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POWER CENTRAL VALUES OF DERIVATIONS ON MULTILINEAR POLYNOMIALS

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Abstract. Let R be a prime ring with extended centroid C, ρ a nonzero right ideal of R, d a nonzero derivation of R, $f(X_1, \dots, X_t)$ a multilinear polynomial over C, $a \in R$ and n a fixed positive integer.

(I) If $ad(f(x_1, \dots, x_t))^n = 0$ ($d(f(x_1, \dots, x_t))^n a = 0$) for all $x_1, \dots, x_t \in \rho$, then either $a\rho = 0$ ($a = 0$ resp.), $d(\rho)\rho = 0$ or $\rho C = eRC$ for some idempotent e in the socle of RC such that $f(X_1, \dots, X_t)$ is central-valued on $eRCe$.

(II) If $ad(f(x_1, \dots, x_t))^n \in C$ ($d(f(x_1, \dots, x_t))^n a \in C$) for all $x_1, \dots, x_t \in \rho$ and $ad(f(y_1, \dots, y_t))^n \neq 0$ ($d(f(y_1, \dots, y_t))^n a \neq 0$) for some $y_1, \dots, y_t \in \rho$, then either $f(\rho)\rho = 0$ or $f(X_1, \dots, X_t)$ is central-valued on RC unless $\dim_C RC = 4$.

Throughout this paper, R always denotes a prime ring with center Z, extended centroid C and two-sided Martindale quotient ring Q. Let d be a derivation of R and n a fixed positive integer. In [7] Giambruno and Herstein proved that if $d(x)^n = 0$ for all $x \in R$, then $d = 0$. In [2] Bresar proved that if R is a semiprime ring and $a \in R$ such that $ad(x)^n = 0$, then $ad(R) = 0$ when R is $(n-1)!$-torsion free. Lee and Lin proved Bresar's result on Lie ideals of prime rings without the $(n-1)!$-torsion free assumption [15]. They proved that if $ad(x)^n = 0$ for all $x \in L$, a Lie ideal of R, then $ad(L) = 0$ unless $\operatorname{char} R = 2$ and $\dim_C RC = 4$. In addition, if $[L, L] \neq 0$, then $ad(R) = 0$. It is well-known that a derivation d on R can be extended uniquely to a derivation on Q. In the first part of this paper, we will generalize above results by proving the following.

Theorem 1. *Let R be a prime ring with extended centroid C, ρ a nonzero right ideal of R, d a nonzero derivation of R, $f(X_1, \dots, X_t)$ a multilinear polynomial over C, $a \in R$ and n a fixed positive integer.*

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(I) *If* $ad(f(x_1,\dots,x_t))^n = 0$ *for all* $x_1,\dots,x_t \in \rho$, *then either* $a\rho = 0$, $d(\rho)\rho = 0$ *or* $\rho C = eRC$ *for some idempotent* e *in the socle of* RC *such that* $f(X_1,\dots,X_t)$ *is central-valued on* $eRCe$.

(II) *If* $d(f(x_1,\dots,x_t))^n a = 0$ *for all* $x_1,\dots,x_t \in \rho$, *then either* $a = 0$, $d(\rho)\rho = 0$ *or* $\rho C = eRC$ *for some idempotent* e *in the socle of* RC *such that* $f(X_1,\dots,X_t)$ *is central-valued on* $eRCe$.

For the central case Herstein showed that if d is a derivation of R such that $d(x)^n \in Z$ for all $x \in R$, then either $d = 0$ or $\dim_C RC \cdot 4$ [8]. In [18], Wong considered the derivation acting on multilinear polynomials. Let R be a prime algebra over a commutative ring K with unit. If $d(f(x_1,\dots,x_t))^n \in Z$, for all $x_1,\dots,x_t \in R$, where $f(X_1,\dots,X_t)$ is a multilinear polynomial over K, then $f(X_1,\dots,X_t)$ is central-valued on R unless R is an order in a 4-dimensional simple algebra. In [3], Chang and Lee consider the derivation acting on one-sided ideals. Let ρ be a right ideal of R such that $[\rho,\rho]\rho \neq 0$ and $\dim_C RC > 4$. They proved that if $ad([x,y])^n \in Z$ ($d([x,y])^n a \in Z$), then either $ad(\rho) = 0$ ($a = 0$ resp.) or d is the inner derivation induced by some element $p \in Q$ such that $p\rho = 0$. In the second part of this paper, we will prove the following.

