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Optimal Control of Second Order Stochastic Evolution Hemivariational Inequalities with Poisson Jumps

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Abstract. The purpose of this article is to study the optimal control problem of second order stochastic evolution hemivariational inequalities with Poisson jumps by virtue of cosine operator theory in the Hilbert space. Initially, the sufficient conditions for existence of mild solution of the proposed system are verified by applying properties of Clarke's subdifferential operator and fixed point theorem in multivalued maps. Further, we formulated and proved the existence results for optimal control of the proposed system with corresponding cost function by using Balder theorem. Finally an example is provided to illustrate the main results.

1. Introduction

Stochastic differential equations (SDEs) are important to modelling the real life phenomena where there is a need for an aspect of randomness. Stochastic evolution equations (SEEs) in infinite dimensional spaces are motivated by the random phenomena studied in the natural sciences like physics, chemistry and in control theory. The existence of mild solutions for various types of SEEs and its optimal control in Hilbert spaces are extensively studied by many authors (see [5, 12, 22, 25, 26]).

An optimal control problem (OCP) describes the path of control variables concerned with minimizing the cost functional or maximizing a payoff to the corresponding system over a set of admissible control functions. Nowadays, optimal control theory has a considerable development and have fruitful applications in many fields like science and engineering (see [7, 9, 11, 14, 15]). Stochastic optimal control problem (SOCP) makes to design the time path of the controlled variables which performs the desired control task with minimum cost despite the presence of noise. SOCPs and its applications have received

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extensive attention in the literature (see [13, 21, 23, 24]). The optimal control of SEEs is an emerging topics in the literature (see [23–26]). For a nonlinear SOCPs, Zhou [25] studied the optimal control in which the controlled state dynamics is governed by a stochastic evolution equations in Hilbert space. Moreover, nonlinear second order SOCPs in infinite dimensions, mainly Hilbert space and Banach space, arise in various applications such as stochastic wave equations and many other physical phenomena in both science and engineering (see [22] and references therein). To model the stochastic phenomena, researchers have employed Wiener process and Poisson jumps. SDEs with Poisson jumps are the most popular systems in modelling and widely used to describe the asset and commodity price dynamics (see [4, 22, 24]).

Hemivariational inequalities represent a class of nonlinear inclusions that are associated with the Clarke's subdifferential operator and has applications in non-convex optimization and structural analysis. Inequality problems can be classified into two main classes, namely; variational inequality and hemivariational inequality. It is known that the variational inequality mainly concerns with the convex energy functions whereas the hemivariational inequalities are focussed with non-smooth and non-convex energy functions. The notion of hemivariational inequality was first proposed by the author Panagiotopoulos in 1981 and in that he represents mechanical problems by using hemivariational inequalities [17]. Many problems from nonsmooth contact mechanics involving multivalued and nonmonotone consitutive laws with boundary conditions can be modelled by means of hemivational inequality or subdifferential inclusions (see [8, 16–18] for more applications). At present the existence of hemivariational inequalities in various fields receives much attention to the authors (see [7, 10, 12, 14, 15, 20]). Specifically, Migorski and Ochal [15] established the OCPs for parabolic hemivariational inequalities. The existence of solutions and its optimal control for hyperbolic hemivariational inequalities are investigated in the literature [20]. The second order nonlinear evolution hemivariational inequalities in Hilbert space with applications to classical wave equation is discussed in [10]. The existence and controllability results of hemivariarional inequalities using stochastic fractional differential equations are discussed in [12]. The intention of present paper is to identify an optimal control for the model generated by second order stochastic hemivariational inequalities with Poisson jumps.

Recently, the authors studied the optimal control results of hemivariational inequalities using infinite-dimensional spaces (see [9, 14, 15, 19, 20]). In particular, optimal control of hemivariational inequalities with delay is studied by Jeong [19]. In [9], Liu et al. studied the existence of feasible pairs and optimal state-control pairs for the feedback control system governed by evolution hemivariational inequalities. However, to the best of our knowledge, there is no work reported on the existence of optimal control study described by second order stochastic hemivariational inequalities driven by Poisson jumps in infinite dimensional spaces. Motivated by [9, 10, 12], we developed the model in terms of second order hemivariational inequality in stochastic sense with Poisson jumps in Hilbert space.

In this paper, the optimal control study of the second order nonlinear stochastic evolution hemivariational inequalities with Poisson jumps as follows:

$$
\begin{cases} \left\langle dx'(t) + [Ax(t) + Bu(t)]\,dt + g(t,x(t))\,dw(t) + \int_Z h(t,x(t),\eta)\widetilde{N}(dt,d\eta), v\right\rangle_H , \\ \qquad +F^0(t,x(t);v) \geq 0, \quad \text{a.e. } t \in J := [0,T] \text{ and } \forall\, v \in H, \\ x(0) = x_0, \quad x'(0) = y_0, \end{cases} \tag{1.1}
$$

where the state variables $x(\cdot)$ takes the values in the separable Hilbert space H with the norm $\|\cdot\|_H$. $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $(t > 0)$ on H. Let U be set of all admissible controls which is also a Hilbert space and u be a control function. Let B be a bounded linear operator from U into H. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space and let K be another separable Hilbert space. Suppose that $\{w(t) : t \geq 0\}$ is a K-Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. We are employing the same notations $\|\cdot\|$ for the norm of $L(K,H)$, where $L(K,H)$ denotes the space of all bounded operators from K into H. Simply as $L(K,H) = L(H)$ if $K = H$. Let $N(dt,d\eta)$ be the Poisson counting measure which is induced by the Poisson point process $r(\cdot)$ in the measurable space $(Z, \mathfrak{B}(Z))$ defined on the complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and the compensated martingale measure is denoted by $\widetilde{N}(dt,d\eta) = N(dt,d\eta) - \lambda(d\eta)dt$. Let $g\colon J \times H \to L_Q(K,H)$, $h\colon J \times H \times (Z - \{0\}) \to H$ be appropriate mappings and specified it in the next section where $L_Q(K,H)$ denotes the space of all Q-Hilbert Schmidt operators from K into H. Let $F^0(t,\cdot\,;\cdot\,)$ be the generalized Clarke's directional derivative [3] of a locally Lipschitz function $F(t,\cdot\,)\colon H \to \mathbb{R}$.

This article is characterized as follows: Section 2 gives some basic definitions and the preliminary results. Section 3 describes the sufficient conditions for the existence of mild solutions of proposed system by utilizing some properties of Clarke's subdifferential operator and multivalued fixed point theorem. Also, the existence of an optimal control is derived by applying Balder theorem. In Section 4, an application is provided to illustrate the theory.

2. Preliminaries

Let H be a separable Hilbert space and its norm be denoted as $\|\cdot\|_H$. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be the complete probability space with the normal filteration $\{\mathfrak{F}_t, t \geq 0\}$. $\mathbb{E}(\cdot)$ denotes the expectation of a random variable. The $\mathfrak{F}_t$-adapted state $x(\cdot)$ and control $u(\cdot)$ variables takes the values in H and U respectively. $L^2(\mathfrak{F}, H) = L^2(\Omega, \mathfrak{F}, \mathbb{P}, H)$ denotes the Hilbert space of all strongly $\mathfrak{F}$-measurable square integrable H-valued random variable satisfying

$\mathbb{E}\,\|x\|_H^2 < \infty$. Let $C(J, L^2(\mathfrak{F}, H))$ be the Banach space of all continuous maps from J into $L^2(\mathfrak{F}, H)$ with the norm $\|x\|_{L^2} = \left[\sup_{t\in[0,T]} \mathbb{E}\,\|x(t)\|_H^2\right]^{1/2} < \infty$. $L^2_{\mathfrak{F}}(J, H)$ will denote the Hilbert space of all stochastic processes $\mathfrak{F}_t$-adapted measurable defined on J with the values in H with the norm $\|x\|_{L^2_{\mathfrak{F}}(J,H)} = \left[\int_0^T \mathbb{E}\,\|x(t)\|_H^2\, dt\right]^{1/2} < \infty$. The space $L^2_{\mathfrak{F}}(J, U)$ will denote the Hilbert space of all stochastic processes $\mathfrak{F}_t$-adapted measurable defined on J with the values in H satisfying with the norm $\|u\|_{L^2_{\mathfrak{F}}(J,U)} = \left[\int_0^T \mathbb{E}\,\|u(t)\|_U^2\, dt\right]^{1/2} < \infty$. (For details see [5, 12] and references therein).

