$$\sup_{\xi\in\mathbb{R}}\{|1-S(\xi)|e^{-\alpha_0\xi}\}<+\infty. \tag{3.11}$$

Proof. Let $(S(\xi),I(\xi),R(\xi))$ be a non-negative and bounded solution of (2.1) satisfying $\lim_{\xi\to-\infty}(S(\xi),I(\xi),R(\xi))=(1,0,0)$. Then $(S(\xi),I(\xi),R(\xi))$ satisfies

$$\begin{aligned} d_2I''(\xi)-cI'(\xi)+\beta S(\xi)g(I(\xi))+\delta R(\xi)-(\mu+\gamma)I(\xi)=0,\\ d_3R''(\xi)-cR'(\xi)+\gamma I(\xi)-(\mu+\delta)R(\xi)=0. \end{aligned} \tag{3.12}$$

And, by the fluctuate lemma [10], we get $\lim_{\xi\to\pm\infty}(S'(\xi),I'(\xi),R'(\xi))=(0,0,0)$. Set $I'=w$, $R'=z$. Then (3.12) can be rewritten as

$$\psi'=C\psi+f(\xi,\psi),$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\beta-\mu-\gamma}{d_2} & \frac{c}{d_2} & -\frac{\delta}{d_2} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\gamma}{d_3} & 0 & \frac{\mu+\delta}{d_3} & \frac{c}{d_3} \end{pmatrix}, \quad \psi = \begin{pmatrix} I \\ w \\ R \\ z \end{pmatrix} \quad \text{and} \quad f(\xi,\psi) = \begin{pmatrix} 0 \\ \frac{\beta}{d_2}(I(\xi) - S(\xi)g(I(\xi))) \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to show that the characteristic equation for the matrix B is given by $H(\lambda)$. It follows from Lemma 2.2(3) that the equation $H(\lambda) = 0$ has no roots with zero parts. Hence the initial equilibrium $(1, 0, 0)$ is hyperbolic. Then it follows from Stable Manifold Theorem [17, p. 107] (see also, the proof of [35, Lemma 3.1]) that there exists a positive constant $\alpha \in \mathbb{R}$ such that (3.9) and (3.10) hold.

Next we show that (3.11) holds. First, we need to show that $0 \le S(\xi) < 1$ for all $\xi \in \mathbb{R}$. In fact, if not, we suppose that there exists ξ_0 such that $S(\xi_0) \ge 1$. If there exists a local maximum value $S(\xi_1)$ of $S(\xi)$ such that $S(\xi_1) \ge 1$, it follows that $S'(\xi_1) = 0$ and $S''(\xi_1) \le 0$. Therefore,

$$0 = d_1 S''(\xi_1) + \mu(1 - S(\xi_1)) - \beta S(\xi_1) g(I(\xi_1)) < 0,$$

a contraction. Otherwise, there exists ξ_2 such that $S(\xi_2) > 1$ and that $S(\xi)$ is increasing on $(\xi_2, +\infty)$. Note that $S(\xi)$ is bounded, then there exists ξ_3 ($\xi_3 > \xi_2$) such that $S'(\xi_3) \ge 0$ and $S''(\xi_3) \le 0$, again getting a contradiction to the first equality of (2.1). Hence, we have shown that $0 \le S(\xi) < 1$ for all $\xi \in \mathbb{R}$.

Using the fact $0 \le S(\xi) < 1$ for all $\xi \in \mathbb{R}$, we see that $0 < \alpha S(\xi) g(I(\xi)) \le \alpha I(\xi)$ for all $\xi \in \mathbb{R}$, and, by the first inequality of (3.9),

$$\sup_{\xi\in\mathbb{R}} \{S(\xi) g(I(\xi)) e^{-\alpha\xi}\} < +\infty. \tag{3.13}$$

Let $\widetilde{S}(\xi) = 1 - S(\xi) \ge 0$ for all $\xi \in \mathbb{R}$. Integrating the first equation of (2.1) from $-\infty$ to $\xi < 0$ yields

$$d_1 \widetilde{S}'(\xi) - c\widetilde{S}(\xi) = -h(\xi), \quad \xi < 0, \tag{3.14}$$

where

$$h(\xi) = \int_{-\infty}^{\xi} [\beta S(\eta) g(I(\eta)) - \mu(1 - S(\eta))]\, d\eta.$$

Note that $0 \le S(\xi) < 1$ for all $\xi \in \mathbb{R}$ again and (3.13), it follows

$$h(\xi) \le \beta \int_{-\infty}^{\xi} S(\eta) g(I(\eta))\, d\eta \le C_0 e^{\alpha\xi}$$

for some constant $C_0 > 0$, that is, $h(\xi) = O(e^{\alpha\xi})$ as $\xi \to -\infty$. Solving (3.14) yields

$$\widetilde{S}(\xi) = \widetilde{S}(0) e^{\frac{c}{d_1}\xi} + \frac{1}{d_1} e^{\frac{c}{d_1}\xi} \int_{\xi}^{0} e^{-\frac{c}{d_1}\eta} h(\eta)\, d\eta, \quad \xi < 0.$$

