

## LOWER-BOUND ESTIMATES FOR EIGENVALUE OF THE LAPLACE OPERATOR ON SURFACES OF REVOLUTION

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**Abstract.** In this paper, we estimate eigenvalues of the Laplace operator on surfaces of revolution. We first reduce our Laplace eigenvalue problems to the corresponding Sturm-Liouville eigenvalue problems. Two variational inequalities are then used to obtain lower-bound estimates for eigenvalues of the corresponding Sturm-Liouville problems. Based on the relationship between eigenvalues of the Laplace problems and the Sturm-Liouville problems, we obtain lower-bound estimates for eigenvalues of the mixed and Neumann problems of the Laplace operator (Theorem 1 and Theorem 2). Indeed, our estimate in the first case is optimal.

### 1. INTRODUCTION AND PRELIMINARY

The classical eigenvalue problems have been studied systematically since 1850's which mainly came from the study of acoustic theory and of vibrating elastic membranes. Pioneer mathematicians are, for example, J.W.S. Rayleigh, R. Courant, G. Polya, G. Szego, L.E. Payne, M.G. Krein, M. Berger, C. Bandle, J.R. Kuttler, V.G. Sigillito, S.Y. Cheng and S.T. Yau. The first remarkable eigenvalue estimate is the lower-bound estimate for the first eigenvalue,  $\lambda_1$ , of the Dirichlet problem to the Laplace operator, the *Faber-Krahn inequality*: The circular region has the smallest  $\lambda_1$  of all regions with the same area, namely,

$$\lambda_1 \geq \frac{2\pi}{A} j_0^2,$$

where  $j_0$  is the first positive zero of the 0th Bessel function and  $A$  is the area. Unlike upper-bound estimates, lower-bound estimates for an eigenvalue is difficult to obtain.

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For details and a literature review, the reader should consult the monograph of Polya and Szego [7], the books of Bandle [1] and Chavel [2], the excellent review articles by Payne [6] and by Kuttler and Sigillito [4], and the reference therein.

In [9], Shen and Shieh studied the first eigenvalue of the Dirichlet problem to the Laplace operator over spherical bands on the unit sphere. Introducing area of the spherical band as a new variable, they can reduce the two-dimensional eigenvalue problem to a one-dimensional Sturm-Liouville eigenvalue problem by method of separation of variables and prove that the first eigenvalue attains its maximum when the spherical band is symmetric to the equator. They also prove the monotonicity of this first eigenvalue when the spherical band moves to the unit sphere, and generalize this result to the surface of revolution by a smooth strictly increasing generating function,  $f(x)$ , on the interval  $[0, \infty]$  with  $f'(0) = 0$ .

Instead of qualitative study, in this paper, we use Shen and Shieh's methodology to study lower-bound estimates for eigenvalues of mixed and Neumann problems to the Laplace operator on surfaces of revolution. At first we reduce the Laplace eigenvalue problems to the eigenvalue problems of corresponding Sturm-Liouville operators via the method of separation of variables. By introducing surface area as a new variable, we obtain a self-adjoint Sturm-Liouville operator. Two Troesch's variational inequalities are then used to obtain lower-bound estimates for eigenvalues of the Sturm-Liouville problems. Based on the correspondence of eigenvalues for the Laplace problems and the Sturm-Liouville problems, we obtain lower-bound estimates for eigenvalues of the Laplace problems. Indeed, we obtain lower-bound estimates for the first eigenvalue of the mixed problems of the Laplace operator (Theorem 1) and lower-bound estimates for the first positive eigenvalue with odd multiplicity of the Neumann problem of the Laplace operator (Theorem 2). In the mixed problem case, the equality is achieved if and only if the surface is a cylinder. Thus our estimate is an optimal one.

The paper is organized as follows. In the end of this section, we introduce some basic facts concerning Laplace eigenvalue problems and Sturm-Liouville eigenvalue problems. In §2, we demonstrate how to reduce the Laplace equation to a corresponding Sturm-Liouville equation through separation of variables and changing variables. We also discuss the relationship between the eigenvalues of the Laplace operator and eigenvalues of the corresponding Sturm-Liouville operator. In §3, we apply Troesch's variational inequalities to get our lower-bound estimates for eigenvalues of mixed and Neumann problems of the Laplace operator. Our estimates are new and is optimal in some case, which is a different research direction to Shen and Shieh's one.

