TAIWANESE JOURNAL OF MATHEMATICS
Vol. 22, No. 2, pp. 325–337, April 2018
DOI: 10.11650/tjm/170806

Remark on Proper Holomorphic Maps Between Reducible Bounded Symmetric Domains

Aeryeong Seo

Abstract. In this paper we study proper holomorphic maps between bounded symmetric domains when the source domain is not irreducible. More precisely, we provide sufficient conditions for semi-product proper holomorphic maps to be product proper. As an application we characterize proper holomorphic maps between equidimensional bounded symmetric domains.

1. Introduction

Proper holomorphic maps between bounded domains in the Euclidean spaces have been studied quite intensively since Alexander proved that any proper holomorphic self-map of the unit ball is an automorphism [1]. Henkin and Tumanov generalized this result to irreducible bounded symmetric domains of rank ≥ 2. One of the most important theorem along these lines was proved by Tsai [18]: *let* Ω *and* ω *be bounded symmetric domains. Suppose that* Ω *is irreducible and* $\operatorname{rank}(\Omega) \geq \operatorname{rank}(\omega) \geq 2$. *Then* $\operatorname{rank}(\Omega) = \operatorname{rank}(\omega)$ *and any proper holomorphic map* $f\colon \Omega \to \omega$ *is a totally geodesic isometric embedding up to normalizing constants with respect to the Bergman metrics.* Furthermore, Tu showed that for equidimensional bounded symmetric domains Ω, ω where Ω is irreducible and $\operatorname{rank}(\Omega) \geq 2$, every proper holomorphic map from Ω to ω is a biholomorphism [19]. On the other hand, in case the source domain is reducible and it does not have an irreducible factor of complex dimension equal to 1, to the author's knowledge it is not known whether or not there exists a proper holomorphic self-map which is not an automorphism.

In this paper we study the proper holomorphic maps between the reducible bounded symmetric domains and prove the following:

Theorem 1.1. *Let* $\Omega_1, \ldots, \Omega_k$, $\omega_1, \ldots, \omega_l$ *be irreducible bounded symmetric domains. Let* $\Omega = \Omega_1 \times \cdots \times \Omega_k$ *and* $\omega = \omega_1 \times \cdots \times \omega_l$. *Suppose that*

(1) $\dim \Omega_{i_1} + \dim \Omega_{i_2} \geq \dim \omega_j$ *for any* $i_1, i_2 \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$, *and*

(2) $\operatorname{rank}(\Omega_1) + \cdots + \operatorname{rank}(\Omega_k) \geq \operatorname{rank}(\omega_1) + \cdots + \operatorname{rank}(\omega_l)$.

Received June 18, 2017; Accepted August 22, 2017.
Communicated by Sai-Kee Yeung.
2010 *Mathematics Subject Classification.* 32M15, 32H35.
Key words and phrases. proper holomorphic map, bounded symmetric domain.

Suppose that there is a proper rational map $f\colon \Omega \to \omega$. Then $k = l$ and f is a product map, i.e., there is a permutation σ of $\{1, \dots, k\}$ such that $f = (f_1, \dots, f_k)$ where $f_j\colon \Omega_{\sigma(j)} \to \omega_j$ is a proper holomorphic map for each $j = 1, \dots, k$.

Theorem 1.2. *Any proper holomorphic map between bounded symmetric domains with the same dimension is a product map. In particular for $\Omega = U^k \times \Omega'$ and $\omega = U^l \times \omega'$ where U is the unit disc in $\mathbb{C}$ and Ω', ω' are products of the unit balls of dimension ≥ 2 and irreducible bounded symmetric domains of rank ≥ 2, if there is a proper holomorphic map f from Ω to ω, then $k = l$ and f is of the form*

$$(z_1, \dots, z_k, z) \mapsto (\phi_1(z_{\sigma(1)}), \dots, \phi_k(z_{\sigma(k)}), f'(z)),$$

where

$$\phi_j(z_{\sigma(j)}) = e^{i\theta_j} \prod_{\mu=1}^{m_j} \frac{a_{j\mu} - z_{\sigma(j)}}{1 - \overline{a}_{j\mu} z_{\sigma(j)}}$$

with some $(\theta_1, \dots, \theta_k) \in [0, 2\pi)^k$, $a_{ij} \in U$ for each i, j, a permutation σ of $\{1, \dots, k\}$ and f' is a biholomorphism from Ω' to ω'.

In Theorem 1.2 the maps ϕ_j, $1 \leq j \leq k$ comes from the known classification result of proper holomorphic self-maps of the polydisc U^k (cf. [16]).

Let $\Omega^I_{r,s}$ be a bounded symmetric domain of type I defined by

$$\Omega^I_{r,s} = \{Z \in M^{\mathbb{C}}_{r,s} : I_{r,r} - ZZ^* > 0\}.$$

Here we denote by > 0 positive definiteness of square matrices, by $M^{\mathbb{C}}_{r,s}$ the set of $r \times s$ complex matrices and by $I_{r,r}$ the $r \times r$ identity matrix. Besides for $Z \in M^{\mathbb{C}}_{r,s}$, denote by Z^* the complex conjugate of Z. Let $f\colon \Omega^I_{2,2} \times \Omega^I_{2,2} \to \Omega^I_{4,4}$ be a proper holomorphic map defined by $f(Z, W) = \left(\begin{smallmatrix} Z & 0 \\ 0 & W \end{smallmatrix}\right)$. By composing proper holomorphic maps from $\Omega^I_{4,4}$ into bounded symmetric domains with higher rank, one can produce a lot of proper holomorphic maps from $\Omega^I_{2,2} \times \Omega^I_{2,2}$ that are not product maps.

The study on the structure of the set of proper holomorphic maps between the given domains along the lines of Theorem 1.1 initiated by Remmert and Stein [14]. They proved that for given domains $\Omega = \Omega_1 \times \Omega_2$ and $\omega = \omega_1 \times \omega_2$ with bounded planar domains Ω_1, Ω_2, ω_1, ω_2, any proper holomorphic map from Ω to ω is a product map. Its generalization can be found in [12]. Recently Janardhanan [9] and Chakrabarti-Verma [5] extended it to the product of compact Riemann surfaces and that of pseudoconvex domains satisfying Condition R.