Suppose that $\{r(t) : t \in J\}$ is the $\mathfrak{F}_t$-adapted Poisson point process taking its values in a measurable space $(Z, \mathcal{B}(Z))$ with a σ-finite intensity measure $\lambda(d\eta)$. We denote $N(ds, d\eta)$ as the Poisson counting measure, which is induced by $r(\cdot)$ and the compensating martingale is given by

$$\widetilde{N}(ds, d\eta) = N(ds, d\eta) - \lambda(d\eta)ds.$$

Let w be the $\mathfrak{F}_t$-adapted Q-Wiener process independent of the Poisson point process $\{r(t) : t \in J\}$ on $(\Omega, \mathfrak{F}, \mathbb{P})$ with linear bounded covariance operator Q such that $\mathrm{Tr}(Q) < \infty$. We assume that there exists a complete orthonormal system $\{e_n\}$ in K, a bounded sequence of nonnegative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda_n e_n$, $n = 1, 2, \dots$ and a sequence $\{\beta_n\}$ of independent Wiener process such that

$$\langle w(t), \vartheta\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,\langle e_n, \vartheta\rangle\,\beta_n(t), \quad \vartheta \in K,\ t \geq 0.$$

Let $\psi \in L(K, H)$ and define

$$\|\psi\|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \left\|\sqrt{\lambda_n}\psi e_n\right\|^2.$$

If $\|\psi\|_Q < \infty$, then ψ is called a Q-Hilbert Schmidt operator. Let $L_Q(K, H)$ be the space of all Q-Hilbert Schmidt operators $\psi\colon K \to H$. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$, with $\|\psi\|_Q^2 = \langle\psi, \psi\rangle$ is a Hilbert space with the above norm topology.

Definition 2.1. [10, 22] The one parameter family $\{C(t) : t \in \mathbb{R}\}$ of operators in $L(H)$ is said to be a strongly continuous cosine family if,

(i) $C(0) = I$, where I is the identity operator in H.

(ii) $C(t + s) + C(t - s) = 2C(t)C(s)$, $\forall\, s, t \in \mathbb{R}$.

(iii) $C(t)x$ is continuous in t on $\mathbb{R}$ for every $x \in H$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined to be

$$S(t)x = \int_0^t C(t)x\, ds, \quad x \in H,\ t \in \mathbb{R}.$$

We define the linear operator as

$$Ax = \frac{d^2}{dt^2}C(t)x\Big|_{t=0} \quad \text{for } x \in D(A),$$

which is the generator of the strongly continuous cosine operator $C(t)$. It is known that A is the closed, linear operator and densely defined on H. We denote by M_c and M_s, a pair of positive constants such that $\|C(t)\| \leq M_c$ and $\|S(t)\| \leq M_s$ for every $t \in J$ (see Proposition 2.3 in [10]). We recall some definitions on multivalued maps (see [8, 9]).

For our convenience, we define the following notations: Let X and Y be Banach spaces and denote $\mathrm{Gr}(F) = \{(x, y) \in X \times Y : x \in X, y \in F(x)\}$. Let $\mathcal{P}(X)$ be the set of all nonempty subsets of X.

$$\mathcal{P}_{cl}(X) = \{\mathcal{A} \in \mathcal{P}(X) : \mathcal{A} \text{ is closed}\}, \qquad \mathcal{P}_{bd}(X) = \{\mathcal{A} \in \mathcal{P}(X) : \mathcal{A} \text{ is bounded}\},$$
$$\mathcal{P}_{cp}(X) = \{\mathcal{A} \in \mathcal{P}(X) : \mathcal{A} \text{ is compact}\}, \qquad \mathcal{P}_{cv}(X) = \{\mathcal{A} \in \mathcal{P}(X) : \mathcal{A} \text{ is convex}\}.$$

Definition 2.2. Given a Banach space X and a multivalued map $F\colon X \to 2^X \setminus \{0\} = \mathcal{P}(X)$. Then

(i) F is convex valued if $F(x)$ is convex for every $x \in X$.

(ii) F is said to be upper semicontinuous (u.s.c) on X if for each $x_0 \in X$, the set $F(x_0)$ is nonempty, closed subset of X and if for each open set U of X containing $F(x_0)$, there exists an open neighborhood V of x_0 such that $F(V) \subseteq U$.

(iii) F is bounded on the bounded sets if $F(B) = \bigcup_{x \in B} F(x)$ is bounded in X for all $B \in \mathcal{P}_{bd}(X)$ (i.e., $\sup_{x \in B}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$).

(iv) F is completely continuous if $F(B)$ is relatively compact for every bounded subset $B \in \mathcal{P}(X)$.

(v) The multimap F is said to be closed if its graph $\mathrm{Gr}(F)$ is a closed subset of $X \times Y$.

(vi) F has a fixed point if there is an element $x \in X$ such that $x \in F(x)$.

Definition 2.3. Let X be a Banach space and X^* be its dual space. The Clarke's generalized directional derivative for a locally Lipschitz function $F\colon X \to \mathbb{R}$ at x in the direction v, denoted by $F^0(x; v)$, is given by

$$F^0(x; v) = \lim_{y \to x} \sup_{\lambda \to 0^+} \frac{F(y + \lambda v) - F(y)}{\lambda}.$$

The generalized Clarke subdifferential of F at x, denoted by ∂F is a subset of X^*, given by

$$\partial F(x) = \left\{x^* \in X^* : F^0(x; v) \geq \langle x^*, v\rangle, \forall\, v \in X\right\}.$$

Lemma 2.4. (see [9]) *Let X and Y be the metric spaces and if $F\colon X \to K(Y)$ is a closed compact multimap then F is u.s.c.*

Lemma 2.5. (see [5]) *Let $G\colon J \times \Omega \to L_Q(K,H)$ be a strongly measurable mapping such that $\int_0^T \mathbb{E}\,\|G(t)\|^2_{L_Q(K,H)}\,dt < \infty$. Then*

$$\mathbb{E}\left\|\int_0^t G(s)\,dw(s)\right\|^2 \leq L_G \int_0^t \mathbb{E}\,\|G(s)\|^2\,ds, \quad \forall\, t \in J,$$

where L_G is a constant involving on T.

Theorem 2.6. (see [6]) *Let U and $\overline{U}$ be the open and closed subsets of a Banach space X. Let $\phi_1\colon \overline{U} \to X$ be a single-valued and $\phi_2\colon \overline{U} \to \mathcal{P}_{cp,cv}(X)$ be a multi-valued operator such that $\phi_1(\overline{U}) + \phi_2(\overline{U})$ is bounded. Suppose that*

(i) *ϕ_1 is a contraction with a contraction constant k and*

(ii) *ϕ_2 is u.s.c and completely continuous.*

Then either

1. *the operator inclusion $\chi x \in \phi_1 x + \phi_2 x$ has a solution for $\chi = 1$, or*
2. *there is an element $\tau \in \partial U$ such that $\chi\tau \in \phi_1\tau + \phi_2\tau$ for some $\chi > 1$, where ∂U is the boundary of U.*

3. Main results

3.1. Existence of mild solutions

This section provides the existence of a mild solution for the proposed system (1.1). Now we examine the existence of mild solution of the semilinear inclusion: (see [12])

$$\begin{cases} dx'(t) \in [Ax(t) + Bu(t)]\,dt + g(t,x(t))\,dw(t) \\ \qquad\qquad + \int_Z h(t,x(t),\eta)\widetilde{N}(dt,d\eta) + \partial F(t,x(t)), \quad \text{a.e. } t \in J, \\ x(0) = x_0, \quad x'(0) = y_0. \end{cases} \tag{3.1}$$

If $x(t) \in C(J, L^2(\mathfrak{F},H))$ is a solution of (3.1), then there exists a $\mathfrak{F}_t$-adapted measurable function $f(t) \in \partial F(t,x(t))$ such that $f \in L^2_{\mathfrak{F}}(J,H)$ and

$$\begin{cases} dx'(t) = [Ax(t) + Bu(t)]\,dt + g(t,x(t))\,dw(t) \\ \qquad\qquad + \int_Z h(t,x(t),\eta)\widetilde{N}(dt,d\eta) + f(t), \quad \text{a.e. } t \in J, \\ x(0) = x_0, \quad x'(0) = y_0. \end{cases}$$