Theorem 2. *Let* R *be a prime ring with extended centroid* C, ρ *a nonzero right ideal of* R, d *a nonzero derivation of* R, $f(X_1,\dots,X_t)$ *a multilinear polynomial over* C, $a \in R$ *and* n *a fixed positive integer.*

(I) *If* $ad(f(x_1,\dots,x_t))^n \in C$ *for all* $x_1,\dots,x_t \in \rho$ *and* $ad(f(y_1,\dots,y_t))^n \neq 0$ *for some* $y_1,\dots,y_t \in \rho$, *then either* $f(\rho)\rho = 0$ *or* $f(X_1,\dots,X_t)$ *is central-valued on* RC *unless* $\dim_C RC = 4$.

(II) *If* $d(f(x_1,\dots,x_t))^n a \in C$ *for all* $x_1,\dots,x_t \in \rho$ *and* $d(f(y_1,\dots,y_t))^n a \neq 0$ *for some* $y_1,\dots,y_t \in \rho$, *then either* $f(\rho)\rho = 0$ *or* $f(X_1,\dots,X_t)$ *is central-valued on* RC *unless* $\dim_C RC = 4$.

For a nonzero right ideal ρ of R and a polynomial $f(X_1,\dots,X_t)$ over C, let $A = \{f(x_1,\dots,x_t) | x_i \in \rho\}$ and denote by $f(\rho)$ the additive subgroup of RC generated by A. We denote by $f^d(X_1,\dots,X_t)$ the polynomial obtained from $f(X_1,\dots,X_t)$ by replacing each coefficient α with $d(\alpha)$. The first step is to discuss the structure of A. The following lemma is well-known.

Lemma 1. *Let* $f(X_1,\dots,X_t)$ *be a polynomial over* C, *not necessarily multilinear. If* $t \in \rho$ *with* $t^2 = 0$, *then* $(1-t)A(1+t) = A$.

Lemma 2. *Let* R *be a prime ring,* ρ *a nonzero right ideal of* R, $f(X_1,\dots,X_t)$ *a multilinear polynomial over* C, $a \in R$ *and* n *a fixed positive integer.*

(I) *If* $af(x_1,\dots,x_t)^n = 0$ *for all* $x_1,\dots,x_t \in \rho$, *then either* $a\rho = 0$ *or* $f(\rho)\rho = 0$.

(II) *If* $f(x_1, \dots, x_t)^n a = 0$ *for all* $x_1, \dots, x_t \in \rho$, *then either* $a = 0$ *or* $f(\rho)\rho = 0$.

Proof. (I) Since $af(x_1, \dots, x_t)^n = 0$ for all $x_1, \dots, x_t \in \rho$, we have $a\rho = 0$ or $f(X_1, \dots, X_t)^n$ ia a PI for ρ by [4, Theorem 4]. In the latter case, $f(\rho)\rho = 0$ by [5, Main Theorem′].

(II) By assumption, $f(x_1, \dots, x_t)^n a = 0$ for all $x_1, \dots, x_t \in \rho$. By [5, Lemma 3], either $f(x_1, \dots, x_t)^n \rho = 0$ for all $x_1, \dots, x_t \in \rho$ or $a = 0$. In the case that $f(x_1, \dots, x_t)^n \rho = 0$ for all $x_1, \dots, x_t \in \rho$, we have $f(x_1, \dots, x_t)^{n+1} = 0$ for all $x_i \in \rho$. Thus $f(\rho)\rho = 0$ by [5, Main Theorem′] again and the lemma is proved.

Now we will discuss the structure of A when $R = \mathrm{M}_m(F)$, the set of $m \times m$ matrices over a field F. We may assume $\rho = eR = (e_{11} + e_{22} + \cdots + e_{ll})R$. Let $E = \{\alpha e_{ij} | \alpha \in F, 1 \cdot\ i \cdot\ l, 1 \cdot\ j \cdot\ m\}$ and $B = \{f(x_1, \dots, x_t) | x_i \in E\}$. Note that $B \subseteq A \subseteq f(\rho)$. For $b \in B$, either b is a diagonal matrix or $b = \alpha e_{ij}$ for some $\alpha \in F$ and $i \neq j$ by [16, Lemma 2]. Since $f(X_1, \dots, X_t)$ is multilinear, for any $x \in A$, $x = \sum b_i$ for some $b_i \in B$. Thus for any $x \in f(\rho)$, $x = \sum b_i$ for some $b_i \in B$.

Lemma 3. *Let* $R = \mathrm{M}_m(F)$, $\rho = eR = (e_{11} + e_{22} + \cdots + e_{ll})R$ *and* $f(X_1, \dots, X_t)$ *a multilinear polynomial over a field* F.

(I) *If* $f(X_1, \dots, X_t)$ *is not an identity of* ρ, *then* $\alpha e_{ij} \in A$ *for all* $\alpha \in F, i \cdot\ l$ *and* $j > l$.

(II) *If* $f(X_1, \dots, X_t)$ *is not central-valued on* eRe, *then*

(i) $\alpha e_{ij} \in A$ *for all* $\alpha \in F, i \cdot\ l$ *and* $j \neq i$.