We now turn to a brief introduction to the theory of eigenvalue problems for the Laplace operator and the Sturm-Liouville operator. The reader should consult [2, 8] for details.

For a general multi-dimensional Riemannian manifold, the eigenvalue problems of the Laplace operator is to find an eigenvalue  $\lambda$  and an associated nontrivial eigenfunction  $\Psi(x)$  which satisfies

$$(1) \quad \Delta\Psi + \lambda\Psi = 0$$

with associated boundary conditions. In general, there are three kinds of homogeneous boundary conditions, namely, Dirichlet, Neumann and mixed boundary conditions.

The  $\lambda$ -eigenspace,  $E_\lambda$ , is the vector space of eigenfunctions corresponding to the eigenvalue  $\lambda$ , and the dimension of each  $\lambda$ -eigenspace is known as the multiplicity of the eigenvalue  $\lambda$ . For each one of the above Laplace eigenvalue problem, the set of eigenvalues consists of an increasing sequence tending to  $\infty$  and bounded below by 0. Each eigenspace is of finite dimension. Therefore it will be convenient to list the eigenvalues as

$$0 \cdot \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \dots \rightarrow \infty$$

with each eigenvalue repeated according to its multiplicity. In particular, for the Dirichlet and mixed eigenvalue problems,  $0 < \lambda_1 < \lambda_2$ , while  $0 = \lambda_1 < \lambda_2$  for the Neumann problems [2].

For the one-dimensional case, let  $(\alpha, \beta)$  be a finite open interval. The eigenvalue problem for the Sturm-Liouville operator is to find an eigenvalue  $\lambda$  and a nontrivial eigenfunction  $f(x)$  satisfy the equation

$$(2) \quad \frac{d}{dx} \left( p(x) \frac{d}{dx} f(x) \right) + (\lambda q(x) - r(x)) f(x) = 0,$$

with associated boundary conditions. Here  $p(x)$ ,  $q(x)$  and  $r(x)$  are continuous and both  $p(x)$  and  $q(x)$  are positive. In general, three kinds of boundary conditions are imposed, namely, Dirichlet, Neumann and mixed boundary conditions. Compared to multi-dimensional cases, results for the Sturm-Liouville eigenvalue problems are simpler: Each eigenspace is of multiplicity one and the eigenvalues can be listed as an increasing sequence bounded below and tending to  $\infty$ ,

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots \rightarrow \infty.$$

Note that  $\lambda_1$  is bounded below by 0 if  $r(x)$  is non-negative.

We end this section by stating the Rayleigh's theorem or the minimum principle which plays an important role in the study for the eigenvalue problems. For simplicity, we state the form for the Sturm-Liouville eigenvalue problems: The  $n$ th

eigenvalue  $\lambda_n$  can be characterized as

$$(3) \quad \lambda_n = \inf_{\substack{u(x) \in W(\alpha, \beta) \\ \int_{\alpha}^{\beta} u(x) u_k(x) dx = 0, \\ k = 1, 2, \dots, n-1.}} \frac{\int_{\alpha}^{\beta} [p(x) \left(\frac{d}{dx} u(x)\right)^2 + r(x) u^2(x)] dx}{\int_{\alpha}^{\beta} q(x) u^2(x) dx}$$

where  $u_k(x)$  is the  $k$ -th eigenfunction,  $W(\alpha, \beta)$  is an appropriate space depending on the boundary condition. The above quotient is called the Rayleigh quotient [3]. Note that an upper-bound for  $\lambda_1$  can be obtained easily by evaluating the Rayleigh quotient with an arbitrary test function  $u(x)$ . Consult, for example, [5].

## 2. REDUCTION TO THE STURM-LIOUVILLE PROBLEM

We now demonstrate the reduction of the Laplace eigenvalue problem to the Sturm-Liouville eigenvalue problem. The main ingredients are the method of separation of variables and changing variables.