This implies,

$$\begin{cases} \left\langle dx'(t) + [Ax(t) + Bu(t)]\, dt + g(t, x(t))dw(t) + \int_Z h(t, x(t), \eta)\widetilde{N}(dt, d\eta), v \right\rangle_H \\ \qquad + \langle f(t), v\rangle = 0, \quad \text{a.e. } t \in J, \\ x(0) = x_0, \quad x'(0) = y_0. \end{cases}$$

Since $f(t) \in \partial F(t, x(t))$ and $\langle f(t), v\rangle \leq F^0(t, x(t); v)$. Then,

$$\begin{cases} \left\langle dx'(t) + [Ax(t) + Bu(t)]\, dt + g(t, x(t))\, dw(t) + \int_Z h(t, x(t), \eta)\widetilde{N}(dt, d\eta), v \right\rangle_H \\ \qquad + F^0(t, x(t); v) \geq 0, \quad \text{a.e. } t \in J \text{ and } \forall\, v \in H, \\ x(0) = x_0, \quad x'(0) = y_0. \end{cases}$$

Thus, inspite of studying the stochastic hemivariational inequality (1.1), we have to discuss with the semilinear stochastic inclusion (3.1).

Definition 3.1. (see [12, 22]) For every $u \in L^2_{\mathfrak{F}}(J, U)$, a function $x(t) \in C(J, L^2(\mathfrak{F}, H))$ is called a mild solution of a system (3.1) if there exists a $\mathfrak{F}_t$-adapted measurable function $f \in L^2(\mathfrak{F}, H)$ such that $f(t) \in \partial F(t, x(t))$ for a.e. $t \in J$ and

$$\begin{aligned} x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)[Bu(s) + f(s)]\, ds + \int_0^t S(t-s)g(s, x(s))\, dw(s) \\ + \int_0^t S(t-s) \int_Z h(s, x(s), \eta(s))\widetilde{N}(ds, d\eta), \quad \forall\, t \in J. \end{aligned}$$

We will make the following hypotheses for proving our main results.

(H1) The sine operator $S(t)$ associated with the operator A is compact for every $t > 0$.

(H2) Let $F\colon J \times H \to \mathbb{R}$ be a function satisfying the following conditions:

(i) $F(\,\cdot\,, x)$ is measurable for every $x \in H$.

(ii) $F(t, \cdot\,)$ is locally Lipschitz continuous for a.e. $t \in J$.

(iii) There exists a function $a_1 \in L^2_{\mathfrak{F}}(J, \mathbb{R}^+)$ and a constant $c \geq 0$ such that

$$\|\partial F(t, x)\|^2 = \sup\left\{\|f(t)\|^2 / f(t) \in \partial F(t, x)\right\} \leq a_1(t) + c\,\|x\|^2$$

for all $x \in H$ and a.e. $t \in J$.

(iv) The mapping $\partial F(t, \cdot\,)$ satisfies the Lipschitz continuity with respect to the second variable, i.e., there exists a positive constant M_f such that

$$\|\partial F(t, x_1(t)) - \partial F(t, x_2(t))\|^2 \leq M_f\,\|x_1(t) - x_2(t)\|^2$$

for every $x_1, x_2 \in H$ and for every $t \in J$.

(H3) $g\colon J\times H\to L_Q(K,H)$ is Lipschitz continuous with respect to the second variable for a.e. $t\in J$ with the Lipschitz constant M_g and there exists a function $a_2\in L^2_{\mathfrak{F}}(J,\mathbb{R}^+)$ and a positive constant d such that

$$\|g(t,x)\|^2_{L_Q(K,H)}\le a_2(t)+d\|x\|^2.$$

(H4) $h\colon J\times H\times Z\backslash\{0\}\to H$ is Lipschitz continuous with respect to the second variable for a.e. $t\in J$ with the Lipschitz constant M_h and there exists a function $a_3\in L^2_{\mathfrak{F}}(J,\mathbb{R}^+)$ and a positive constant e such that

$$\int_Z\|h(t,x,\eta)\lambda(d\eta)\|^2_H\le a_3(t)+e\|x\|^2.$$

Define the multivalued operator $\mathcal{N}\colon L^2_{\mathfrak{F}}(J,H)\to P(L^2_{\mathfrak{F}}(J,H))$ by

$$\mathcal{N}=\left\{z\in L^2_{\mathfrak{F}}(J,H)/z(t)\in\partial F(t,x(t)),\text{a.e. } t\in J\right\},\quad \forall x\in L^2_{\mathfrak{F}}(J,H).$$

Lemma 3.2. (see Lemma 3.3 in [12]) *If the hypotheses* (H1)–(H4) *are satisfied. Then for each $x\in L^2_{\mathfrak{F}}(J,H)$ the set $\mathcal{N}(x)$ is nonempty, convex and have weekly compact values.*

Lemma 3.3. [10] *Suppose* (H1)–(H4) *hold and the operator $\mathcal{N}$ satisfies that, if $x_n\to x$ in $L^2_{\mathfrak{F}}(J,H)$, $z_n\to z$ weakly in $L^2_{\mathfrak{F}}(J,H)$ and $z_n\in\mathcal{N}(x_n)$ then we have that $z\in\mathcal{N}(x)$.*

Theorem 3.4. *If the hypotheses* (H1)–(H4) *and Lemma* 2.5 *hold then for every $u\in L^2_{\mathfrak{F}}(J,U)$, the stochastic controlled system* (3.1) *has a mild solution on J provided $M_hM_s^2<1$.*

Proof. Consider the multivalued map $\mathcal{F}\colon C(J,L^2(\mathfrak{F},H))\to 2^{C(J,L^2(\mathfrak{F},H))}$ defined by

$$\begin{aligned}\mathcal{F}(x)=\Big\{y\in C(J,L^2(\mathfrak{F},H)):y(t)=C(t)x_0+S(t)y_0+\int_0^t S(t-s)[Bu(s)+f(s)]\,ds\\ +\int_0^t S(t-s)g(s,x(s))\,dw(s)\\ +\int_0^t S(t-s)\int_Z h(s,x(s),\eta(s))\widetilde{N}(ds,d\eta)\Big\},\end{aligned}$$

where $f\in\mathcal{N}(x)$. Now the problem of finding mild solutions for (3.1) is equivalent to attaining fixed points of $\mathcal{F}(x)$. It is enough to show that the operator $\mathcal{F}(x)$ should satisfy all the conditions in Theorem 2.6. We will characterize the proof into several steps. First we decompose $\mathcal{F}(x)=\mathcal{F}_1(x)+\mathcal{F}_2(x)$.

Define $\mathcal{F}_1\colon C(J,L^2(\mathfrak{F},H))\to 2^{C(J,L^2(\mathfrak{F},H))}$ by

$$\begin{aligned}\mathcal{F}_1(x)=\Big\{y\in C(J,L^2(\mathfrak{F},H)):y(t)=C(t)x_0+S(t)y_0\\ +\int_0^t S(t-s)\int_Z h(s,x(s),\eta(s))\widetilde{N}(ds,d\eta)\Big\}\end{aligned}$$

and define $\mathcal{F}_2\colon C(J, L^2(\mathfrak{F}, H)) \to 2^{C(J,L^2(\mathfrak{F},H))}$ by

$$\mathcal{F}_2(x) = \Big\{ y \in C(J, L^2(\mathfrak{F}, H)) : y(t) = \int_0^t S(t-s)[Bu(s) + f(s)]\, ds \\ + \int_0^t S(t-s) g(s, x(s))\, dw(s) \Big\},$$

where $f \in \mathcal{N}(x)$.