(ii) $\alpha e_{ij} + \beta e_{ik} \in A$ *for all* $\alpha, \beta \in F, i, j \cdot\ l, i \neq j$ *and* $k \neq i, j$.

(iii) $\alpha e_{ik} + \beta e_{jk} \in A$ *for all* $\alpha, \beta \in F, i, j \cdot\ l, i \neq j$ *and* $k \neq i, j$.

Proof. (I) By [4, Theorem 1 (II) (i)], $\alpha e_{ij} \in f(\rho)$ for all $\alpha \in F, i \cdot\ l$ and $j > l$. Thus $\alpha e_{ij} \in A$ by the observation in previous paragraph.

(II) (i) By [4, Theorem 1 (II) (iii)], $\alpha e_{ij} \in f(\rho)$ for all $\alpha \in F, i \cdot\ l$ and $j \neq i$. Thus $\alpha e_{ij} \in A$ by observation in the previous paragraph.

(ii) In the case that $\alpha = 0$, we have $\beta e_{ij} \in A$ by (i). Thus we may assume that $\alpha \neq 0$. For any $\beta \in F, i, j \cdot\ l, i \neq j$, and $k \neq i, j$, $\alpha e_{ij} + \beta e_{ik} = (1 - \alpha^{-1}\beta e_{jk})(\alpha e_{ij})(1 + \alpha^{-1}\beta e_{jk}) \in A$ by Lemma 1 and (i).

(iii) We may assume that $\alpha \neq 0$ as in (ii). For any $\beta \in F$, $i, j \cdot\ l, i \neq j$, and $k \neq i, j$, $\alpha e_{ik} + \beta e_{jk} = (1 + \alpha^{-1}\beta e_{ji})(\alpha e_{ik})(1 - \alpha^{-1}\beta e_{ji}) \in A$ by Lemma 1 and (i).

The following lemma is a special case of the Theorem 1.

Lemma 4. *If $R \cong \mathrm{M}_m(F)$ and d is inner, where F is a field, then Theorem 1 holds.*

Proof. We may assume $R = \mathrm{M}_m(F)$, $\rho = eR = (e_{11} + e_{22} + \cdots + e_{ll})R$ and $d(r) = pr - rp$ for some $p \in R$. Write $p = \sum p_{ij}e_{ij}$, where $p_{ij} \in F$.

(I) Suppose that $f(X_1, \ldots, X_t)$ is not central-valued on eRe. Suppose first that $p_{ij} \neq 0$ for some $j \cdot \ l$ and $i \neq j$. Since $e_{ji} \in A$ by Lemma 3, $a(pe_{ji} - e_{ji}p)^n = 0$. Now $0 = a(pe_{ji} - e_{ji}p)^n e_{jj} = (-p_{ij})a(pe_{ji} - e_{ji}p)^{n-1}e_{jj} = \cdots = (-p_{ij})^n ae_{jj}$. Hence $ae_{jj} = 0$. For other $j' \cdot \ l, j' \neq i$, if $p_{ij'} \neq 0$, then we obtain that $ae_{j'j'} = 0$ as above. If $p_{ij'} = 0$, let $x = e_{ji} + e_{j'i}$, then $x \in A$ by Lemma 3 and $a(p(e_{ji} + e_{j'i}) - (e_{ji} + e_{j'i})p)^n = 0$. Now $0 = a(p(e_{ji} + e_{j'i}) - (e_{ji} + e_{j'i})p)^n(e_{jj'} + e_{j'j'}) = (-p_{ij})a(p(e_{ji} + e_{j'i}) - (e_{ji} + e_{j'i})p)^{n-1}(e_{jj'} + e_{j'j'}) = \cdots = (-p_{ij})^n a(e_{jj'} + e_{j'j'})$. Since $ae_{jj'} = ae_{jj}e_{jj'} = 0$, then $ae_{j'j'} = 0$. Thus we have shown that $ae_{jj} = 0$ for all $j \cdot \ l$, $j \neq i$. If $i > l$, then $ae_{jj} = 0$ for all $j \cdot \ l$, and $a\rho = 0$. We are done in this case. Now assume that $i \cdot \ l$. If $p_{ki} \neq 0$ for some $k \neq i$, then we obtain that $ae_{ii} = 0$ as above and $a\rho = 0$. Next suppose that $p_{ki} = 0$ for all $k \neq i$. If $p_{ks} \neq 0$ for some $s \cdot \ l$ and $s \neq k, i$, then we obtain that $ae_{ii} = 0$ as above. Hence $a\rho = 0$ and we are done. Thus we may assume that $p_{i'j'} = 0$ for all $j' \cdot \ l$ and $i' \neq j'$ unless $i' = i$ and $j' = j$. Replacing p with $p - p_{ii}1$, we may assume that $p_{ii} = 0$. Note that $pe_{ji} = p_{ij}e_{ii}$. Now $0 = a(pe_{ji} - e_{ji}p)^n e_{ii} = a(p_{ij}e_{ii} - e_{ji}p)^n e_{ii} = (p_{ij})a(p_{ij}e_{ii} - e_{ji}p)^{n-1}e_{ii} = \cdots = (p_{ij})^n ae_{ii}$. Hence $ae_{ii} = 0$. Thus $a\rho = 0$ and we are done.