Let  $y = f(x)$  with  $f(x) > 0$ ,  $x \in (\alpha, \beta)$  be a  $C^2$  curve and be the surface of revolution generated by this curve, with a parameterization

$$\{(x, f(x)\cos(\theta), f(x)\sin(\theta)) : x \in (\alpha, \beta), 0 \leq \theta \leq 2\pi\}.$$

Then the Laplace operator on the surface  $S$  can be formulated by

$$(4) \quad \Delta^s \Psi(x, \theta) = \frac{1}{w(x)f(x)} \left( \frac{\partial}{\partial x} \left( \frac{f(x)}{w(x)} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial \theta} \left( \frac{w(x)}{f(x)} \frac{\partial}{\partial \theta} \right) \right) \Psi(x, \theta),$$

where  $w(x) = \sqrt{1 + (f'(x))^2}$  [2].

By separation of variables, let  $\Psi(x, \theta) = u(x)v(\theta)$  and substitute it into the Laplace eigenvalue problem (1), we obtain

$$(5) \quad \frac{1}{w(x)f(x)} \frac{d}{dx} \left( \frac{f(x)}{w(x)} \frac{d}{dx} u(x) \right) + \left( \lambda - \frac{k^2}{f(x)^2} \right) u(x) = 0 \quad \alpha < x < \beta,$$

$$(6) \quad \frac{d^2}{d\theta^2} v(\theta) + k^2 v(\theta) = 0, \quad 0 < \theta < 2\pi \text{ and } v(0) = v(2\pi), v'(0) = v'(2\pi).$$

in which  $k$  is a nonnegative integer. Note that for each  $k$ ,  $v(\theta)$  is a linear combination of  $\sin(k\theta)$  and  $\cos(k\theta)$  and equation (5) is of Sturm-Liouville type.

Introduce the surface area as a new variable, [9],

$$(7) \quad y(x) = 2\pi \int_{\alpha}^x w(t)f(t)dt;$$

and define

$$h(y) = u(x(y)); \quad g(y) = f(x(y)); \quad z(y) = 4\pi^2 g(y)^2.$$

Then (5) becomes a self-adjoint Sturm-Liouville equation

$$(8) \quad \frac{d}{dy} \left( z(y) \frac{d}{dy} h(y) \right) + \left( \lambda - \frac{k^2}{g(y)^2} \right) h(y) = 0.$$

**Remark 1.** The function  $z(y)$  preserves several geometric properties of the generating function  $f(x)$ . For example, if  $f(x)$  is decreasing  $z(y)$  is also decreasing; if  $f(x)$  is concave, so is  $z(y)$ . These properties are important for our lower-bound estimates for eigenvalues.

When a boundary condition is imposed on the Laplace eigenvalue problem (1), there is a corresponding boundary condition to the Sturm-Liouville equation (8). For example, the corresponding boundary conditions of the Sturm-Liouville equation (8) to the mixed boundary condition

$$(9) \quad \Psi(\alpha, \theta) = 0 = \frac{\partial}{\partial \nu} \Psi(\beta, \theta) \quad \theta \in R$$

and the Neumann boundary condition

$$(10) \quad \frac{\partial}{\partial \nu} \Psi(\alpha, \theta) = 0 = \frac{\partial}{\partial \nu} \Psi(\beta, \theta) \quad \theta \in R$$

of the Laplace eigenvalue problem (1) are  $h(0) = 0 = h'(A)$ . and  $h'(0) = 0 = h'(A)$  respectively. Here  $\frac{\partial}{\partial \nu}$  denotes the outer normal derivative.

Before we discuss the relationship between eigenvalues of the Laplace operator (1) and the eigenvalues of the corresponding Sturm-Liouville operator (8), let us consider a simple example. Let  $S$  be the cylinder generated by  $f(x) = c$ ,  $x \in (\alpha, \beta)$ , and  $\lambda_n$  be the  $n$ th eigenvalue of the Laplace operator (1) with the Neumann boundary conditions (10). Then the corresponding Sturm-Liouville eigenvalue problem is

$$(11) \quad \begin{cases} 4\pi^2 c^2 h''(x) + \left( \lambda - \frac{k^2}{c^2} \right) h(x) = 0, \\ h'(0) = 0 = h'(A), \end{cases}$$

where  $\gamma = \beta - \alpha$  and  $A = 2\pi c\gamma$  is the surface area of the cylinder. For each  $k$ , let  $\nu_n^k$  be the  $n$ th eigenvalue of the above eigenvalue problem (11). Then  $\nu_n^k = \frac{k^2}{c^2} + \frac{(n-1)^2 \pi^2}{\gamma^2}$  with eigenspace spanned by  $\cos(\frac{(n-1)\pi}{A} x)$ . By completeness, each eigenvalue for the Laplace eigenvalue problem,  $\lambda_m$ , corresponds to one eigenvalue of the Sturm-Liouville eigenvalue problem,  $\nu_n^k$  for some  $n$  and  $k$ . Thanks to the