Step 1*: $\mathcal{F}_1$ is a contraction mapping.*
Let $x_1, x_2 \in C(J, L^2(\mathfrak{F}, H))$. Using (H2)–(H4),

$$\begin{aligned}
&\mathbb{E}\, \|(\mathcal{F}_1 x_1)(t) - (\mathcal{F}_1 x_2)(t)\|^2 \\
&\leq \mathbb{E} \Bigg\| C(t)x_0 + S(t)y_0 + \int_0^t S(t-s) \int_Z h(s, x_1(s), \eta(s)) \widetilde{N}(ds, d\eta) \\
&\quad - \left[C(t)x_0 + S(t)y_0 + \int_0^t S(t-s) \int_Z h(s, x_2(s), \eta(s)) \widetilde{N}(ds, d\eta) \right] \Bigg\|^2 \\
&\leq \int_0^t \mathbb{E}\, \|S(t-s)\|^2 \int_Z \mathbb{E}\, \|[h(s, x_1(s), \eta(s)) - h(s, x_2(s), \eta(s))]\lambda(d\eta)\|^2\, ds \\
&\leq M_h M_s^2 \int_0^t \mathbb{E}\, \|x_1(s) - x_2(s)\|^2\, ds \\
&\leq M_h M_s^2 \sup_{s \in (0,t)} \mathbb{E}\, \|x_1(s) - x_2(s)\|_H^2 \leq M_h M_s^2 \|x_1 - x_2\|_{L^2}^2 .
\end{aligned}$$

We have $M_h M_s^2 < 1$, thus $\mathcal{F}_1$ is a contraction mapping.

Step 2*: $\mathcal{F}_2(x)$ is convex for every $x \in C(J, L^2(\mathfrak{F}, H))$.*
If $y_1, y_2 \in \mathcal{F}_2(x)$ then there exists $f_1, f_2 \in \mathcal{N}(x)$ such that for every $t \in J$ we have,

$$\begin{aligned}
y_1(t) &= \int_0^t S(t-s)[Bu(s) + f_1(s)]\, ds + \int_0^t S(t-s) g(s, x(s))\, dw(s), \\
y_2(t) &= \int_0^t S(t-s)[Bu(s) + f_2(s)]\, ds + \int_0^t S(t-s) g(s, x(s))\, dw(s).
\end{aligned}$$

Let $0 \leq r \leq 1$, then for any $t \in J$ we have

$$(ry_1 + (1-r)y_2)(t) = \int_0^t S(t-s)[Bu(s) + (rf_1 + (1-r)f_2)(s)]\, ds \\ + \int_0^t S(t-s) g(s, x(s))\, dw(s).$$

By Lemma 3.2, $\mathcal{N}(x)$ is convex then we have $rf_1 + (1-r)f_2 \in \mathcal{N}(x)$. Hence $(ry_1 + (1 - r)y_2)(t) \in \mathcal{F}_2(x)$. Therefore $\mathcal{F}_2(x)$ is convex. Also by Lemma 3.2, it is clear that the operator $\mathcal{F}_2(x)$ is nonempty and has weakly compact values for every $x \in C(J, L^2(\mathfrak{F}, H))$.

Step 3*: The operator $\mathcal{F}_2(x)$ maps bounded sets into bounded sets in $C(J, L^2(\mathfrak{F}, H))$.*
For any $l > 0$, define $\mathbb{B}_l = \big\{ x \in C(J, L^2(\mathfrak{F}, H)) : \|x\|_C^2 \leq l \big\}$. For every $x \in \mathbb{B}_l$ and $t \in J$.

Then applying Hölder inequality and (H2)–(H3) we have

$$
\begin{aligned}
&\mathbb{E}\left\|\mathcal{F}_2(x)\right\|^2 \\
&\leq 3\left[\mathbb{E}\left\|\int_0^t S(t-s)Bu(s)\,ds\right\|^2 + \mathbb{E}\left\|\int_0^t S(t-s)f(s)\,ds\right\|^2\right. \\
&\qquad \left. + \mathbb{E}\left\|\int_0^t S(t-s)g(s,x(s))\,dw(s)\right\|^2\right], \\
&\leq 3M_s^2\left[T\int_0^t \mathbb{E}\left\|Bu(s)\right\|^2 ds + \int_0^t \mathbb{E}\left\|f(s)\right\|^2 ds + L_G\int_0^t \mathbb{E}\left\|g(s,x(s))\right\|^2 ds\right] \\
&\leq 3M_s^2\left[T\left\|B\right\|^2\left\|u\right\|^2_{L^2_{\mathfrak{F}}(J,U)} + \int_0^t (a_1(s) + c\mathbb{E}\left\|x\right\|^2)\,ds + L_G\int_0^t (a_2(s) + d\mathbb{E}\left\|x\right\|^2)\,ds\right] \\
&\leq 3M_s^2\left[T\left\|B\right\|^2\left\|u\right\|^2_{L^2_{\mathfrak{F}}(J,U)} + \left\|a_1\right\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}T^{1/2} + clT + L_G(\left\|a_2\right\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}T^{1/2} + dlT)\right] \\
&\leq 3M_s^2\left[T\left\|B\right\|^2\left\|u\right\|^2_{L^2_{\mathfrak{F}}(J,U)} + \left\|a_1\right\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)} + L_G\left\|a_2\right\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)} + (c+L_Gd)lT\right] \\
&= l_0.
\end{aligned}
$$

Hence, there exists a positive constant l_0 such that for each $y \in \mathcal{F}_2(x)$, $\mathbb{E}\left\|y\right\|^2 \leq l_0$. Therefore, $\mathcal{F}_2(\mathbb{B}_l)$ is bounded in $C(J, L^2(\mathfrak{F}, H))$.

Step 4: $\{\mathcal{F}_2(x) : x \in \mathbb{B}_l\}$ is equicontinuous.

Initially, for every $x \in \mathbb{B}_l$, when $t_1 = 0$, $0 < t_2 \leq \delta_0$ and δ_0 is sufficiently small, we have that

$$
\begin{aligned}
&\mathbb{E}\left\|(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)\right\|^2 \\
&\leq 2\mathbb{E}\left\|\int_0^{t_2} S(t_2-s)[Bu(s)+f(s)]\,ds\right\|^2 + 2\mathbb{E}\left\|\int_0^{t_2} S(t_2-s)g(s,x(s))\,dw(s)\right\|^2 \\
&\leq 2M_s^2\mathbb{E}\left(\left\|B\right\|^2\left\|u\right\|^2_{L^2_{\mathfrak{F}}(J,U)}\delta_0 + \left\|a_1\right\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}\delta_0^{1/2} + cl\delta_0\right) + 2L_GM_s^2\left(\left\|a_2\right\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}\delta_0^{1/2} + dl\delta_0\right).
\end{aligned}
$$

Thus, $\mathbb{E}\left\|(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)\right\|^2 \to 0$ independently of $x \in \mathbb{B}_l$ as $\delta_0 \to 0$. Likewise, for any $x \in \mathbb{B}_l$, $\delta_0/2 < t_1 < t_2 \leq T$,

$$
\begin{aligned}
&\mathbb{E}\left\|(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)\right\|^2 \\
&= \mathbb{E}\left\|\int_0^{t_2} S(t_2-s)(Bu(s)+f(s))\,ds - \int_0^{t_1} S(t_1-s)(Bu(s)+f(s))\,ds\right. \\
&\qquad \left. + \int_0^{t_2} S(t_2-s)g(s,x(s))\,dw(s) - \int_0^{t_1} S(t_1-s)g(s,x(s))\,dw(s)\right\|^2 \\
&\leq 2\left[\mathbb{E}\left\|\int_0^{t_1}[S(t_2-s)-S(t_1-s)](Bu(s)+f(s))\,ds + \int_{t_1}^{t_2} S(t_2-s)(Bu(s)+f(s))\,ds\right\|^2\right. \\
&\qquad \left. + \mathbb{E}\left\|\int_0^{t_1}[S(t_2-s)-S(t_1-s)]g(s,x(s))\,dw(s) + \int_{t_1}^{t_2} S(t_2-s)g(s,x(s))\,dw(s)\right\|^2\right] \\
&= I_1 + I_2.
\end{aligned}
$$