2. Preliminaries

In [4], Cartan introduced the notion of Riemannian symmetric spaces. Among them, Hermitian symmetric spaces of non-compact type are realized as bounded domains in the complex Euclidean spaces and those are called bounded symmetric domains. All irreducible bounded symmetric domains are consisted of 4 classical types and 2 exceptional types. Here is the list [4, 7]:

(1) $\Omega^{I}_{r,s} = \{Z \in M^{\mathbb{C}}_{r,s} : I_r - ZZ^* > 0\}$,

(2) $\Omega^{II}_{n} = \{Z \in M^{\mathbb{C}}_{n,n} : I_n - ZZ^* > 0, Z^t = -Z\}$,

(3) $\Omega^{III}_{n} = \{Z \in M^{\mathbb{C}}_{n,n} : I_n - ZZ^* > 0, Z^t = Z\}$,

(4) $\Omega^{IV}_{n} = \{Z = (z_1, \ldots, z_n) \in \mathbb{C}^n : ZZ^* < 1, 0 < 1 - 2ZZ^* + |ZZ^t|^2\}$,

(5) $\Omega^{V}_{16} = \{z \in M^{\mathbb{O}_{\mathbb{C}}}_{1,2} : 1 - (z|z) + (z^{\#}|z^{\#}) > 0, 2 - (z|z) > 0\}$, and

(6) $\Omega^{VI}_{27} = \{z \in H_3(\mathbb{O}_{\mathbb{C}}) : 1 - (z|z) + (z^{\#}|z^{\#}) - |\det z|^2 > 0, 3 - 2(z|z) + (z^{\#}|z^{\#}) > 0, 3 - (z|z) > 0\}$.

The notation to define bounded symmetric domains of type V and VI will be given in Section 2.2. From now on, we recall boundary components of Hermitian symmetric spaces of non-compact type. For more detail, refer to [11, 20].

2.1. Boundary components of irreducible bounded symmetric domains

Let X_0 be a Hermitian symmetric space of non-compact type. Let G_0 be the identity component of the isometry group of X_0 with respect to the Bergman metric of X_0 and $K_0 \subset G_0$ the isotropy subgroup at $o \in X_0$. Then X_0 is biholomorphic to G_0/K_0. Denote by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ the Lie algebras of G_0 and K_0 respectively. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{m}_0$ be the Cartan decomposition. Let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{m} = \mathfrak{m}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathfrak{g}_c = \mathfrak{k}_0 + \sqrt{-1}\mathfrak{m}_0$ be a Lie algebra of compact type and G_c the corresponding connected Lie group of $\mathfrak{g}_c$. Then $X_c = G_c/K_0$ is the compact dual of X_0. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. Note that $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and let $\mathfrak{g}^\alpha$ denote the root space with respect to a root $\alpha \in \Delta$. Let $\Delta_{\mathfrak{k}}$, $\Delta_{\mathfrak{m}}$ denote the set of compact, non-compact roots of $\mathfrak{g}$ with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ respectively and choose an order of Δ such that the set of positive non-compact roots $\Delta^+_{\mathfrak{m}}$ satisfies that $\sum_{\alpha \in \Delta^+_{\mathfrak{m}}} \mathfrak{g}^\alpha = T^{1,0}_o X_0$. Here $T^{1,0}X_0$ denotes the holomorphic tangent bundle of X_0.

For $\alpha, \beta \in \Delta$, one says that α and β are strongly orthogonal if and only if $\alpha \pm \beta \notin \Delta$. Let $\Pi := \{\alpha_1, \ldots, \alpha_r\}$ denote a maximal set of strongly orthogonal positive non-compact roots of $\mathfrak{g}$. Then X_0 is of rank r. Let $\Lambda \subset \Pi$. Denote $\mathfrak{g}_\Lambda$ the derived algebra of $\mathfrak{h} + \sum_{\alpha \perp \Pi \setminus \Lambda} \mathfrak{g}^\alpha$ where $\perp$ is the orthogonality with respect to the inner product induced by the Killing form of $\mathfrak{g}$. Let G_Λ denote the Lie subgroup of G corresponding to $\mathfrak{g}_\Lambda$ and $G_{\Lambda,0}$ denote

$G_0 \cap G_\Lambda$. Let $X_\Lambda = G_\Lambda \cdot o$ and $X_{\Lambda,0} = G_{\Lambda,0} \cdot o \subset X_0$. If $\Lambda = \Pi - \{\alpha\}$ for $\alpha \in \Pi$, then X_Λ and $X_{\Lambda,0}$ are called *maximal characteristic subspaces* of X_c and X_0 respectively.

Let ∂X_0 be the topological boundary of X_0 in X_c and $U = \{z \in \mathbb{C} : |z| < 1\}$ the unit disc. A holomorphic map $g\colon U \to X_c$ such that $g(U) \subset \partial X_0$ is called a holomorphic arc in ∂X_0. A finite sequence $\{g_1, \dots, g_s\}$ of holomorphic arcs in ∂X_0 is called a chain of holomorphic arcs in ∂X_0 if $f_j(U) \cap f_{j+1}(U) \neq \emptyset$ for any $j = 1, \dots, s-1$. One can give an equivalence class on ∂X_0 such that for $z_1, z_2 \in \partial X_0$, $z_1 \sim z_2$ if and only if there is a chain of holomorphic arcs $\{g_1, \dots, g_s\}$ in ∂X_0 with $z_1 \in g_1(U)$ and $z_2 \in g_s(U)$. The equivalence classes are the boundary components of ∂X_0 in X_c.

Theorem 2.1 (Wolf, [20])**.** *The G_0 orbits on the topological boundary of X_0 in its compact dual are the sets*

$$G_0(c_{\Pi-\Lambda}o) = \bigcup_{k \in K_0} kc_{\Pi-\Lambda}X_{\Lambda,0}, \quad \Lambda \subsetneq \Pi$$

where $c_{\Pi-\Lambda}$ is the Cayley transformation with respect to $\Pi - \Lambda$. Furthermore the boundary components of X_0 in X are the sets $kc_{\Pi-\Lambda}X_{\Lambda,0}$ with $k \in K_0$ and $\Lambda \subsetneq \Pi$. These are Hermitian symmetric spaces of non-compact type and rank is given by

$$\operatorname{rank}(kc_{\Pi-\Lambda}X_{\Lambda,0}) = |\Lambda|.$$

The boundary components with $|\Lambda| = |\Pi| - 1$ are called the maximal faces of X_0 (cf. [11, Definition 1.5.2]). Note that any proper holomorphic map $f\colon \Omega \to \omega$ between irreducible bounded symmetric domains with $\operatorname{rank}\Omega \geq \operatorname{rank}\omega \geq 2$ has a rational extension over the compact duals of Ω and ω by Mok and Tsai in [11].