Choose $\alpha_0 < \min\{c/d_1, \alpha\}$, we get

$$\widetilde{S}(\xi)e^{-\alpha_0\xi} = \widetilde{S}(0)e^{(c/d_1-\alpha_0)\xi} + \frac{1}{d_1}e^{(c/d_1-\alpha_0)\xi}\int_\xi^0 e^{-\frac{c}{d_1}\eta}h(\eta)\,d\eta, \quad \xi < 0.$$

Note that $h(\xi) = O(e^{\alpha\xi})$ as $\xi \to -\infty$, it is easy to see that $\widetilde{S}(\xi) = 0(e^{\alpha_0\xi})$ as $\xi \to -\infty$. In view of the fact $0 \le \widetilde{S}(\xi) \le 1$ for all $\xi \in \mathbb{R}$, we have $\lim_{\xi\in\mathbb{R}}\{\widehat{S}(\xi)e^{-\alpha_0\xi}\} < +\infty$. Thus, (3.11) holds. The proof is completed. □

To show the non-existence of traveling wave solutions of (1.2), we now define the two-sided Laplace transform for a non-negative and bounded function $F(\xi)$ by

$$\mathcal{L}_F(\lambda) = \int_{\mathbb{R}} e^{-\lambda\xi}F(\xi)\,d\xi$$

for $\lambda \ge 0$. Obviously, $\mathcal{L}_F(\lambda)$ is defined in $[0, \lambda_F^*)$ such that $\lambda_F^* < +\infty$ satisfying $\lim_{\lambda\to\lambda_F^*-0}$ $\mathcal{L}_F(\lambda)$ or $\lambda_F^* = +\infty$.

Theorem 3.6. *Assume that $\mathcal{R}_0 > 1$ holds. Then, for any $c \in (0, c^*)$, system* (1.2) *has no non-negative and bounded traveling solutions $(S(x+ct), I(x+ct), R(x+ct))$ with speed c satisfying the boundary condition* (2.2)*.*

Proof. Suppose that there exists a non-negative and bounded traveling wave solution $(S(\xi), I(\xi), R(\xi))$ of system (2.1) satisfying (2.2). Set

$$\mathcal{L}_I(\lambda) = \int_{\mathbb{R}} e^{-\lambda\xi}I(\xi)\,d\xi, \ \ \lambda \in [0, \lambda_I^*), \quad \mathcal{L}_R(\lambda) = \int_{\mathbb{R}} e^{-\lambda\xi}R(\xi)\,d\xi, \ \ \lambda \in [0, \lambda_R^*).$$

By Lemma 3.5, it is easy to see that $\lambda_I^* \ge \alpha$ and $\lambda_R^* \ge \alpha$.

Now we rewrite the second and third equations of (2.1) as follows:

$$\begin{aligned} d_2I''(\xi) - cI'(\xi) + (\beta - \mu - \gamma)I(\xi) + \delta R(\xi) &= Q(\xi), \\ d_3R''(\xi) - cR'(\xi) - (\mu + \delta)R(\xi) &= -\gamma I(\xi), \end{aligned} \tag{3.15}$$

where $Q(\xi) = \beta(I(\xi) - S(\xi)g(I(\xi))$. Taking the two-sided Laplace transform for (3.15) yields

$$h_2(\lambda)\mathcal{L}_I(\lambda) + \delta\mathcal{L}_R(\lambda) = \mathcal{L}_Q(\lambda), \quad h_3(\lambda)\mathcal{L}_R(\lambda) = -\gamma\mathcal{L}_I(\lambda), \tag{3.16}$$

where $\mathcal{L}_Q(\lambda) = \int_{\mathbb{R}} e^{-\lambda\xi}Q(\xi)\,d\xi$.

We claim that $\lambda_I^* = \lambda_R^* < +\infty$. Indeed, we first show that $\lambda_I^* < +\infty$ and $\lambda_R^* < +\infty$. In fact, by the second equation of (2.1), we get

$$\Delta_2(\lambda) := (d_2\lambda^2 - c\lambda - (\mu+\gamma))\mathcal{L}_I(\lambda) + \beta\int_{\mathbb{R}} e^{-\lambda\xi}S(\xi)g(I(\xi))\,d\xi + \delta\mathcal{L}_R(\lambda) = 0.$$