Now we may assume that $p_{ij} = 0$ for all $j \cdot \ l$ and $i \neq j$. For $i, j \cdot \ l, i \neq j$, define $\phi(x) = (1 + e_{ij})x(1 - e_{ij})$, an automorphism of R. By assumption, $\phi(a)(\phi(p)\phi(x) - \phi(x)\phi(p))^n = 0$ for all $x \in A$. By Lemma 1, $\phi(a)(\phi(p)x - x\phi(p))^n = 0$ for all $x \in A$. The (i, j)-entry of $\phi(p)$ is $p_{jj} - p_{ii}$. If $p_{ii} \neq p_{jj}$ for some $i, j \cdot \ l$, then we obtain $\phi(a)\rho = 0$ as above. It is clear that $\phi(\rho) = \rho$. Hence $\phi(a\rho) = \phi(a)\phi(\rho) = \phi(a)\rho = 0$. So $a\rho = 0$. If $p_{ii} = p_{jj}$ for all $i, j \cdot \ l$, then $d(\rho)\rho = 0$ and so we are done.

(II) Suppose that $f(X_1, \ldots, X_t)$ is not central–valued on eRe. First we assume that $p_{ij} \neq 0$ for some $j \cdot \ l$ and $i \neq j$. Since $e_{ji} \in A$ by Lemma 3, $0 = e_{ii}(pe_{ji} - e_{ji}p)^n a = p_{ij}e_{ii}(pe_{ji} - e_{ji}p)^{n-1}a = \cdots = (p_{ij})^n e_{ii}a$. Thus $e_{ii}a = 0$. For other $i' \neq j$, if $p_{i'j} \neq 0$, then we obtain that $e_{i'i'}a = 0$ as above. In the case that $p_{i'j} = 0$, we have $0 = (e_{ii} + e_{ii'})(p(e_{ji} + e_{ji'}) - (e_{ji} + e_{ji'})p)^n a = p_{ij}(e_{ii} + e_{ii'})(p(e_{ji} + e_{ji'}) - (e_{ji} + e_{ji'})p)^{n-1}a = \cdots = (p_{ij})^n(e_{ii} + e_{ii'})a = (p_{ij})^n e_{ii'}a$ for $e_{ji} + e_{ji'} \in A$ by Lemma 2. Hence $e_{ii'}a = 0$ and so $e_{i'i'}a = e_{i'i}e_{ii'}a = 0$. Thus $a = 1a = (e_{11} + e_{22} + \cdots + e_{mm})a = e_{jj}a$. Now $0 = (pe_{ji} - e_{ji}p)^n a = (pe_{ji} - e_{ji}p)^n e_{jj}a = (-p_{ij})(pe_{ji} - e_{ji}p)^{n-1}e_{jj}a = \cdots = (-p_{ji})^n e_{jj}a$, so $e_{jj}a = 0$ and $a = e_{jj}a = 0$. We are done in this case.

Next we assume that $p_{ij} = 0$ for all $j \cdot \ l$ and $i \neq j$. For $i, j \cdot \ l, i \neq j$, define $\phi(x) = (1 + e_{ij})x(1 - e_{ij})$, an automorphism of R. By assumption, $(\phi(p)\phi(x) -$

$\phi(x)\phi(p))^n\phi(a) = 0$ for all $x \in A$. From Lemma 1, $(\phi(p)x - x\phi(p))^n\phi(a) = 0$ for all $x \in A$. The (i,j)-entry of $\phi(p)$ is $p_{ii} - p_{jj}$. If $p_{ii} \neq p_{jj}$ for some $i, j \cdot \ l$, then we obtain $\phi(a) = 0$ as above and so $a = 0$. In the case that $p_{ii} = p_{jj}$ for all $i, j \cdot \ l$, we have $d(\rho)\rho = 0$. Thus the lemma is proved.