monotonicity of eigenvalues for Sturm-Liouville eigenvalue problems, it is not hard to see that  $\lambda_1 = 0 = \nu_1^0$ . For  $\lambda_2$ , it could be  $\nu_2^0 = (\frac{\pi}{\gamma})^2$  or  $\nu_1^1 = (\frac{1}{c})^2$ . If  $\lambda_2 = \nu_2^0 < \nu_1^1$ , then the multiplicity of  $\lambda_2$  is 1 or 3. Otherwise it is of multiplicity 2 with the eigenspace generated by  $\cos(\frac{\pi}{A}x) \cos \theta$  and  $\cos(\frac{\pi}{A}x) \sin \theta$ .

By a similar argument, we can conclude

**Lemma 1.** *The first eigenvalue of the Laplace eigenvalue problem (1) with Dirichlet's or mixed boundary condition is exactly the first eigenvalue of the Sturm-Liouville eigenvalue problem (8) with  $k = 0$  and the corresponding boundary condition.*

*The first positive eigenvalue with odd multiplicity for the Laplace eigenvalue problem (1) with the Neumann boundary condition (10) is the first positive eigenvalue  $\nu_2^0$  of the corresponding Sturm-Liouville eigenvalue problem (8) with  $k = 0$  and the Neumann boundary condition.*

We now turn to investigate on lower-bound estimates for eigenvalues of the Laplace eigenvalue problems (1).

### 3. LOWER-BOUND ESTIMATES FOR EIGENVALUES

In this section, we derive our main theorems concerning the lower-bound estimates for eigenvalues of Laplace eigenvalue problems (1). According to Lemma 1, all we need to do is to obtain a lower-bound estimate of the corresponding Sturm-Liouville eigenvalue problems (8). These estimates can be obtained through variational inequalities. In this paper, two inequalities due to Troesch [10] are used to obtain lower-bound estimates for eigenvalues of Laplace eigenvalue problems (1) with mixed and Neumann boundary conditions. For completeness, we state Troesch's inequalities below.

**Lemma 2.** *Let  $g(x)$  be a continuous, piecewise smooth function on  $[0, 1]$  and  $h(x)$  be a positive, concave function with piecewise smooth derivative  $h_x(x)$ . In addition, if we assume that  $g(0) = 0$  and  $h_x(0) \cdot 0$ , then the inequality*

$$\frac{\int_0^1 h(x)(g_x(x))^2 dx}{\int_0^1 h(x) dx \int_0^1 g(x)^2 dx} \geq \frac{\pi^2}{4},$$

*holds. Moreover, the equality holds if and only if  $h(x)$  is a constant and  $g(x) = \text{const.} \sin(\frac{\pi x}{2})$ .*

**Lemma 3.** *Let  $g(x)$  be a continuous, piecewise smooth function on  $[0, 1]$  and  $h(x)$  be a positive, concave function with piecewise smooth derivative  $h_x(x)$ . In*

addition, if we assume that  $\int_0^1 g(x)dx = 0$  then the inequality

$$\frac{\int_0^1 h(x)(g_x(x))^2 dx}{\int_0^1 h(x)dx \int_0^1 g(x)^2 dx} \geq \frac{j_1^2}{2}$$

holds, where  $j_1 \approx 3.832$  is the first positive zero of the first-order Bessel function  $J_1(x)$ . Moreover, the equality holds if and only if  $h(x) = x$  and  $g(x) = J_0(j_1\sqrt{x})$ , or  $h(x) = 1 - x$  and  $g(x) = J_0(j_1\sqrt{1-x})$ .