Lemma 2.2. *Let $f, g\colon \Omega \to \omega$ be proper holomorphic maps between irreducible bounded symmetric domains with $\operatorname{rank}\Omega \geq \operatorname{rank}\omega \geq 2$. Suppose that for any maximal face $X \subset \partial\Omega$, f and g map X into the same maximal face of $\partial\omega$. Then $f \equiv g$.*

Proof. The lemma is due to a result of Mok and Tsai in [11]. Here is a summary. Let Ω_c and ω_c denote the compact duals of Ω and ω respectively. Let $\mathcal{D}(\Omega)$, $\mathcal{D}(\Omega_c)$, $\mathcal{D}(\omega)$ and $\mathcal{D}(\omega_c)$ be the moduli spaces of maximal characteristic symmetric spaces contained in Ω, Ω_c, ω and ω_c respectively. Under the condition that $\operatorname{rank}\Omega \geq \operatorname{rank}\omega$, f maps the maximal characteristic symmetric spaces of Ω into those of ω. This phenomenon induces a meromorphic map $f^\#\colon \mathcal{D}(\Omega) \to \mathcal{D}(\omega)$ and $f^\#$ admits a rational extension $\widetilde{f}^\#\colon \mathcal{D}(\Omega_c) \to \mathcal{D}(\omega_c)$. Furthermore $f^\#$ induces a rational extension $\widetilde{f}\colon \Omega_c \to \omega_c$ of f. In this process let us assume that for each maximal face $X \subset \partial\Omega$, the images of f and g are contained in the same maximal face. This implies that $\widetilde{f}^\# \equiv \widetilde{g}^\#$ on the collection of all maximal faces, which is a maximal totally real subset of $\mathcal{D}(\Omega_c)$. Hence $\widetilde{f}^\# \equiv \widetilde{g}^\#$ on $\mathcal{D}(\Omega_c)$ and consequently $f \equiv g$ on Ω (on Ω_c). $\square$

Corollary 2.3. *Let $\Omega \subset \mathbb{C}^n$ and ω be irreducible bounded symmetric domains of* $\operatorname{rank}\Omega \geq \operatorname{rank}\omega$ *and M a connected bounded domain in complex Euclidean space. Let $f\colon \Omega\times M \to \omega$ be a holomorphic map such that $f(\,\cdot\,,z)\colon \Omega \to \omega$ is a proper holomorphic map for each $z \in M$. Then f does not depend on $z \in M$.*

Proof. Suppose that $\operatorname{rank}\omega \geq 2$. Then $f(\,\cdot\,,z)$ has a rational extension over the boundary and for each $p \in \partial\Omega$, $f(p,\cdot)\colon M \to \mathbb{C}^n$ is a holomorphic map such that $f(p,M) \subset \partial\omega$. Let $X \subset \partial\Omega$ be a boundary component. Suppose that the boundary components of ω containing $f(X,z_1)$ and $f(X,z_2)$ are different. In particular for $p \in X$, $f(p,z_1)$ and $f(p,z_2)$ belong to the different boundary components. However if we consider the holomorphic map $f(p,\cdot)\colon M \to \mathbb{C}^n$, $f(p,z_1)$ and $f(p,z_2)$ should belong to the same boundary component. By Lemma 2.2, we obtain the result.

In case $\operatorname{rank}\Omega = 1$, f is a holomorphic map from $\mathbb{B}^n \times M$ to $\mathbb{B}^N$ for some $n \leq N$. Fix $p \in \partial\mathbb{B}^n$ and let $U_p = \{\lambda p \in \mathbb{B}^n : \lambda \in \mathbb{C}\}$ which is biholomorphic to the unit disc in $\mathbb{C}$. By Fatou's theorem, for generic $\theta \in [0,2\pi)$, $f(e^{i\theta}p,\cdot) := \lim_{r\to 1} f(re^{i\theta}p,\cdot)\colon M \to \mathbb{C}^N$ exists and it is a holomorphic function. Since $f(e^{i\theta}p,M) \subset \partial\mathbb{B}^N$, we obtain that $f(e^{i\theta}p,\cdot)$ is a constant map for each generic $\theta \in [0,2\pi)$. In particular, $f(e^{i\theta}p,z)$ does not depend on $z \in M$ and $\frac{\partial f}{\partial z_l}(e^{i\theta}p,z) = \frac{\partial f}{\partial \bar{z}_l}(e^{i\theta}p,z) = 0$ for each l. Since this holds for each $p \in \partial\mathbb{B}^n$ and generic $\theta \in [0,2\pi)$, we obtain that $\frac{\partial f}{\partial z_l} \equiv \frac{\partial f}{\partial \bar{z}_l} \equiv 0$ for each l on $\mathbb{B}^n \times M$. This implies that f does not depend on $z \in M$. □

Remark 2.4. An alternative proof of the case $\operatorname{rank}(\Omega) = 1$ can be obtained through [13, Proposition 2.3].