Proof of Theorem 1. (I) For any $x_1, \dots, x_t \in \rho, r \in R$,

$$\begin{aligned} 0 &= a\Big(d\big(f(x_1r, x_2, \dots, x_t)\big)\Big)^n \\ &= a\big(f^d(x_1r, x_2, \dots, x_t) + f(d(x_1)r + x_1d(r), x_2, \dots, x_t) \\ &\quad + f(x_1r, d(x_2), x_3, \dots, x_t) + \cdots + f(x_1r, x_2, \dots, d(x_t))\big)^n. \end{aligned}$$

If d is X-outer, then

$$\begin{aligned} &a\big(f^d(x_1r, x_2, \dots, x_t) + f(d(x_1)r + x_1s, x_2, \dots, x_t) \\ &+ f(x_1r, d(x_2), x_3, \dots, x_t) + \cdots + f(x_1r, x_2, \dots, d(x_t))\big)^n = 0 \end{aligned}$$

for all $r, s \in R, x_1, \dots, x_t \in \rho$ by Kharchenko's theorem [9]. In particular, setting $r = 0$, we have $af(x_1s, x_2, \dots, x_t)^n = 0$ for all $s \in R, x_1, \dots, x_t \in \rho$, and for all $s \in Q, x_1, \dots, x_t \in \rho$ by [1]. Now $af(x_1, x_2, \dots, x_t)^n = 0$ for all $x_1, \dots, x_t \in \rho$. By Lemma 2, either $a\rho = 0$ or $f(\rho)\rho = 0$. In the case that $f(\rho)\rho = 0$, $\rho C = eRC$ for some idempotent e in the socle of RC by [13, Proposition] and $f(X_1, \dots, X_t)$ is central-valued on $eRCe$.

Now we may assume that $d(r) = pr - rp$ for some $p \in Q$. Suppose first that R does not satisfy any nontrivial GPI. Write

$$f(X_1, \dots, X_t) = \alpha X_1X_2 \cdots X_t + \sum_{\sigma \neq 1} \alpha_\sigma X_{\sigma(1)}X_{\sigma(2)} \cdots X_{\sigma(t)},$$

where $\alpha, \alpha_\sigma \in C$ and we may assume $\alpha \neq 0$. For any $x \in \rho$,

$$a\big(pf(xX_1, xX_2, \dots, xX_t) - f(xX_1, xX_2, \dots, xX_t)p\big)^n$$

is a GPI for R, so

$$a\big(pf(xX_1, xX_2, \dots, xX_t) - f(xX_1, xX_2, \dots, xX_t)p\big)^n = 0 \tag{1}$$

in $Q *_C C\{X_1, \dots, X_t\}$. If apx and ax are C-independent for some $x \in \rho$, then

$$a(\alpha pxX_1xX_2 \cdots xX_t)(pf(xX_1, xX_2, \dots, xX_t) - f(xX_1, xX_2, \dots, xX_t)p)^{n-1} = 0,$$

in $Q *_C C\{X_1, \dots, X_t\}$. Since px and x are C-independent,

$$a(\alpha pxX_1xX_2 \cdots xX_t)^2(pf(xX_1, xX_2, \dots, xX_t) - f(xX_1, xX_2, \dots, xX_t)p)^{n-2} = 0,$$

in $Q *_C C\{X_1, \dots, X_t\}$. Repeating the same process yields

$$a(\alpha p x X_1 x X_2 \cdots x X_t)^n = 0,$$

in $Q *_C C\{X_1, \dots, X_t\}$, a contradiction. Thus for any $x \in \rho$, apx and ax are C-dependent. Then $a(p-\alpha)\rho = 0$ for some $\alpha \in C$ by [12, Lemma 3]. Replacing p with $p-\alpha$, we may assume that $ap\rho = 0$. Now (1) implies that

$$\begin{aligned} &-a(f(xX_1, xX_2, \dots, xX_t)p)(pf(xX_1, xX_2, \dots, xX_t) \\ &-f(xX_1, xX_2, \dots, xX_t)p)^{n-1} = 0, \end{aligned}$$

in $Q *_C C\{X_1, \dots, X_t\}$ for any $x \in \rho$. Since $p \notin C$, the above equation implies that

$$\begin{aligned} &-a(f(xX_1, xX_2, \dots, xX_t)p)(pf(xX_1, xX_2, \dots, xX_t) \\ &-f(xX_1, xX_2, \dots, xX_t)p)^{n-2}(-f(xX_1, xX_2, \dots, xX_t)p) = 0, \end{aligned} \tag{2}$$

in $Q *_C C\{X_1, \dots, X_t\}$. For $x \in \rho$, if px and x are C-independent, then (2) implies that

$$\begin{aligned} &-a(f(xX_1, xX_2, \dots, xX_t)p)(pf(xX_1, xX_2, \dots, xX_t) \\ &-f(xX_1, xX_2, \dots, xX_t)p)^{n-3}(-f(xX_1, xX_2, \dots, xX_t)p)^2 = 0, \end{aligned}$$

in $Q *_C C\{X_1, \dots, X_t\}$. Repeating this process, we can yield

$$a(-f(xX_1, xX_2, \dots, xX_t)p)^n = 0,$$

in $Q *_C C\{X_1, \dots, X_t\}$, and so $ax = 0$. In the case that $px = \alpha x$ for some nonzero element $\alpha \in C$, we also have $0 = apx = a\alpha x$, and so $ax = 0$. Thus $a\rho = 0$ and we are done.