Consider

$$
\begin{aligned}
I_1 &\le 4\left[\int_0^{t_1} \mathbb{E}\,\|[S(t_2-s)-S(t_1-s)](Bu(s)+f(s))\|^2\,ds + \int_{t_1}^{t_2} \mathbb{E}\,\|S(t_2-s)(Bu(s)+f(s))\|^2\,ds\right]\\
&\le 4\Big[\int_0^{t_1-\delta} \mathbb{E}\,\|S(t_2-s)-S(t_1-s)\|^2\,\|Bu(s)+f(s)\|^2\,ds\\
&\qquad + \int_{t_1-\delta}^{t_1} \mathbb{E}\,\|S(t_2-s)-S(t_1-s)\|^2\,\|Bu(s)+f(s)\|^2\,ds\\
&\qquad + \int_{t_1}^{t_2} \mathbb{E}\,\|S(t_2-s)\|^2\,\|(Bu(s)+f(s))\|^2\,ds\Big]\\
&\le 4\Big[\sup_{s\in[0,t_1-\delta]} \mathbb{E}\,\|[S(t_2-s)-S(t_1-s)]\|^2\\
&\qquad\quad \times \left(\|B\|^2\,\|u\|^2_{L^2_{\mathfrak{F}}(J,U)}\,(t_1-\delta) + \|a_1\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}\,(t_1-\delta)^{1/2} + cl(t_1-\delta)\right)\\
&\qquad + 2M_s^2\left(\|B\|^2\,\|u\|^2_{L^2_{\mathfrak{F}}(J,U)}\,\delta + \|a_1\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}\,\delta^{1/2} + cl\delta\right)\\
&\qquad + M_s^2\left(\|B\|^2\,\|u\|^2_{L^2_{\mathfrak{F}}(J,U)}\,(t_2-t_1) + \|a_1\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}\,(t_2-t_1)^{1/2} + cl(t_2-t_1)\right)\Big].
\end{aligned}
$$

Also,

$$
\begin{aligned}
I_2 &\le 4L_G\Big[\int_0^{t_1-\delta} \mathbb{E}\,\|S(t_2-s)-S(t_1-s)\|^2\,\|g(s,x(s))\|^2\,ds\\
&\qquad + \int_{t_1-\delta}^{t_1} \mathbb{E}\,\|S(t_2-s)-S(t_1-s)\|^2\,\|g(s,x(s))\|^2\,ds\\
&\qquad + \int_{t_1}^{t_2} \mathbb{E}\,\|S(t_2-s)\|^2\,\|g(s,x(s))\|^2\,ds\Big]\\
&\le 4L_G\Big[\sup_{s\in[0,t_1-\delta]} \mathbb{E}\,\|[S(t_2-s)-S(t_1-s)]\|^2\left(\|a_2\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}\,(t_1-\delta)^{1/2} + dl(t_1-\delta)\right)\\
&\qquad + 2M_s^2\left(\|a_2\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}\,\delta^{1/2} + dl\delta\right) + M_s^2\left(\|a_2\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}\,(t_2-t_1)^{1/2} + dl(t_2-t_1)\right)\Big].
\end{aligned}
$$

We noted down that the continuity of $S(t)$, $(t>0)$ in t in the uniform operator topology the right-hand side of the above inequalities I_i $(i=1,2)$ are independent of x and tends to zero as $t_2 \to t_1$ and $\delta \to 0$. Therefore $\mathbb{E}\,\|(\mathcal{F}_2x)(t_2)-(\mathcal{F}_2x)(t_1)\|^2 \to 0$ independently of $x \in \mathbb{B}_l$ as $\delta \to 0$ which implies that the family $\{\mathcal{F}_2(x) : x \in \mathbb{B}_l\}$ is equicontinuous.

Step 5: $\mathcal{F}_2(x)$ is completely continuous.

Let $t \in J$ be fixed. Our claim is to show that the set $\Pi(t) = \{(\mathcal{F}_2x)(t) : x \in \mathbb{B}_l\}$ is relatively compact in H. If $t=0$, then $\Pi(0) = \{(\mathcal{F}_2x)(0) : x \in \mathbb{B}_l\} = \{0\}$ which is compact.

Let $0 < t \le T$ be fixed. For any $x \in \mathbb{B}_l$ and for every $\epsilon \in (0,t)$, define the operator $\mathcal{F}_2^\epsilon$ on $\mathbb{B}_l$ as,

$$
\begin{aligned}
(\mathcal{F}_2^\epsilon x)(t) &= \int_0^{t-\epsilon} S(t-s)[Bu(s)+f(s)]\,ds + \int_0^{t-\epsilon} S(t-s)g(s,x(s))\,dw(s)\\
&= \int_0^{t-\epsilon} S(\epsilon)S(t-s-\epsilon)[Bu(s)+f(s)]\,ds + \int_0^{t-\epsilon} S(\epsilon)S(t-s-\epsilon)g(s,x(s))\,dw(s)
\end{aligned}
$$

$$= S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)[Bu(s)+f(s)]\,ds + S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,x(s))\,dw(s),$$

where $f \in \mathcal{N}(x)$. Noted down that $\mathbb{E}\left\|\int_0^{t-\epsilon} S(t-s-\epsilon)[Bu(s)+f(s)]\,ds\right\|^2$ is bounded, since by using the definition of $\mathbb{B}_l$

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^{t-\epsilon} S(t-s-\epsilon)[Bu(s)+f(s)]\,ds\right\|^2 \\
&\leq \int_0^{t-\epsilon} \mathbb{E}\left\|S(t-s-\epsilon)[Bu(s)+f(s)]\right\|^2 ds \\
&\leq M_s^2\mathbb{E}\left(\|B\|^2\|u\|^2_{L^2_{\mathfrak{F}}(J,U)}(t-\epsilon) + \|a_1\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}(t-\epsilon)^{1/2} + cl(t-\epsilon)\right).
\end{aligned}$$

Already we have assumed that $\mathbb{E}\|x\|_H^2 < \infty$, then $\mathbb{E}\left\|\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,x(s))\,dw(s)\right\|^2$ is bounded. Since by Lemma 2.5 and again by using the definition of $\mathbb{B}_l$

$$\begin{aligned}
\mathbb{E}\left\|\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,x(s))\,dw(s)\right\|^2 &\leq \int_0^{t-\epsilon} \mathbb{E}\left\|S(t-s-\epsilon)g(s,x(s))\right\|^2 ds \\
&\leq L_G M_s^2\left(\|a_2\|_{L^2_{\mathfrak{F}}(J,\mathbb{R}^+)}(t-\epsilon)^{1/2} + dl(t-\epsilon)\right) < \infty.
\end{aligned}$$

By using the compactness of $S(\epsilon)$, $(\epsilon > 0)$ we obtain that the set $\Pi(t) = \{y(t) : \mathcal{F}(\mathbb{B}_l)\}$ is relatively compact in H for every $\epsilon \in (0,t)$. Moreover, for every $x \in \mathbb{B}_l$ we have that

$$\begin{aligned}
&\mathbb{E}\left\|(\mathcal{F}_2 x)(t) - (\mathcal{F}_2^{\epsilon} x)(t)\right\|^2 \\
&\leq 2\int_{t-\epsilon}^{t} \mathbb{E}\|S(t-s)\|^2\|Bu(s)+f(s)\|^2 ds + 2L_G\int_{t-\epsilon}^{t}\|S(t-s)\|^2\mathbb{E}\|g(s,x(s))\|^2 ds \\
&\leq 2M_s^2\left[\|B\|^2\|u\|^2_{L^2_{\mathfrak{F}}(J,U)}\epsilon + \|a_1\|_{L^2_{\mathfrak{F}}(J,R^+)}\epsilon^{1/2} + cl\epsilon + L_G\left(\|a_2\|_{L^2_{\mathfrak{F}}(J,R^+)}\epsilon^{1/2} + dl\epsilon\right)\right],
\end{aligned}$$

which implies that the set $\Pi(t)$ is totally bounded (i.e., relatively compact in H). By Arzela-Ascoli's theorem, we conclude that $\mathcal{F}_2(x)$ is completely continuous.