2.2. Irreducibility of generic norms

Let us briefly introduce the notation for the exceptional cases Ω_{16}^{V} and Ω_{27}^{VI}. Refer to [15] for more details. Let $\mathbb{O}_{\mathbb{C}}$ denote the complex 8-dimensional algebra of complex octonions. For $a = (a_0,a_1,\ldots,a_7) \in \mathbb{O}_{\mathbb{C}}$ with $a_i \in \mathbb{C}$, let $a \mapsto \widetilde{a} := (a_0,-a_1,\ldots,-a_7)$ denote the Cayley conjugation and $a \mapsto \overline{a} := (\overline{a}_0,\overline{a}_1,\ldots,\overline{a}_7)$ the complex conjugation. The Hermitian scalar product is given by $(a|b) = a\widetilde{\overline{b}}+\widetilde{a}\overline{b}$. Let $H_3(\mathbb{O}_{\mathbb{C}})$ be the complex vector space of 3×3 matrices with entries in $\mathbb{O}_{\mathbb{C}}$ which are Hermitian with respect to the Cayley conjugation in $\mathbb{O}_{\mathbb{C}}$. Explicitly $A \in H_3(\mathbb{O}_{\mathbb{C}})$ can be expressed as

$$A = \begin{pmatrix} \alpha_1 & a_3 & \widetilde{a}_2 \\ \widetilde{a}_3 & \alpha_2 & a_1 \\ a_2 & \widetilde{a}_1 & \alpha_3 \end{pmatrix} \quad \text{with } a_i \in \mathbb{O}_{\mathbb{C}} \text{ and } \alpha_i \in \mathbb{C} \text{ for all } i = 1,2,3. \tag{2.1}$$

For $A \in H_3(\mathbb{O}_{\mathbb{C}})$ expressed by (2.1), let $A^{\#} \in H_3(\mathbb{O}_{\mathbb{C}})$ be the adjoint matrix of A expressed by

$$A^{\#} = \begin{pmatrix} \alpha_2\alpha_3 - a_1\widetilde{a}_1 & \widetilde{a}_2\widetilde{a}_1 - \alpha_3 a_3 & \widetilde{\widetilde{a}_1\widetilde{a}_3 - \alpha_2 a_2} \\ \widetilde{\widetilde{a}_2\widetilde{a}_1 - \alpha_3 a_3} & \alpha_3\alpha_1 - a_2\widetilde{a}_2 & \widetilde{a}_3\widetilde{a}_2 - \alpha_1 a_1 \\ \widetilde{a}_1\widetilde{a}_3 - \alpha_2 a_2 & \widetilde{\widetilde{a}_3\widetilde{a}_2 - \alpha_1 a_1} & \alpha_1\alpha_2 - a_3\widetilde{a}_3 \end{pmatrix}. \tag{2.2}$$

The Hermitian scalar product on $H_3(\mathbb{O}_{\mathbb{C}})$ is given by $(A|B) = \sum_{i=1}^{3} \alpha_i\overline{\beta}_i + \sum_{i=1}^{3}(a_i|b_i)$. Explicitly,

$$\begin{aligned} (A|A) &= \sum_{i=1}^{3} |\alpha_i|^2 + 2\sum_{i=1}^{3}(|a_{i0}|^2 + \cdots + |a_{i7}|^2), \\ (A^{\#}|A^{\#}) &= |\alpha_2\alpha_3 - a_1\widetilde{a}_1|^2 + |\alpha_3\alpha_1 - a_2\widetilde{a}_2|^2 \\ &\quad + |\alpha_1\alpha_2 - a_3\widetilde{a}_3|^2(\widetilde{a}_3\widetilde{a}_2 - \alpha_1 a_1|\widetilde{a}_3\widetilde{a}_2 - \alpha_1 a_1) \\ &\quad + (\widetilde{a}_1\widetilde{a}_3 - \alpha_2 a_2|\widetilde{a}_1\widetilde{a}_3 - \alpha_2 a_2) + (\alpha_1\alpha_2 - a_3\widetilde{a}_3|\alpha_1\alpha_2 - a_3\widetilde{a}_3), \\ |\det A|^2 &= \left|\alpha_1\alpha_2\alpha_3 - \sum_{i=1}^{3} \alpha_i a_i\widetilde{a}_i + a_1(a_2a_3) + (\widetilde{a}_3\widetilde{a}_2)\widetilde{a}_1\right|^2 \end{aligned}$$

with $a_i = (a_{i0}, \ldots, a_{i7}) \in \mathbb{O}_{\mathbb{C}}$ for $i = 1, 2, 3$. Let $M_{1,2}^{\mathbb{O}_{\mathbb{C}}}$ denote the set of 1×2 complex octonion matrices. For $z = (z_1, z_2) \in M_{1,2}^{\mathbb{O}_{\mathbb{C}}}$, we identify z with

$$\begin{pmatrix} 0 & z_2 & \widetilde{z}_1 \\ \widetilde{z}_2 & 0 & 0 \\ z_1 & 0 & 0 \end{pmatrix} \in H_3(\mathbb{O}_{\mathbb{C}})$$

and apply the same notation $\#$, $(\cdot, \cdot)$ and so on.

Denote $\mathrm{SM}_{n,n}^{\mathbb{C}}$ (resp. $\mathrm{ASM}_{n,n}^{\mathbb{C}}$) the set of symmetric (resp. antisymmetric) $n \times n$ complex matrices. Let $S_{r,s}^{I}$, S_n^{II}, S_n^{III}, S_n^{IV}, S^{V} and S^{VI} be generic norms to the corresponding domains (cf. [10]) defined by

$$\begin{aligned} S_{r,s}^{I}(Z, \overline{Z}) &= \det(I_r - ZZ^*) && \text{for } Z \in M_{r,s}^{\mathbb{C}}, \\ S_n^{II}(Z, \overline{Z}) &= s_n^{II}(Z) && \text{for } Z \in \mathrm{ASM}_{n,n}^{\mathbb{C}}, \\ S_n^{III}(Z, \overline{Z}) &= \det(I_n - ZZ^*) && \text{for } Z \in \mathrm{SM}_{n,n}^{\mathbb{C}}, \\ S_n^{IV}(Z, \overline{Z}) &= 1 - 2ZZ^* + |ZZ^t|^2 && \text{for } Z \in \mathbb{C}^n, \\ S^{V}(Z, \overline{Z}) &= 1 - (Z|Z) + (Z^{\#}|Z^{\#}) && \text{for } Z \in M_{1,2}^{\mathbb{O}_{\mathbb{C}}}, \\ S^{VI}(Z, \overline{Z}) &= 1 - (Z|Z) + (Z^{\#}|Z^{\#}) - |\det Z|^2 && \text{for } Z \in H_3(\mathbb{O}_{\mathbb{C}}) \end{aligned}$$

with $\det(I_n - ZZ^*) = s_n^{II}(Z)^2$ for some polynomial $s_n^{II}(Z)$ and $Z \in \mathrm{ASM}_{n,n}^{\mathbb{C}}$ (cf. [10]). Note that the topological boundary of an irreducible bounded symmetric domain is contained in the zero set of the generic norm of the domain.