Now we may assume that R is a GPI-ring. By [17, Theorem 3], RC is a primitive ring with nonzero socle H. We may regard d is a derivation on H and $ad(f(x_1, \dots, x_t))^n = 0$ for all $x_1, \dots, x_t \in \rho H$ by [11, Theorem 2]. Replacing a with a nonzero element in Ha, we may assume that $a \in H$. Choose F to be the algebraic closure of C or C according to $|C| = \infty$ or $|C| < \infty$. Then we have $(a$ $1)[p \quad 1, f(x_1, \dots, x_t)]^n = 0$ for all $x_1, \dots, x_t \in \rho H \quad {}_C F$ by [10, Proposition]. Suppose $a\rho \neq 0$ and $d(\rho)\rho \neq 0$. Choose $w, z, z' \in \rho$ such that $aw \neq 0$ and $d(z)z' \neq 0$. For any $x, x_1, \dots, x_t, y \in \rho$, there exists an idempotent $g \in \rho H \quad {}_C F$ such that $x, x_1, \dots, x_t, y, w, z, z' \in g(\rho H \quad {}_C F)$. By Litoff's theorem [6], there exists an idempotent $g' \in H \quad {}_C F$ such that $g, gp, pg, a, pz, zp, x, x_1, \dots, x_t, y, w, z, z' \in g'(H \quad {}_C F)g'$ and $g'(H \quad {}_C F)g' \cong \mathrm{M}_m(F)$ for some $m \geq 2$. For $y_1, \dots, y_t \in g\mathrm{M}_m(F) \subseteq (\rho H \quad {}_C F) \cap \mathrm{M}_m(F)$, we have

$$0 = a[p, f(y_1, \dots, y_t)]^n g' = g' a g' [g' p g', f(y_1, \dots, y_t)]^n.$$

Since $(g'ag')g = ag \neq 0$ and $d(z)z' \neq 0$, $f(X_1, \dots, X_t)$ is central-valued on $g\mathrm{M}_m(F)g$ by Lemma 4 and so $[x, f(x_1, \dots, x_t)]y = 0$. Thus ρ satisfies the PI $[X, f(X_1, \dots, X_t)]Y$. By [13, Proposition], $\rho C = eRC$ for some idempotent $e \in \mathrm{soc}(RC)$ and $f(X_1, \dots, X_t)$ is central-valued on $eRCe$.

(II) For any $x_1, \dots, x_t \in \rho, r \in R$,

$$\begin{aligned}&\Big(d\big(f(x_1r, x_2, \dots, x_t)\big)\Big)^n a\\ &= \big(f(d(x_1)r + x_1d(r), x_2, \dots, x_t) + f(x_1r, d(x_2), x_3, \dots, x_t) + \cdots\\ &+ f(x_1r, x_2, \dots, d(x_t))\big)^n a = 0.\end{aligned}$$

If d is X-outer, then

$$\begin{aligned}&\big(f(d(x_1)r + x_1s, x_2, \dots, x_t) + f(x_1r, d(x_2), x_3, \dots, x_t) + \cdots\\ &+ f(x_1r, x_2, \dots, d(x_t))\big)^n a = 0,\end{aligned}$$

for all $r, s \in R, x_1, \dots, x_t \in \rho$ by Kharchenko's theorem [9]. In particular, setting $r = 0$, we have $f(x_1s, x_2, \dots, x_t)^n a = 0$ for all $s \in R, x_1, \dots, x_t \in \rho$, and so for all $s \in Q, x_1, \dots, x_t \in \rho$ by [1]. Now $f(x_1, x_2, \dots, x_t)^n a = 0$ for all $x_1, \dots, x_t \in \rho$. By Lemma 2, either $a = 0$ or $f(\rho)\rho = 0$. In the case that $f(\rho)\rho = 0$, $\rho C = eRC$ for some idempotent e in the socle of RC by [13, Proposition] and $f(X_1, \dots, X_t)$ is central-valued on $eRCe$.