Step 6: $\mathcal{F}_2(x)$ has a closed graph.
Let $x_n \to x^*$ in $C(J, L^2(\mathfrak{F}, H))$, $y_n \in \mathcal{F}(x_n)$ and $y_n \to y^*$ in $C(J, L^2(\mathfrak{F}, H))$. Then we have to show that $y^* \in \mathcal{F}_2(x^*)$. Let $y_n \in \mathcal{F}_2(x_n)$ then there exists $f_n \in \mathcal{N}(x_n)$ such that

$$y_n(t) = \int_0^t S(t-s)[Bu(s)+f_n(s)]\,ds + \int_0^t S(t-s)g(s,x_n(s))\,dw(s). \tag{3.2}$$

Using (H2)(iii) and (H3), we can show that

$$\{(f_n, g(\,\cdot\,,x_n))\}_{n\geq 1} \subseteq L^2_{\mathfrak{F}}(J,H)\times L_Q(K,H)$$

is bounded. By passing to a subsequence if necessary that

$$(f_n, g(\,\cdot\,,x_n)) \to (f^*, g(\,\cdot\,,x^*)) \quad \text{weakly in } L^2_{\mathfrak{F}}(J,H)\times L_Q(K,H). \tag{3.3}$$

Since we have $S(t)$ is compact, also with the hypothesis (H3), equating (3.2) and (3.3) we get

$$y_n(t) \to \int_0^t S(t-s)[Bu(s)+f^*(s)]\,ds + \int_0^t S(t-s)g(s,x^*(s))\,dw(s). \tag{3.4}$$

We note that $y_n \to y^*$ in $C(J,L^2(\mathfrak{F},H))$ and $f_n \in \mathcal{N}(x_n)$. From Lemma 3.2 and (3.4), we obtain that $f^* \in \mathcal{N}(x^*)$. Therefore $y^* \in \mathcal{F}_2(x^*)$. Thus, $\mathcal{F}_2(x)$ has a closed graph. By proposition 3.3.12(2) in [16], $\mathcal{F}_2$ is u.s.c.

Step 7: The operator inclusion $\chi x \in \mathcal{F}_1(x)+\mathcal{F}_2(x)$ has a solution for $\chi = 1$. According to the Theorem 2.6, it suffices to show that no element $x \in C(J,L^2(\mathfrak{F},H))$ exists such that $\chi x \in \mathcal{F}_1(x)+\mathcal{F}_2(x)$ for some $\chi > 1$. Suppose $\chi x \in \mathcal{F}_1(x)+\mathcal{F}_2(x)$ for some $\chi > 1$ and there exists $f \in \mathcal{N}(x)$ such that

$$\begin{aligned}\chi x(t) &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)[Bu(s)+f(s)]\,ds + \int_0^t S(t-s)g(s,x(s))\,dw(s)\\ &\quad + \int_0^t S(t-s)\int_Z h(s,x(s),\eta(s))\widetilde{N}(ds,d\eta).\end{aligned}$$

Using (H2)–(H4) we get

$$\begin{aligned}\mathbb{E}\left\|x(t)\right\|_H^2 &\le 6\Big[M_c^2\mathbb{E}\left\|x_0\right\|^2 + M_s^2\mathbb{E}\left\|y_0\right\|^2 + M_s^2\int_0^t \mathbb{E}\left\|Bu(s)\right\|^2 ds + M_s^2\int_0^t \mathbb{E}\left\|f(s)\right\|^2 ds\\ &\qquad + L_G M_s^2\int_0^t \mathbb{E}\left\|g(s,x(s))\right\|^2 ds + M_s^2\int_0^t\int_Z \mathbb{E}\left\|h(s,x(s),\eta(s))\lambda(d\eta)\right\|^2 ds\Big]\\ &\le 6\Big[M_c^2\mathbb{E}\left\|x_0\right\|^2 + M_s^2\mathbb{E}\left\|y_0\right\|^2 + M_s^2\left\|B\right\|^2\left\|u\right\|^2_{L^2_{\mathfrak{F}}(J,U)} t + M_s^2\left\|a_1\right\|_{L^2_{\mathfrak{F}}(J,R^+)} t^{1/2}\\ &\qquad + M_s^2 c\int_0^t \mathbb{E}\left\|x(s)\right\|^2 ds + L_G M_s^2\left(\left\|a_2\right\|_{L^2_{\mathfrak{F}}(J,R^+)} t^{1/2} + d\int_0^t \mathbb{E}\left\|x(s)\right\|^2 ds\right)\\ &\qquad + M_s^2\left(\left\|a_3\right\|_{L^2_{\mathfrak{F}}(J,R^+)} t^{1/2} + e\int_0^t \mathbb{E}\left\|x(s)\right\|^2 ds\right)\Big]\\ &\le \rho + 6(M_s^2 c + L_G M_s^2 d + M_s^2 e)\int_0^t \mathbb{E}\left\|x(s)\right\|^2 ds,\end{aligned}$$

where

$$\begin{aligned}\rho &= 6M_c^2\mathbb{E}\left\|x_0\right\|^2 + M_s^2\Big\{\mathbb{E}\left\|y_0\right\|^2 + \left\|B\right\|^2\left\|u\right\|^2_{L^2_{\mathfrak{F}}(J,U)} t + \left\|a_1\right\|_{L^2_{\mathfrak{F}}(J,R^+)} t^{1/2}\\ &\quad + L_G\left\|a_2\right\|_{L^2_{\mathfrak{F}}(J,R^+)} t^{1/2} + \left\|a_3\right\|_{L^2_{\mathfrak{F}}(J,R^+)} t^{1/2}\Big\}\end{aligned}$$

Put $C_1 = 6M_s^2(c+L_G d+e)$. Thus, $\mathbb{E}\left\|x(t)\right\|^2 \le \rho + C_1\int_0^t \mathbb{E}\left\|x(s)\right\|^2 ds$. By using Gronwall's inequality

$$\mathbb{E}\left\|x(t)\right\|^2 \le \rho e^{C_1 t},$$

which implies $\mathbb{E}\left\|x\right\|^2_{L^2} \le \rho e^{C_1 t} = r$, with ρ and C_1 are the positive constants.

Set $V_r = \left\{x \in C(J, L^2(\mathfrak{F}, H)) : \mathbb{E}\,\|x\|^2_{L^2} < r+1\right\}$. Clearly, V_r is an open subset of the Banach space $C(J, L^2(\mathfrak{F}, H))$. From the choice of V_r, there is no $x \in C(J, L^2(\mathfrak{F}, H))$ satisfying $\chi x \in \mathcal{F}_1(x) + \mathcal{F}_2(x)$ for some $\chi > 1$. Hence by the Theorem 2.6 we conclude that the operator inclusion $x \in \mathcal{F}(x) = \mathcal{F}_1(x) + \mathcal{F}_2(x)$ has a solution which is the mild solution of the system (3.1) in J. □

3.2. Existence of optimal control

In this section, we derive the existence of optimal control for the problem (3.1) with the corresponding cost function.

Let Y be the reflexive Banach space in which the control u takes values. Define the multivalued map $\Lambda\colon J \to P(V)$ and V is closed, convex and bounded such that $\Lambda(\cdot)$ is measurable. Let X be a bounded subset of Y such that $\Lambda(\cdot) \subseteq X$. Let U be the set of all admissible controls and

$$U = \left\{u(\cdot) \in L^2_{\mathfrak{F}}(J, X) \text{ such that } u(t) \in \Lambda(t) \text{ a.e.}\right\}.$$

Clearly U is nonempty and $U \subset L^2_{\mathfrak{F}}(J, Y)$ is bounded, closed and convex (see [21]). We consider the stochastic optimal control problem (P) as follows:

Find a control $(x^0, u^0) \in C(J, L^2(\mathfrak{F}, H)) \times U$ such that $\mathcal{J}(x^0, u^0) \le \mathcal{J}(x, u)$, $\forall\, u \in U$, where $\mathcal{J}(x, u) = \mathbb{E}\int_0^T \mathcal{L}(t, x(t), u(t))\,dt$ is the performance index.

We make the following assumptions (H5):

(i) The functional $\mathcal{L}\colon J \times H \times Y \to \mathbb{R} \cup \{\pm\infty\}$ is Borel measurable.

(ii) $\mathcal{L}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $H \times Y$ for a.e. $t \in J$. That is, $\forall\, x \in H$, $u \in Y$, $\{x^n\} \subset H$ and $\{u^n\} \subset Y$ such that $x^n \to x$ in H and $u^n \to u$ in Y we have $\lim_{x^n \to x} \inf \mathcal{L}(t, x^n, u^n) \ge \mathcal{L}(t, x, u)$.

(iii) $\mathcal{L}(t, x, \cdot)$ is convex on Y for every $x \in H$ and $t \in J$ a.e.