Lemma 2.5. *Generic norms of irreducible bounded symmetric domains are irreducible.*

Proof. In case of the classical bounded symmetric domains, it is proved in [17]. Since the same method can be applied to type V, we only prove the lemma for the bounded symmetric domains of type VI. By the explicit expression (2.2), the total degree of $S^{VI}(A)$ is 6 which come from $|\det A|^2$ and the maximal degrees in variables $\mathrm{Re}\, a_{ij}$ and $\mathrm{Im}\, a_{ij}$ are 4 for any $i = 1, 2, 3$ and $j = 0, 1, \ldots, 7$ which come from $|\det A|^2$ and the first line of the expression of $(A^{\#}|A^{\#})$. Note that if we rearrange the equation in descending power of variable $\mathrm{Re}\, a_{ij}$, the coefficient of $(\mathrm{Re}\, a_{ij})^4$ is $|\alpha_i|^2 + 1$. Suppose that $S^{VI}(A)$ is reducible, that is, $S^{VI}(A) = P_1(A)P_2(A)$ with nonzero polynomial $P_1(A)$ and $P_2(A)$.

Suppose that $(\mathrm{Re}\, a_{10})^4$ term belongs to P_1. This implies that all other $(\mathrm{Re}\, a_{ij})^4$ variables should appear in P_1 but not in P_2 and $P_2(A)$ should contain $|\alpha_1|^2 + 1$. However there is no $(\mathrm{Re}\, a_{ij})|\alpha_1|^2$ term in $S^{VI}(A)$ for $i \neq 1$ and $j \neq 0$, $P_1(A)$ and hence $P_2(A)$ cannot contain $(\mathrm{Re}\, a_{10})^4$. Hence $S^{VI}(A) = P_1(A)P_2(A)$ and $P_1(A)$, $P_2(A)$ should contain $(\mathrm{Re}\, a_{10})^k$, $(\mathrm{Re}\, a_{10})^{4-k}(1 + |\alpha_1|^2)$ respectively or vice versa for some $k \in \{1, 2, 3\}$.

Input $a_i = 0$ and $\alpha_j = 0$ for all $i = 2, 3$ and $j = 1, 2, 3$. Then $S^{VI}(A)$ equals to S^{IV} in $\mathbb{C}^8$ which is irreducible. This gives us a contradiction and the lemma is proved. □

3. Proof of Theorem 1.1

Definition 3.1. Let $\Omega_1, \ldots, \Omega_k, \omega_1, \ldots, \omega_l$ be bounded domains in $\mathbb{C}^{\mu_1}, \ldots, \mathbb{C}^{\mu_k}, \mathbb{C}^{\nu_1}, \ldots, \mathbb{C}^{\nu_l}$ respectively. Let $\Omega = \Omega_1 \times \cdots \times \Omega_k$ and $\omega = \omega_1 \times \cdots \times \omega_l$. Denote

$$\partial_i\Omega := \Omega_1 \times \cdots \times \Omega_{i-1} \times \partial\Omega_i \times \Omega_{i+1} \times \cdots \times \Omega_k$$

and

$$\Omega_{\widehat{i_1}\ldots\widehat{i_\mu}} = \Omega_1 \times \cdots \times \widehat{\Omega}_{i_1} \times \cdots \times \widehat{\Omega}_{i_\mu} \times \cdots \times \Omega_k$$

for $1 \le i_1 < i_2 < \cdots < i_\mu \le k$. Here the circumflex over a term means that it is to be omitted. Let $f\colon \Omega \to \omega$ be a proper holomorphic map. Denote $f = (f_1, \ldots, f_l)$ where f_j is $\pi_j \circ f$ with the projection $\pi_j\colon \Omega \to \Omega_j$. For $w = (w_1, \ldots, \widehat{w}_{i_1}, \ldots, \widehat{w}_{i_\mu}, \ldots, w_k) \in \Omega_{\widehat{i_1}\ldots\widehat{i_\mu}}$, define a holomorphic map $f_{i_1\ldots i_\mu,j,w}\colon \Omega_{i_1} \times \cdots \times \Omega_{i_\mu} \to \omega_j$ by

$$f_{i_1\ldots i_\mu,j,w}(z_1, \ldots, z_\mu) = f_j(w_1, \ldots, z_1, \ldots, z_\mu, \ldots, w_k).$$

We say that f is a *semi-product* proper holomorphic map, if for each $i \in \{1, \ldots, k\}$ there exists $j \in \{1, \ldots, l\}$ such that $f_{i,j,w}$ is a proper holomorphic map for all $w \in \mathcal{U}$ where $\mathcal{U}$ is an open dense subset of $\Omega_{\widehat{i}}$.

Example 3.2. Let $\Omega_1, \ldots, \Omega_k, \omega_1, \ldots, \omega_l$ be bounded domains with $k \le l$ in $\mathbb{C}^{\mu_1}, \ldots, \mathbb{C}^{\mu_k}$, $\mathbb{C}^{\nu_1}, \ldots, \mathbb{C}^{\nu_l}$ respectively. Let $f_j\colon \Omega_j \to \omega_j$ be a proper holomorphic map for each $j =$

$1, \ldots, k$. Let $f_j \colon \Omega_1 \times \cdots \times \Omega_k \to \omega_j$ be a holomorphic map for each $j = k+1, \ldots, l$. Then the holomorphic map $f = (f_1, \ldots, f_l) \colon \Omega_1 \times \cdots \times \Omega_k \to \omega_1 \times \cdots \times \omega_l$ is a semi-product proper holomorphic map.

Definition 3.3. For given domains $\Omega = \Omega_1 \times \cdots \times \Omega_k$, $\omega = \omega_1 \times \ldots \times \omega_k$, we say that a proper holomorphic map $f \colon \Omega \to \omega$ is a *product* map if f is of the form in Example 3.2 up to the permutation of the set $\{1, \ldots, k\}$.

Proposition 3.4. *Let $\Omega_1, \ldots, \Omega_k$, $\omega_1, \ldots, \omega_l$ be irreducible bounded symmetric domains. Let $\Omega = \Omega_1 \times \cdots \times \Omega_k$ and $\omega = \omega_1 \times \cdots \times \omega_l$. Suppose that*

(1) $\dim \Omega_{i_1} + \dim \Omega_{i_2} \geq \dim \omega_j$ *for any $i_1, i_2 \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$, and*

(2) $\mathrm{rank}(\Omega_1) + \cdots + \mathrm{rank}(\Omega_k) \geq \mathrm{rank}(\omega_1) + \cdots + \mathrm{rank}(\omega_l)$.