Now we may assume that $d(r) = pr - rp$ for some $p \in Q$. If px and x are C-dependent for all $x \in \rho$, then $(p - \beta)\rho = 0$ for some $\beta \in C$ by [12, Lemma 3] and so $d(\rho)\rho = 0$. We are done in this case. Next assume that px and x are C-independent for some $x \in \rho$. Suppose first that R does not satisfy any nontrivial GPI. Write

$$f(X_1, \dots, X_t) = \alpha X_1X_2 \cdots X_t + \sum_{\sigma \neq 1} \alpha_\sigma X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(t)},$$

where $\alpha, \alpha_\sigma \in C$ and we may assume that $\alpha \neq 0$. For any $y \in \rho$,

$$\big(pf(xX_1, yX_2, \dots, yX_t) - f(xX_1, yX_2, \dots, yX_t)p\big)^n a,$$

is a GPI for R, so

$$\big(pf(xX_1, yX_2, \dots, yX_t) - f(xX_1, yX_2, \dots, yX_t)p\big)^n a = 0,$$

in $Q *_C C\{X_1, \dots, X_t\}$. Since px and x are C-independent,

$$\begin{aligned}&\alpha pxX_1yX_2 \cdots yX_t\big(pf(xX_1, yX_2, \dots, yX_t)\\ &- f(xX_1, yX_2, \dots, yX_t)p\big)^{n-1} a = 0,\end{aligned}$$

in $Q *_C C\{X_1, \dots, X_t\}$. Repeating the same process yields

$$(\alpha p x X_1 y X_2 \cdots y X_t)^n a = 0,$$

so $a = 0$ and we are done.

Now we may assume that R is a GPI ring. By [17, Theorem 3], RC is a primitive ring with nonzero socle H. We may regard d is a derivation on H and $d(f(x_1, \dots, x_t))^n a = 0$ for all $x_1, \dots, x_t \in \rho H$ by [11, Theorem 2]. Replacing a with a nonzero element in aH, we may assume that $a \in H$. Choose F to be the algebraic closure of C or C according to $|C| = \infty$ or $|C| < \infty$. Then we have $[p \quad 1, f(x_1, \dots, x_t)]^n (a \quad 1) = 0$ for all $x_1, \dots, x_t \in \rho H \quad {}_C F$ by [10, Proposition]. Suppose $a \neq 0$ and $d(\rho)\rho \neq 0$. Choose $z, z' \in \rho$ such that $d(z)z' \neq 0$. For any $x, x_1, \dots, x_t, y \in \rho$, there exists an idempotent $g \in \rho H \quad {}_C F$ such that $x, x_1, \dots, x_t, y, z, z' \in g(\rho H \quad {}_C F)$. By Litoff's theorem [6], there exists an idempotent $g' \in H \quad {}_C F$ such that $g, gp, pg, a, pz, zp, x, x_1, \dots, x_t, y, z, z' \in g'(H \quad {}_C F)g'$ and $g'(H \quad {}_C F)g' \cong \mathrm{M}_m(F)$ for some $m \geq 2$. For $y_1, \dots, y_t \in g\mathrm{M}_m(F) \subseteq \rho H \quad {}_C F \cap \mathrm{M}_m(F)$, we have

$$0 = g'[p, f(y_1, \dots, y_t)]^n a = [g'pg', f(y_1, \dots, y_t)]^n g'ag'.$$

Since $g'ag' = a \neq 0$ and $d(z)z' \neq 0$, $f(X_1, \dots, X_t)$ is central-valued on $g\mathrm{M}_m(F)g$ by Lemma 4 and so $[x, f(x_1, \dots, x_t)]y = 0$. Thus ρ satisfies the PI $[X, f(X_1, \dots, X_t)]Y$. By [13, Proposition], $\rho C = eRC$ for some idempotent $e \in \mathrm{soc}(RC)$ and $f(X_1, \dots, X_t)$ is central-valued on $eRCe$.

Proof of Theorem 2. Since $ad(f(X_1, \dots, X_t))^n$ is a central DI for ρ, R is a PI-ring by [3, Theorem 1] and so RC is a finite-dimensional central simple C-algebra by Posner's theorem. By Wedderburn-Artin theorem, $RC \cong \mathrm{M}_s(D)$ for some s and some finite-dimensional central division C-algebra D. It follows from [11, Theorem 2] that $ad(f(x_1, \dots, x_t))^n \in C$ for all $x_1, \dots, x_t \in \rho C$. Replacing R with RC, we may assume, without loss of generality, that $R = \mathrm{M}_s(D)$. Since $ad(f(y_1, \dots, y_t))^n \neq 0$ for some $y_1, \dots, y_t \in \rho$, a is invertible. Thus $ad(f(x_1, \dots, x_t))^n \in C$ if and only if $d(f(x_1, \dots, x_t))^n a \in C$. So, it suffices to prove the case $d(f(x_1, \dots, x_t))^n a \in C$ for all $x_1, \dots, x_t \in \rho$.