(iv) there exists constants $p, q \ge 0$, κ is nonnegative and $\kappa \in L^1(J, \mathbb{R})$ such that $\mathcal{L}(t, x(t), u(t)) \ge \kappa(t) + p\mathbb{E}\,\|x\|^2_H + q\mathbb{E}\,\|u\|^2_Y > -\infty$.

Theorem 3.5. *Suppose the hypotheses* (H1)–(H5) *are satisfied. Let B be a strongly continuous operator then the optimal control problem (P) admits at least one optimal pair.*

Proof. Suppose $\inf\{\mathcal{J}(x, u) : u \in U\} = \infty$ then the result is obvious. Assuming that $\inf\{\mathcal{J}(x, u) : u \in U\} = m < \infty$. Using the hypotheses on $\mathcal{L}$ we have that $\mathcal{J}(x, u) \ge m > -\infty$.

By the definitions of infimum, there exists a minimizing sequences of feasible pair $\{x^n, u^n\} \subset A$ where $A = \{(x, u) : x$ is a mild solution of the proposed system corresponding to $u \in U\}$ such that $\mathcal{J}(x^n, u^n) \to m$ as $n \to +\infty$. Since $\{u^n\} \subset U$, $n = 1, 2, \ldots$ and $\{u_n\}$ is a bounded subset of a reflexive Banach space, there exists a subsequence denoted by $\{u^n\}$ again such that $u^n \to u^0$ weakly in $L^2_{\mathfrak{F}}(J, Y)$. We have that U is closed and convex, by using Marzur lemma $u^0 \in U$. Suppose $\{x^n\}$ is the sequence of mild solution of the proposed system corresponding to u^n and $f^n \in \mathcal{N}(x^n)$ then,

$$\begin{aligned} x^n(t) = C(t)x_0 + S(t)y_0 &+ \int_0^t S(t-s)[Bu^n(s) + f^n(s)]\, ds \\ &+ \int_0^t S(t-s) g(s, x^n(s))\, dw(s) \\ &+ \int_0^t S(t-s) \int_Z h(s, x^n(s), \eta(s)) \widetilde{N}(ds, d\eta), \quad \forall\, t \in J. \end{aligned}$$

Also x^0 is the mild solution corresponding to u^0 and $f^0 \in \mathcal{N}(x^0)$ then

$$\begin{aligned} x^0(t) = C(t)x_0 + S(t)y_0 &+ \int_0^t S(t-s)[Bu^0(s) + f^0(s)]\, ds \\ &+ \int_0^t S(t-s) g(s, x^0(s))\, dw(s) \\ &+ \int_0^t S(t-s) \int_Z h(s, x^0(s), \eta(s)) \widetilde{N}(ds, d\eta), \quad \forall\, t \in J. \end{aligned}$$

From the boundedness of $\{u^n\}$, $\{u^0\}$ and Theorem 3.4, it follows that there exists a positive number M such that $\mathbb{E}\, \|x^n\|^2 \le M$, $\mathbb{E}\, \|x^0\|^2 \le M$.

For every $t \in J$, we obtain that

$$\begin{aligned} &\mathbb{E}\, \left\|x^n(t) - x^0(t)\right\|^2 \\ &\le 4\Big[\int_0^t \mathbb{E}\, \|S(t-s)\|^2 \left\|Bu^n(s) - Bu^0(s)\right\|^2 ds + \int_0^t \mathbb{E}\, \|S(t-s)\|^2 \left\|f^n(s) - f^0(s)\right\|^2 ds \\ &\quad + L_G \int_0^t \mathbb{E}\, \|S(t-s)\|^2 \left\|g(s, x^n(s)) - g(s, x^0(s))\right\|^2 ds \\ &\quad + \int_0^t \mathbb{E}\, \|S(t-s)\|^2 \int_Z \mathbb{E}\, \left\|h(s, x^n(s), \eta) - h(s, x^0(s), \eta)\lambda(d\eta)\right\|^2 ds\Big] \\ &\le 4M_s^2\Big[\int_0^t \mathbb{E}\, \left\|Bu^n(s) - Bu^0(s)\right\|^2 ds + \int_0^t \sup\left(\mathbb{E}\, \left\|f^n(s) - f^0(s)\right\|^2\right) ds \\ &\quad + L_G M_g \int_0^t \mathbb{E}\, \left\|x^n(s) - x^0(s)\right\|^2 ds + M_h \mathbb{E} \int_0^t \mathbb{E}\, \left\|x^n(s) - x^0(s)\right\|^2 ds\Big] \\ &= 4M_s^2\Big[\int_0^t \mathbb{E}\, \left\|Bu^n(s) - Bu^0(s)\right\|^2 ds + \int_0^t \mathbb{E}\, \left\|\partial F(s, x^n(s)) - \partial F(s, x^0(s))\right\|^2 ds \\ &\quad + L_G M_g \int_0^t \mathbb{E}\, \left\|x^n(s) - x^0(s)\right\|^2 ds + M_h \mathbb{E} \int_0^t \mathbb{E}\, \left\|x^n(s) - x^0(s)\right\|^2 ds\Big] \\ &\le 4M_s^2 \int_0^t \mathbb{E}\, \left\|Bu^n(s) - Bu^0(s)\right\|^2 ds + 4M_s^2(M_f + L_G M_g + M_h) \int_0^t \mathbb{E}\, \left\|x^n(s) - x^0(s)\right\|^2 ds. \end{aligned}$$

By the virtue of singular version of Gronwall's inequality, there exists a constant $\widetilde{M} > 0$ such that

$$\mathbb{E}\left\|x^n(t) - x^0(t)\right\|^2 \leq \widetilde{M}\mathbb{E}\left\|Bu^n - Bu^0\right\|^2_{L^2_{\mathfrak{F}}(J,Y)},$$

where $\widetilde{M} = 4M_s^2Te^{\alpha T}$ and α is a positive constant. Also we have that B is strongly continuous and $u^n \to u^0$ weakly then $\mathbb{E}\left\|Bu^n - Bu^0\right\|^2_{L^2_{\mathfrak{F}}(J,Y)} \to 0$ weakly as $n \to \infty$. Therefore, $\mathbb{E}\left\|x^n(t) - x^0(t)\right\|^2 \to 0$ weakly as $n \to \infty$. This provides $x^n \to x^0$ weakly in $C(J, L^2(\mathfrak{F}, H))$ as $n \to \infty$ [1].

Note that the assumptions (H5) implies that all the hypotheses of Balder's theorem [2] are satisfied. Thus, $(x,u) \to \mathbb{E}\left(\int_0^t \mathcal{L}(t, x^0(t), u^0(t))\,dt\right)$ is sequentially lower semicontiunous in $L^2_{\mathfrak{F}}(J,Y)$. Hence $\mathcal{J}$ is weakly lower semicontinuous. Using the condition (H5)(iv), $\mathcal{J}$ attains its infimum at $u^0 \in U$, i.e.,

$$m = \lim_{n\to\infty} \mathbb{E}\int_0^t \mathcal{L}(t, x^n(t), u^n(t))\,dt \geq \mathbb{E}\int_0^t \mathcal{L}(t, x^0(t), u^0(t))\,dt = \mathcal{J}(x^0, u^0) \geq m,$$

i.e.,

$$\mathbb{E}\int_0^t \mathcal{L}(t, x^0(t), u^0(t))\,dt = \mathcal{J}(x^0, u^0) = m = \inf_{u\in U} \mathcal{J}(x,u).$$

Hence (x^0, u^0) is the required optimal control pair. $\square$

4. Example

We consider the nonlinear stochastic wave equation with initial and boundary conditions driven by Poisson jumps as

(4.1)

$$\begin{aligned}
\partial\left(\frac{\partial y(t,x)}{\partial t}\right) &= \left[\frac{\partial^2 y(t,x)}{\partial x^2} + f(t,x)\right]\partial t + h_1\left(t, \int_0^1 p_1(s)y(t,x)\,ds\right)dw(t) \\
&\quad + \left(\int_0^1 b(x,s)u(s,t)\,ds\right)\partial t + \int_Z y(t,x)\eta\widetilde{N}(dt,d\eta), \quad t \in [0,1] = J,\ x \in [0,\pi], \\
y(t,0) &= y(t,\pi) = 0 \quad \text{and} \quad y(0,x) = y_0(x), \quad 0 \leq t \leq 1, \\
\frac{\partial y(0,x)}{\partial t} &= y_1(x), \quad 0 < x < \pi.
\end{aligned}$$