Let $f \colon \Omega \to \omega$ be a semi-product proper holomorphic map. Then $k = l$ and f is a product map.

Proof. Suppose that $\dim \Omega_{i_1} + \dim \Omega_{i_2} > \dim \omega_j$ for any $i_1, i_2 \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$. Since f is semi-product proper, for each $i_1 < i_2 \in \{1, \ldots, k\}$ there are $j_1, j_2 \in \{1, \ldots, l\}$ such that $f_{i_1, j_1, w} \colon \Omega_{i_1} \to \omega_{j_1}$ and $f_{i_2, j_2, \zeta} \colon \Omega_{i_2} \to \omega_{j_2}$ are proper for each $w \in \Omega_{\widehat{i_1}}$ and $\zeta \in \Omega_{\widehat{i_2}}$. If $j_1 = j_2$ then

$$f_{j_1}(z_1, \ldots, z_{i_1-1}, \cdot, z_{i_1+1}, \ldots, z_{i_2-1}, \cdot, z_{i_2+1}, \ldots, z_k) \colon \Omega_{i_1} \times \Omega_{i_2} \to \omega_{j_1}$$

is a proper holomorphic map. If $\dim \Omega_{i_1} + \dim \Omega_{i_2} > \dim \omega_{j_1}$, this yields a contradiction because the source domain's dimension is bigger than that of the target domain. If $\dim \Omega_{i_1} + \dim \Omega_{i_2} = \dim \omega_{j_1}$, we apply [3, Theorem 1.1]: *if $\nu \colon D_1 \to D_2$ is a proper holomorphic map between bounded symmetric domains D_1 and D_2 of the same complex dimension ≥ 2, and either D_1 or D_2 is irreducible, then ν is a biholomorphism.* This implies that f_{j_1} is a biholomorphism. However since $\Omega_{i_1} \times \Omega_{i_2}$ is reducible while ω_j is irreducible, it is also a contradiction. Hence $j_1 \neq j_2$ and $k \leq l$.

Up to permutation of $\{1, \ldots, l\}$, without loss of generality, we may assume that $f \colon \Omega \to \omega$ is a proper holomorphic map such that $f_{i,i,w} \colon \Omega_i \to \omega_i$ is proper for each $i \in \{1, \ldots, k\}$ and $w \in \Omega_{\widehat{i}}$. Besides by Tsai's theorem [18], $\mathrm{rank}(\Omega_i) \leq \mathrm{rank}(\omega_i)$ for each $i \in \{1, \ldots, k\}$. Hence we obtain that $\mathrm{rank}(\Omega_i) = \mathrm{rank}(\omega_i)$ for each $i \in \{1, \ldots, k\}$ and $k = l$.

If we apply Corollary 2.3 to $f_{i,i,w}$ for each i, we may obtain that f is product proper. $\square$

To prove Theorem 1.1, we only need to prove that f is semi-product proper by Proposition 3.4.

Proposition 3.5. *Let $\Omega_1, \ldots, \Omega_k$, $\omega_1, \ldots, \omega_l$ be irreducible bounded symmetric domains. Let $\Omega = \Omega_1 \times \cdots \times \Omega_k$ and $\omega = \omega_1 \times \cdots \times \omega_l$. Then any proper holomorphic map $f \colon \Omega \to \omega$ which has a rational extension to the ambient Euclidean space is a semi-product map.*

Remark 3.6. In Proposition 3.5, we don't need the assumption about rank or dimension of the domains.

Proof of Proposition 3.5. Let S_i, s_j be generic norms of Ω_i, ω_j respectively. Let $f = (f_1, \ldots, f_l)$ where $f_j = \pi_j \circ f$ with the projection $\pi_j \colon \omega \to \omega_j$ onto the j-th component. Fix $i \in \{1, \ldots, k\}$. Since f is proper,

$$s_1(f_1(Z), \overline{f_1(Z)}) \ldots s_l(f_l(Z), \overline{f_l(Z)}) = 0$$

whenever $Z = (Z_1, \ldots, Z_k) \in \overline{\Omega}$ with $S_i(Z_i, \overline{Z}_i) = 0$. Choose a point $z \in \partial\Omega$ such that $z_i := \pi_i(z) \in \partial\Omega_i$ and $dS_i(z_i, \overline{z}_i) \neq 0$ (z_i is a smooth boundary point of Ω_i). Since $S_i(Z_i, \overline{Z}_i)$ and $s_j(f_j(Z), \overline{f_j(Z)})$ are rational functions, there exists an open neighborhood U of z in $\mathbb{C}^{\dim \Omega}$ and an real analytic function Q_i on U such that

$$S_i(Z_i, \overline{Z}_i) Q_i(Z, \overline{Z}) = s_1(f_1(Z), \overline{f_1(Z)}) \ldots s_l(f_l(Z), \overline{f_l(Z)}).$$

This induces the polarized holomorphic equation

$$S_i(Z_i, W_i) Q_i(Z, W) = s_1(f_1(Z), \overline{f}_1(W)) \ldots s_l(f_l(Z), \overline{f}_l(W))$$

on $U \times \overline{U}$ and hence whenever $S_i(Z_i, W_i) = 0$ on $U \times \overline{U}$, we obtain

$$s_1(f_1(Z), \overline{f}_1(W)) \ldots s_l(f_l(Z), \overline{f}_l(W)) = 0$$

on $U \times \overline{U}$. Let V be the maximal connected set of regular points of $\widetilde{V} := \{(Z_i, W_i) \in \mathbb{C}^{\dim \Omega_i} \times \overline{\mathbb{C}^{\dim \Omega_i}} : S_i(Z_i, W_i) = 0\}$ containing $(z_i, \overline{z}_i)$. Then

$$V \subset \{(Z, W) \in \mathbb{C}^{\dim \Omega} \times \overline{\mathbb{C}^{\dim \Omega}} : s_1(f_1(Z), \overline{f}_1(W)) \ldots s_l(f_l(Z), \overline{f}_l(W)) = 0\}$$

by the identity theorem for analytic sets (cf. [6]). Since the set of regular points of $\widetilde{V}$ is open dense subset of $\widetilde{V}$, we can obtain that the irreducible polynomial $S_i(Z_i, W_i)$ is a factor of the numerator of $s_1(f_1(Z), \overline{f}_1(W)) \ldots s_l(f_l(Z), \overline{f}_l(W))$ which is a polynomial. Hence there exists j such that $S(Z_i, W_i)$ divides $s_j(f_j(Z), \overline{f}_j(W))$. By applying $W = \overline{Z}$, we obtain that $S_i(Z_i, \overline{Z}_i)$ is a factor of $s_j(f_j(Z), \overline{f_j(Z)})$. This implies that f is a semi-product proper holomorphic map. $\square$