For any $x_1, \dots, x_t \in \rho, r \in R$,

$$\begin{aligned}
&\Big(d\big(f(x_1 r, x_2, \dots, x_t)\big)\Big)^n a \\
&= \big(f^d(x_1 r, x_2, \dots, x_t) + f(d(x_1)r + x_1 d(r), x_2, \dots, x_t) \\
&+ f(x_1 r, d(x_2), x_3, \dots, x_t) + \cdots + f(x_1 r, x_2, \dots, d(x_t))\big)^n a \in C.
\end{aligned}$$

If d is X-outer, then

$$\begin{aligned}
&\big(f^d(x_1 r, x_2, \dots, x_t) + f(d(x_1)r + x_1 s, x_2, \dots, x_t) \\
&+ f(x_1 r, d(x_2), x_3, \dots, x_t) + \cdots + f(x_1 r, x_2, \dots, d(x_t))\big)^n a \in C,
\end{aligned}$$

for all $r, s \in R, x_1, \ldots, x_t \in \rho$ by Kharchenko's theorem [9]. In particular, setting $r = 0$, we have

$$f(x_1 s, x_2, \ldots, x_t)^n a \in C,$$

for all $s \in R, x_1, \ldots, x_t \in \rho$. Thus $f(x_1, x_2, \ldots, x_t)^n a \in C$ for all $x_1, \ldots, x_t \in \rho$. If $f(x_1, x_2, \ldots, x_t)^n a \neq 0$ for some $x_1, \ldots, x_t \in \rho$, then $f(X_1, \ldots, X_t)^n$ is central valued on R by [14, Theorem 1] and so $a \in C$. Hence $\rho = R$ and $d(f(x_1, \ldots, x_t))^n \in C$ for all $x_1, \ldots, x_t \in R$. Now we are done by [18, Theorem 2]. Suppose next that $f(x_1, x_2, \ldots, x_t)^n a = 0$ for all $x_1, \ldots, x_t \in \rho$. By Lemma 2, $f(\rho)\rho = 0$ since $a \neq 0$. We are done in this case.

Now we may assume that $d(r) = pr - rp$ for some $p \in R$. Choose K a maximal subfield of D; then $\mathrm{M}_s(D) \quad {}_C K \cong \mathrm{M}_m(K)$, where $m = s[K : C]$. Now, the derivation d can be canonically extended to $\mathrm{M}_s(D) \quad {}_C K$ and $d(f(x_1, \ldots, x_t))^n a \in Z(\mathrm{M}_m(K))$ for all $x_1, \ldots, x_t \in \rho \quad K$ by [12, Lemma 2] and [10, Proposition]. Thus we may assume that $R = \mathrm{M}_m(K)$ and $\rho = eR = (e_{11} + \cdots + e_{ll})R$, where $m \geq 2$.

If $m = 2$, then R satisfies S_4 and we are done. Thus we may assume that $m > 2$. For any $i \cdot \ l$ and $j > l$, $e_{ij} \in A$ by Lemma 3 and rank$((pe_{ij} - e_{ij}p)^n a) \cdot \ 2$, hence $(pe_{ij} - e_{ij}p)^n a = 0$ and so $(pe_{ij} - e_{ij}p)^n = 0$ since a is invertible. Now $0 = (pe_{ij} - e_{ij}p)^n e_{ii} = (-p_{ji})(pe_{ij} - e_{ij}p)^{n-1} e_{ii} = \cdots = (-p_{ji})^n e_{ii}$. Hence $p_{ji} = 0$. Then $p\rho \subseteq \rho$ and so $d(\rho) \subseteq \rho$. Since $0 \neq d(f(y_1, \ldots, y_t))^n a \in \rho \cap K$ for some $y_1, \ldots, y_t \in \rho$, $\rho = R$. Suppose that $f(X_1, \ldots, X_t)$ is not central-valued on R, then $e_{ij} \in A$ for all $i \neq j$ by Lemma 3. We can get that $p_{ji} = 0$ as above. For $i \neq j$, define $\phi(x) = (1 + e_{ij})x(1 - e_{ij})$, an automorphism of R. By assumption, $(\phi(p)\phi(x) - \phi(x)\phi(p))^n \phi(a) \in K$ for all $x \in A$. By Lemma 1, $(\phi(p)x - x\phi(p))^n \phi(a) \in C$ for all $x \in A$. Since rank$((\phi(p)e_{ij} - e_{ij}\phi(p))^n \phi(a)) \cdot \ 2$ and $\phi(a)$ is invertible, $(\phi(p)e_{ij} - e_{ij}\phi(p))^n = 0$. We can get that the (i, j)-entry of $\phi(p)$ is 0 as above. The (i, j)-entry of $\phi(p)$ is $p_{jj} - p_{ii}$, so $p_{jj} = p_{ii}$. Hence $p \in K$ and $d = 0$, a contradiction to the assumption. Thus Theorem 2 is proved.

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