Let $H = U = L^2([0,\pi])$. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be the complete probability space. Let f be the known measurable $\mathfrak{F}_t$-adapted multivalued function of y of the form $-f(t,x) \in \partial F(x,t,y(t,x))$ a.e. Here $\partial F(x,t,\xi)$ denotes the Clarke's generalized gradient with respect to the last variable of the function F and $F\colon [0,\pi]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ is a locally Lipschitz energy function in ξ which is generally nonsmooth and nonconvex. The multivalued function $\partial F(x,t,\xi)\colon \mathbb{R} \to 2^{\mathbb{R}}$ is non-monotone and it includes the vertical jumps. Here $h_1\colon [0,\infty)\times\mathbb{R} \to L_Q(\mathbb{R})$, $p_1\colon [0,\infty) \to \mathbb{R}$ be the continuous functions. Let $w(t)$ be the $\mathfrak{F}_t$-adapted standard Wiener process in H. Let $\{k(t) : t \in J\}$ is a $\mathfrak{F}_t$-adapted Poisson point process, independent of the Wiener process $w(t)$, taking its values in the space $[0,\infty)$ with σ-finite measure $\lambda(d\eta)$. Let $N(ds,d\eta)$ be the Poisson counting measure induced by $k(\cdot)$ and $\widetilde{N}(ds,d\eta) = N(ds,d\eta) - \lambda(d\eta)ds$ be its compensating martingale. The operator A is defined as

$$A\zeta = \zeta''$$

with the domain $D(A) = \{\zeta \in H : \zeta, \zeta'$ are absolutely continuous $\zeta'' \in H, \zeta(0) = \zeta(1) = 0\}$. The spectrum of A consists of eigenvalues $-n^2$ for $n \in N$ with associated eigenvectors $e_n(\zeta) = \sqrt{2/\pi}\sin(n\zeta)$. Moreover, the set $\{e_n : n \in N\}$ is an orthonormal basis of H.

$$Ax = \sum_{n=1}^{\infty} -n^2 \langle x, e_n \rangle e_n, \quad x \in D(A).$$

The operators

$$C(t)x = \sum_{n=1}^{\infty} \cos(nt) \langle x, e_n \rangle e_n, \quad e \in \mathbb{R},$$
$$S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle x, e_n \rangle e_n, \quad e \in \mathbb{R}.$$

From [22], $\forall\, x \in H$ and $t \in \mathbb{R}$, $\|S(t)\| \le 1$ and $\|C(t)\| \le 1$. Now consider the function $F\colon (0,1) \to \mathbb{R}$ defined by

$$F(t,y) = \int_0^T \sigma(x,t,y(x))\,dx \quad \text{for a.e. } t \in (0,1),\ y \in H,$$

where $\sigma(x,t,z) = \int_0^z \phi(x,t,\theta)\,d\theta$, $(x,t) \in (0,\pi) \times (0,1)$, $z \in \mathbb{R}$. Assume that $\phi\colon (0,\pi) \times (0,1) \times \mathbb{R} \to \mathbb{R}$ is a function satisfying

(i) for every $x \in (0,\pi)$ and $z \in \mathbb{R}$, $\phi(\,\cdot\,,x,z)\colon (0,1) \to \mathbb{R}$ is measurable.

(ii) for every $t \in (0,1)$ and $z \in \mathbb{R}$, $\phi(t,\cdot\,,z)\colon (0,\pi) \to \mathbb{R}$ is continuous.

(iii) for all $z \in \mathbb{R}$, there exists a positive constant r_1 such that $|\phi(\,\cdot\,,\cdot\,,z)| \le r_1(1+|z|)$.

(iv) for all $z \in \mathbb{R}$, $\phi(\,\cdot\,,\cdot\,,z \pm 0)$ exists.

If ϕ satisfies the conditions (iii), then we have that $\partial\sigma(z) \subset [\underline{\phi}(z), \overline{\phi}(z)]$ for $z \in R$ (omit (x,t) here), where $\underline{\phi}(z)$ and $\overline{\phi}(z)$ denote the essential supremum and essential infimum of ϕ at z (see p. 34 in [3]).

If ϕ satisfies the conditions (i)–(iv), then the function σ defined above is such that

(i) for every $x \in (0,\pi)$, $z \in \mathbb{R}$, $\sigma(\,\cdot\,,x,z)\colon (0,1) \to \mathbb{R}$ is measurable and $\sigma(\,\cdot\,,\cdot\,,0) \in L^2((0,\pi) \times (0,1))$.

(ii) for every $t \in (0,1)$, $z \in \mathbb{R}$, $\sigma(t, \cdot, z)\colon (0,\pi) \to \mathbb{R}$ is continuous.

(iii) for all $(x,t) \in (0,\pi) \times (0,1)$, $\sigma(x,t,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.

(iv) there exists $r_2 > 0$ such that $|\gamma| \leq r_2(1 + |z|)$ for all $\gamma \in \partial\sigma(x,t,z,)$, $(x,t) \in (0,\pi) \times (0,1)$.

(v) there exists $r_3 > 0$ such that $\sigma^0(x,t,z;-z) \leq r_3(1+|z|)$ for all $(x,t) \in (0,\pi) \times (0,1)$.

We take the function $u\colon \Psi x([0,1]) \to \mathbb{R}$ as the control such that $u \in L^2(\Psi x([0,1]))$ and $t \to u(t)$ is measurable. The set $\mathcal{C} = \{u \in U : \|u\|_U \leq \mu\}$ where $\mu \in L^2(J,\mathbb{R}^+)$. We restricted that the admissible controls to be all $u \in L^2(\Psi x([0,1]))$ such that $\|u(\cdot,t)\|^2 \leq \mu(t)$ a.e. $t \in J$.

Now consider

$$\begin{gathered} y(t)(x) = y(t,x), \quad t \in [0,1],\ x \in [0,\pi], \\ B(t)u(t,x) = B(t)u(t)(x) = \int_0^1 b(x,s)u(s,t)\,ds, \\ Q_1(\phi(\xi)) = \int_0^1 p_1(\theta)\phi(\theta,\xi)\,d\theta, \\ \sigma(t,x) = h_1(t,Q_1(t)(x)), \quad h(t,y,\eta) = y(t,x)\eta = y(t)(x)\eta. \end{gathered}$$

By assuming that the nonlinear functions satisfying the hypotheses (H2)–(H4) and the boundedness of the above functions, we observe that the system (4.1) can be rewritten in the abstract form of (3.1). Since all the hypotheses of Theorem 3.4 are satisfied, there exists a mild solution for the system (4.1). Consider the following cost function: $\mathcal{J}(u) = \mathbb{E}\left\{\int_0^1 L(t,y(t),u(t))\,dt\right\}$ where $L(t,y(t),u(t))(x) = \int_0^1\int_0^1 |y(t,x)|^2\,dxdt + \int_0^1\int_0^1 |u(t,x)|^2\,dxdt$ and it is easy to see that the hypotheses of the Theorem 3.5 are satisfied. Therefore, there exists at least one optimal pair for the problem (4.1).

5. Conclusion

This paper has investigated that the optimal control study of second order stochastic evolution hemivariational inequalities with Poisson jumps in Hilbert space. The existence of mild solution for the proposed system has been formulated and proved by utilizing the semigroup of operators theory, stochastic analysis techniques, properties of generalized Clarke subdifferential operators and a fixed point theorem of multivalued maps. Additionally, the existence of optimal control for the considered system has been discussed. Finally, the obtained results have been verified through an example. Due to the importance of SEEs and hemivariational inequalities in both theoretical and real-life applications to mechanical problems, it is significant to find its existence, optimal control results and other quantitative and qualitative properties in infinite dimensional spaces. Some kinds of dynamical systems requires both Poisson jumps and fractional Brownian motion to model its dynamics. Hence in the forthcoming paper, we will consider the optimal control results for fractional differential equations with delays and having mixed fractional Brownian motion in Hilbert spaces.

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