Since $\mathcal{L}_I(\lambda) > 0$, $\mathcal{L}_R(\lambda) > 0$ and $\int_{\mathbb{R}} e^{-\lambda\xi} S(\xi) g(I(\xi))\, d\xi > 0$ for $\lambda \in [0, \lambda_I^*)$, then $\lambda_I^* = +\infty$ implies that $\lim_{\lambda\to+\infty} \Delta_2(\lambda) = \infty$, a contradiction. Thus, $\lambda_I^* < +\infty$. Similarly, we also get $\lambda_R^* < +\infty$. Secondly, we prove $\lambda_I^* = \lambda_R^*$. On the contrary, if $\lambda_I^* < \lambda_R^*$, then $\lim_{\lambda\to\lambda_I^*-0} \mathcal{L}_I(\lambda) = +\infty$ and $\lim_{\lambda\to\lambda_I^*-0} \mathcal{L}_R(\lambda) = \mathcal{L}_R(\lambda_I^*) < +\infty$. This contradicts to the second equality of (3.16). Thus, $\lambda_I^* \geq \lambda_R^*$. On the other hand, if $\lambda_I^* > \lambda_R^*$, then $\lim_{\lambda\to\lambda_R^*-0} \mathcal{L}_R(\lambda) = +\infty$ and $\lim_{\lambda\to\lambda_R^*-0} \mathcal{L}_I(\lambda) = \mathcal{L}_I(\lambda_R^*) < +\infty$. Noth that

$$|Q(\xi)| = \beta|I(\xi) - S(\xi)g(I(\xi)| \leq \beta(|1 - S(\xi)| + \alpha|I(\xi)|)|I(\xi)|, \tag{3.17}$$

here we have used the fact $0 \leq S(\xi) < 1$ for all $\xi \in \mathbb{R}$. By Lemma 3.5, we get

$$\lim_{\lambda\to\lambda_R^*-0} \mathcal{L}_Q(\lambda) = \mathcal{L}_Q(\lambda_R^*) < +\infty,$$

which contradicts to the first equality of (3.16). Thus $\lambda_I^* \leq \lambda_R^*$. Hence, we get $\lambda^* := \lambda_I^* = \lambda_R^*$.

We next show that $h_2(\lambda^*) < 0$ and $h_3(\lambda^*) < 0$ hold. In fact, if $h_2(\lambda^*) \geq 0$, then

$$h_2(\lambda^*)\mathcal{L}_I(\lambda^*) + \delta\mathcal{L}_R(\lambda^*) = +\infty > |\mathcal{L}_Q(\lambda^*)|,$$

which contradicts the first equality of (3.16). Hence, we get $h_2(\lambda^*) < 0$. Similarly, we can prove $h_3(\lambda^*) < 0$. Also, by (3.17) and Lemma 3.5, we see $|\mathcal{L}_Q(\lambda^*)| < +\infty$. By (3.16), we get

$$H(\lambda)\mathcal{L}_I(\lambda) = h_3(\lambda)\mathcal{L}_Q(\lambda),$$

which implies

$$H(\lambda^*) = \lim_{\lambda\to\lambda^*-0} \frac{h_3(\lambda)\mathcal{L}_Q(\lambda)}{\mathcal{L}_I(\lambda)} = 0,$$

contradicting Lemma 2.2(2). The proof is completed. □

4. Numerical simulations and summary

In this section, we first carry out numerical simulations to illustrate the existence of traveling waves for (1.2) obtained by Theorem 3.4 for the two cases, $\beta > \mu + \gamma$ and $\beta < \mu + \gamma$, respectively. For convenience, we only present the diagrams in the domain of $x \in [-50, 50]$. Here most of the values of parameters are taken in [9, 18] and the rest of the parametric values are assumed for numerical computation.

Example 4.1 (Case (C1): $\beta > \mu + \gamma$)**.** For system (1.2), we set the parameter values as: $d_1 = 2$, $d_2 = 1.2$, $d_3 = 1.5$, $\mu = 0.0000351$ (per day), $\beta = 0.4$ (per day), $\delta = 0.805$ (per day), $\gamma = 0.03521$ (per day) and $\alpha = 2$. By a direct calculation, we can get $\mathcal{R}_0 \approx 10918 > 1$, $(S^*, I^*, R^*) \approx (0.000267, 0.958, 0.0419)$, and $\beta > \mu + \gamma$ holds. Therefore, by Theorem 3.4, we conclude that system (1.2) has a positive traveling wave connecting the disease-free equilibrium $E_0 = (1, 0, 0)$ and the endemic equilibrium $E^* \approx (0.000267, 0.958, 0.0419)$. See Figure 4.1 for the simulation diagram of traveling waves.