Let  $f(x)$  be a positive, concave and non-increasing function on  $(\alpha, \beta)$  and  $\lambda_1$  be the first eigenvalue of the Laplace eigenvalue problems (1) with the mixed boundary condition (9) on the surface of revolution  $S$  generated by  $f(x)$ . Let  $h(y)$  be an eigenfunction of the first eigenvalue of the corresponding Sturm-Liouville eigenvalue problem (8) with the corresponding mixed boundary condition. By Lemma 2, we have the following lower-bound estimate:

$$\begin{aligned} \lambda_1 &= \frac{\int_0^A z(y)(h'(y))^2 dy}{\int_0^A h(y)^2 dy} \\ &\geq \frac{1}{4}\pi^2 A^{-3} \int_0^A z(y) dy \\ &= \frac{1}{4}\pi^2 A^{-3} \int_\alpha^\beta (2\pi f(x))^2 (2\pi)w(x) f(x) dx \\ &= 2\pi^5 A^{-3} \int_\alpha^\beta f(x)^3 \sqrt{1 + (f'(x))^2} dx, \end{aligned}$$

where  $A$  is the area of the band region. Note that the equality holds if and only if  $z(y)$  is constant, say  $\gamma^2$  with  $\gamma > 0$ , and  $h(y) = \sin(\frac{\pi y}{2A})$  i.e., the generating function  $f(x) = \frac{\gamma}{2\pi}$  and  $u(x) = \sin(\frac{\pi(x-\alpha)}{2(\beta-\alpha)})$ . In that case, the surface is a cylinder. In other words, the optimality,  $\frac{\pi^2}{4(\beta-\alpha)^2}$ , is achieved if and only if  $S$  is a cylinder.

**Theorem 1.** *Let  $f(x)$  be a positive, concave and non-increasing function on  $(\alpha, \beta)$  and  $\lambda_1$  be the eigenvalue of the Laplace eigenvalue problems (1) with the mixed boundary condition (7) on the surface of revolution  $S$  generated by  $f(x)$  with surface area  $A$ . Then*

$$(12) \quad \lambda_1 \geq 2\pi^5 A^{-3} \int_\alpha^\beta f(x)^3 \sqrt{1 + (f'(x))^2} dx$$

Moreover, equality holds if and only if  $S$  is a cylinder. In that case,  $\lambda_1 = \frac{\pi^2}{4(\beta-\alpha)^2}$ .

If we drop the assumption on non-increasing of the generating function  $f(x)$  and let  $\lambda$  be the first positive eigenvalue with odd multiplicity of the Laplace eigenvalue problems (1) with the Neumann boundary condition (10) on the surface  $S$ , then by Lemma 1 and Troesch's second inequality (Lemma 3), we get a lower-bound for the eigenvalue  $\lambda$ .

$$\begin{aligned}\lambda &= \frac{1}{2}j_1^2 A^{-3} \int_0^A z(y) dy \\ &= 4\pi^3 j_1^2 A^{-3} \int_\alpha^\beta f(x)^3 \sqrt{1 + (f'(x))^2} dx\end{aligned}$$

We state this result as our second theorems:

**Theorem 2.** *Let  $f(x)$  be positive and concave and  $\lambda$  be the first positive eigenvalue with odd multiplicity of the Laplace eigenvalue problems (1) with the Neumann boundary condition (10) on the surface  $S$  with surface area  $A$ . Then*

$$(13) \quad \lambda > 4\pi^3 j_1^2 A^{-3} \int_\alpha^\beta f(x)^3 \sqrt{1 + (f'(x))^2} dx$$

where  $j_1 \approx 3.832$  is the first positive zero of the first-order Bessel function  $J_1(x)$ .

We end this paper by a final remark on optimal case in Theorem 2.

**Remark 2.** The optimality of (13) in the Theorem 2 is not achievable. An counterexample is: For the cylinder generated by  $f(x) = \gamma$ , the first positive eigenvalue with odd multiplicity of the Laplace eigenvalue problems (1) with the Neumann boundary condition (10) is  $\lambda = (\frac{\pi}{\gamma})^2$  which is larger than  $\frac{1}{2}(\frac{j_1}{\gamma})^2$ , the lower-bound obtained in theorem 2. Indeed, in Troesch's Lemma 3, the optimality was obtained by  $z(y) = ay$  and  $h(y) = bJ_0(j_1\sqrt{y})$  which contradicts to the positivity of  $z(y) = f(x(y))$ . Actually, one can easily check that,  $h(y) = J_0(j_1\sqrt{y})$  does satisfy the Sturm-Liouville equation (8) with  $\lambda = \frac{j_1^2}{2}$  but fails to satisfy the boundary condition  $h'(0) = 0$ .

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