Remark 3.7. By the proof of Theorem 1.1, we can obtain that the following: Let $\Omega, \omega_1, \ldots, \omega_l$ be irreducible bounded symmetric domains. Let $f \colon \Omega \to \omega_1 \times \cdots \times \omega_l$ be a proper rational map. Then there should be at least one $j \in \{1, \ldots, l\}$ such that f_j is a proper holomorphic map from Ω into ω_i.

In [2], Bell proved the following: let Ω be a bounded domain in $\mathbb{C}^n$ whose associated Bergman kernel function is a rational function and ω a bounded circular domain in $\mathbb{C}^n$ that contains the origin. Then any proper holomorphic map $f \colon \Omega \to \omega$ must be rational. Hence if $\dim \Omega = \dim \omega$, any proper holomorphic map is rational.

Proof of Theorem 1.2. By a theorem of Bell [2] and Proposition 3.5, we obtain that $f\colon \Omega \to \omega$ is semi-product proper. Let $\Omega = \Omega_1 \times \cdots \times \Omega_k$ and $\omega = \omega_1 \times \cdots \times \omega_l$ with irreducible factors $\Omega_1, \dots, \Omega_k$, $\omega_1, \dots, \omega_l$. Let $f = (f_1, \dots, f_l)$.

For each $j \in \{1, \dots, k\}$, choose $i_j \in \{1, \dots, l\}$ such that $f_{i,i_j,w}\colon \Omega_i \to \omega_{j_i}$ is a proper holomorphic map. Suppose that $\{j_1, \dots, j_k\} \subsetneq \{1, \dots, l\}$. Then for $\mu \in \{1, \dots, l\} \setminus \{j_1, \dots, j_k\}$, $(f_1, \dots, \widehat{f_\mu}, \dots, f_l)\colon \Omega \to \omega_1 \times \cdots \times \widehat{\omega}_\mu \times \cdots \times \omega_l$ is also proper holomorphic map, a plain contradiction since the dimension of the source domain should be smaller than or equal to that of the target domain. This implies that $\{j_1, \dots, j_k\} = \{1, \dots, l\}$ and hence $k \geq l$. Furthermore by the permutation of $\{1, \dots, k\}$ we may assume that there is a partition of $\{1, \dots k\}$, $1 \leq i_1 < i_2 < \cdots < i_{l-1} < i_l = k$ such that

$$
\begin{aligned}
f_{1\dots i_1,1,w_1}\colon \Omega_1 \times \cdots \times \Omega_{i_1} &\to \omega_1 &&\text{with } w_1 \in \Omega_{\widehat{1\dots i_1}},\\
f_{i_1+1\dots i_2,2,w_2}\colon \Omega_{i_1+1} \times \cdots \times \Omega_{i_2} &\to \omega_2 &&\text{with } w_2 \in \Omega_{\widehat{i_1+1\dots i_2}},\\
&\vdots\\
f_{i_{l-1}+1\dots k,l,w_l}\colon \Omega_{i_{l-1}+1} \times \cdots \times \Omega_k &\to \omega_l &&\text{with } w_l \in \Omega_{\widehat{i_{l-1}+1\dots k}}
\end{aligned}
\tag{3.1}
$$

are proper holomorphic maps. Since $\dim \Omega = \dim \omega$, we obtain that

$$\sum_{i=i_\mu+1}^{i_{\mu+1}} \dim \Omega_i = \dim \omega_{\mu+1}$$

for each $i_\mu = i_1, \dots, i_{l-1}$. Then when $\dim \omega_j \geq 2$, [3, Theorem 1.1] yields that $i_j = j$ in (3.1) and when $\dim \omega_j = 1$, $\Omega_{i_{j-1}+1} \times \cdots \times \Omega_{i_j}$ also has dimension 1. Hence we obtain that $f_{i,i,w_i}\colon \Omega_i \to \omega_i$ is a proper holomorphic map and $\dim \Omega_i = \dim \omega_i$ for each $i = 1, \dots, k$ and $w_i \in \Omega_{\widehat{i}}$.

Now by Corollary 2.3 f is a product map and by the classification of proper holomorphic maps between polydiscs in [16] and that between equidimensional irreducible bounded symmetric domains in [19], we obtain the theorem. □

4. Remark on the proper holomorphic self-maps of pseudoconvex flag domains

Let G be a complex semisimple Lie group and G/Q a flag manifold with a parabolic subgroup Q of G. Let G_0 be a real form of G and D a flag domain in G/Q, that is, an open G_0-orbit in G/Q.

Assume that $\mathcal{O}(D) \neq \mathbb{C}$ and give an equivalence relation on D:

$$x \sim y \quad \Longleftrightarrow \quad f(x) = f(y) \quad \text{for all } f \in \mathcal{O}(D).$$

In general $D/\sim$ is a complex homogeneous manifold $G_0/\widehat{V}_0$ and the projection $D = G_0/V_0 \to G_0/\widehat{V}_0$ is a holomorphic mapping. Let's take Q to be an isotropy group of G at

$z_0 \in D = G_0 z_0$. Then there exists $\widehat{Q}$ containing Q and the following diagram is commute:

$$\begin{array}{ccc} z_0 \in G/Q = Z & \supset & D = G_0/V_0 \\ \pi \downarrow & & \downarrow (*) \\ \widehat{z}_0 \in G/\widehat{Q} = \widehat{Z} & \supset & \widehat{D} = G_0/\widehat{V}_0 \end{array}$$

Furthermore fiber of π is $F = Kz_0$ and $\widehat{D}$ is a Hermitian symmetric space of non-compact type where K_0 is a maximal compact subgroup of G_0 and K is a complexification of K_0. Since $\widehat{D}$ is a Stein manifold and $\widehat{D}$ is contractible, we obtain that $(*)$ is topologically trivial. Furthermore by the Grauert-Oka principle, $(*)$ is holomorphically trivial. This implies that $D = \widehat{D} \times F$ with the flag manifold F.