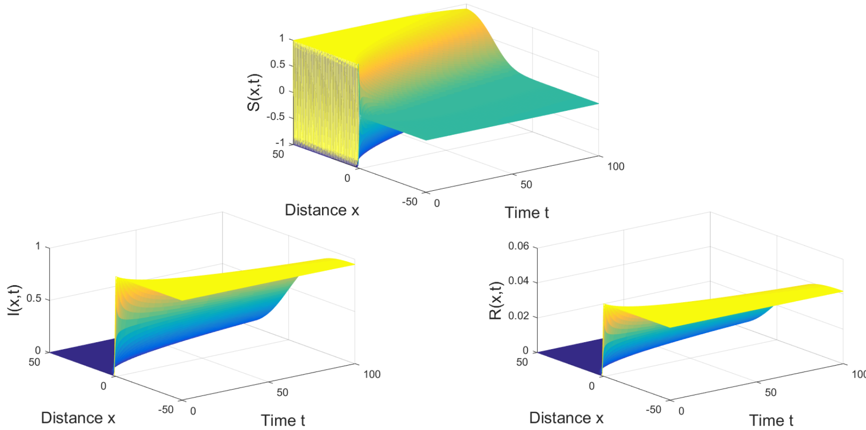

Figure 4.1: The traveling wave solution of (1.2) found with the parameters satisfying $\beta > \mu + \gamma$ in Example 4.1.

Example 4.2 (Case (C3): $\beta < \mu + \gamma$**.** For system (1.2), we set the parameter values as: $d_1 = 2$, $d_2 = 1.2$, $d_3 = 1.5$, $\mu = 0.0000351$ (per day), $\beta = 0.4$ (per day), $\delta = 0.5$ (per day), $\gamma = 0.4$ (per day) and $\alpha = 3$. By a direct calculation, one can get $\mathcal{R}_0 \approx 6331 > 1$, $(S^*, I^*, R^*) \approx (0.000421, 0.555, 0.444)$, and $\beta < \mu + \gamma$ holds. Hence, by Theorem 3.4, we conclude that system (1.2) has a positive traveling wave connecting the disease-free equilibrium $E_0 = (1, 0, 0)$ and the endemic equilibrium $E^* \approx (0.000421, 0.555, 0.444)$. See Figure 4.2 for the simulation diagram of traveling waves.

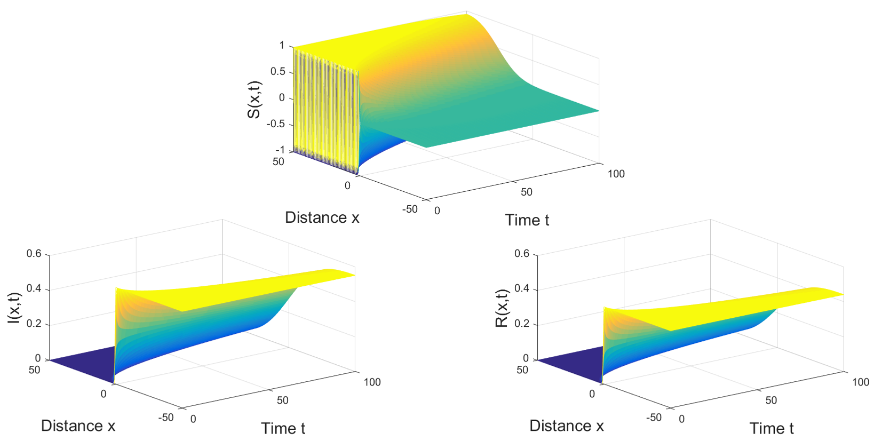

Figure 4.2: The traveling wave solution of (1.2) found with the parameters satisfying $\beta < \mu + \gamma$ in Example 4.2.

In this paper, based on the SIRI epidemic model proposed by [9,14,19], we incorporate the diffusion of individuals to the system (1.1) and then introduce an spatial SIRI epidemic model with nonlinear incidence rate. For this mathematical model, we construct the upper-lower solutions and Lyapunov function for (2.1), together with the Schauder fixed point theorem, establish existence of traveling wave solutions for the model connecting the two equilibria E_0 and E^* with speed $c > c^*$. Furthermore, based on the two-sided Laplace transform, we show that the model has no such a traveling wave solution with speed $c < c^*$. And also, we give two numerical simulations to illustrate our analytic results. Biologically, a traveling wave solution connecting the two equilibria E_0 and E^* accounts for the transition from disease uninfected equilibrium E_0 to the endemic infected equilibrium E^* as time goes, and the wave speed c may explain the spatial spread speed of the disease, which may measure how fast the disease invades geographically. Hence, the study of the traveling waves is a very important topic for disease models with spatial heterogeneity.

Acknowledgments

The authors are grateful to the three anonymous referees for their helpful suggestions and correcting many grammatical mistakes for improving the original manuscript.

Z. Xu was supported by the NSF of Guangdong Province (2016A030313426) and the HLUCF of South China Normal University (2016YN30). Y. Huang was supported by Tianyuan special funds for the NSF of China (A11621401).

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