Theorem 4.1 (Huckleberry, [8])**.** *Let D be a flag domain. The followings are equivalent:*

(1) $\mathcal{O}(D) \neq \mathbb{C}$,

(2) *D is pseudoconvex, i.e., there is a continuous exhaustion function $\rho\colon D \to \mathbb{R}^{\geq 0}$ which is plurisubharmonic on the complement $D \setminus S$ for a compact subset $S \subset D$.*

(3) *$D = \widehat{D} \times F$ with a Hermitian symmetric space of non-compact type $\widehat{D}$ and a flag manifold F.*

Theorem 4.2. *Let $D_1 = \widehat{D}_1 \times F_1$, $D_2 = \widehat{D}_2 \times F_2$ be pseudoconvex flag domains with $\widehat{D}_1 = \widehat{D}_2$. Then $f\colon D_1 \to D_2$ is a proper holomorphic map if and only if f is of the form (f_1, f_2) where $f_1\colon \widehat{D}_1 \to \widehat{D}_2$ is a proper holomorphic map and $f_2\colon D_1 \to F_2$ is a holomorphic map.*

Proof. Let $f = (f_1, f_2)\colon \widehat{D}_1 \times F_1 \to \widehat{D}_2 \times F_2$ be a proper holomorphic map. For each $p \in F_1$, $f_1(\,\cdot\,, p)\colon \widehat{D}_1 \to \widehat{D}_2$ is a proper holomorphic map. By Theorem 1.2, $f_1(\,\cdot\,, p)$ is a product map, i.e., we can express $f_1(\,\cdot\,, p) = (f_{11}(\,\cdot\,, p), \ldots, f_{1k}(\,\cdot\,, p))$ for some k. Then if we apply Corollary 2.3 to each $f_{1j}(\,\cdot\,, p)$, we obtain that it does not depend on p-variable. In particular, f_1 is a proper holomorphic map from $\widehat{D}_1$ to $\widehat{D}_2$ and the proof completed. □

Acknowledgments

The author would like to thank Ng, Sui-Chung for helpful discussion and comments on the subject, and to Huckleberry, Alan for explaining pseudoconvex flag domains to her. Thanks are also due to the anonymous referee, whose careful comments lead to an improvement of the main theorems. This research was partially supported by "Overseas Research Program for Young Scientists" through Korea Institute for Advanced Study (KIAS).

References

[1] H. Alexander, *Holomorphic mappings from the ball and polydisc*, Math. Ann. **209** (1974), 249–256.

[2] S. Bell, *Proper holomorphic mappings that must be rational*, Trans. Amer. Math. Soc. **284** (1984), no. 1, 425–429.

[3] G. Bharali and J. Janardhanan, *Proper holomorphic maps between bounded symmetric domains revisited*, Pacific J. Math. **271** (2014), no. 1, 1–24.

[4] E. Cartan, *Sur les domaines bornés homogènes de l'espace den variables complexes*, Abh. Math. Sem. Univ. Hamburg **11** (1935), no. 1, 116–162.

[5] D. Chakrabarti and K. Verma, *Condition R and proper holomorphic maps between equidimensional product domains*, Adv. Math. **248** (2013), 820–842.

[6] K. Fritzsche and H. Grauert, *From holomorphic functions to complex manifolds*, Graduate Texts in Mathematics **213**, Springer-Verlag, New York, 2002.

[7] Harish-Chandra, *Representations of semisimple Lie groups VI: Integrable and square-integrable representations*, Amer. J. Math. **78** (1956), 564–628.

[8] A. Huckleberry, *Remarks on homogeneous complex manifolds satisfying Levi conditions*, Boll. Unione Mat. Ital. (9) **3** (2010), no. 1, 1–23.

[9] J. Janardhanan, *Proper holomorphic mappings between hyperbolic product manifolds*, Internat. J. Math. **25** (2014), no. 4, 1450039, 10 pp.

[10] O. Loos, *Jordan Pairs*, Lecture Notes in Mathematics **460**, Springer-Verlag, Berlin-New York, 1975.

[11] N. Mok and I. H. Tsai, *Rigidity of convex realizations of irreducible bounded symmetric domains of rank* ≥ 2, J. Reine Angew. Math. **431** (1992), 91–122.

[12] R. Narasimhan, *Several Complex Variables*, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, Ill.-London, 1971.

[13] S.-C. Ng, *On proper holomorphic mappings among irreducible bounded symmetric domains of rank at least* 2, Proc. Amer. Math. Soc. **143** (2015), no. 1, 219–225.

[14] R. Remmert and K. Stein, *Eigentliche holomorphe Abbildungen*, Math. Z. **73** (1960), 159–189.

[15] G. Roos, *Exceptional symmetric domains*, in *Symmetries in Complex Analysis*, 157–189, Contemp. Math. **468**, Amer. Math. Soc., Providence, RI, 2008.

[16] W. Rudin, *Function Theory in Polydiscs*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[17] A. Seo, *Proper holomorphic polynomial maps between bounded symmetric domains of classical type*, Proc. Amer. Math. Soc. **144** (2016), no. 2, 739–751.

[18] I. H. Tsai, *Rigidity of proper holomorphic maps between symmetric domains*, J. Differential Geom. **37** (1993), no. 1, 123–160.

[19] Z.-H. Tu, *Rigidity of proper holomorphic mappings between equidimensional bounded symmetric domains*, Proc. Amer. Math. Soc. **130** (2002), no. 4, 1035–1042.

[20] J. A. Wolf, *Fine structure of Hermitian symmetric spaces*, Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970), 271–357, Pure and App. Math. **8**, Dekker, New York, 1972.

Aeryeong Seo
School of Mathematics, Korea Institute for Advanced Study (KIAS), 85 Hoegiro
(Cheongnyangni-dong 207-43), Dongdaemun-gu, Seoul 130-722, Republic of Korea
E-mail address: `aeryeongseo@kias.re.kr`