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# INEQUALITIES BETWEEN DIRICHLET AND NEUMANN EIGENVALUES FOR DOMAINS IN SPHERES

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**Abstract.** Let $M$ be a domain in the unit $n$-sphere with smooth boundary. The purpose of this paper is to describe some inequalities between Dirichlet and Neumann eigenvalues for $M$ under certain convex restrictions on the boundary. We prove that if the mean curvature of the boundary is nonpositive, then the $k$th nonzero Neumann eigenvalue is less than or equal to the $k$th Dirichlet eigenvalue for $k = 1, 2, \cdots$. Furthermore, if the second fundamental form of the boundary is nonpositive, then the $(k + \left[\frac{n-1}{2}\right])$th nonzero Neumann eigenvalue is less than or equal to the $k$th Dirichlet eigenvalue for $k = 1, 2, \cdots$.

## 1. Introduction

Let $M$ be an $n$-dimensional compact Riemannian manifold with smooth boundary $\partial M$. We consider the eigenvalues of the Laplace-Beltrami operator acting on functions. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the corresponding Dirichlet eigenvalues and $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots$ the corresponding Neumann eigenvalues. We shall compare the Dirichlet eigenvalues with the Neumann eigenvalues.

There are vast inequalities between Dirichlet and Neumann eigenvalues. For $M$ being a domain in a Riemannian manifold, it is a simple consequence of the variation principle of the variation principle of the eigenvalue problems that

$$\mu_{k-1} \leq \lambda_k$$

for $k = 1, 2, \cdots$ [4]. For $M$ being a domain in the Euclidean space, significant work in this direction has been obtained by Aviles [1], Friedlander [5], Payne [8], Polya [9], and Levine and Weinberger [6]. The remarkable result of Friedlander proved that for $M$ being a domain in the Euclidean space,

$$\mu_k \leq \lambda_k$$

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for $k = 1, 2, \cdots$ [5]. This result gives the affirmative answer to a question raised by Payne [8]. Mazzeo proved that the analogue of Friedlander's result is valid for all compact domains in a symmetric space of noncompact type [7]. However, these inequalities are in general false for manifolds with boundary. Indeed,

$$\mu_1 > \lambda_1$$

for a geodesic ball $B(r)$ of radius $r$ in the unit sphere if $B(r)$ is bigger than a semisphere [3].

The aim of this paper is to study domains in the unit $n$-sphere, an exceptional case of the Mazzeo's result. We agree that the second fundamental form $(h_{ij})$ of the boundary of a geodesic ball in the unit sphere with radius $r$ is $(-(\cot r)\delta_{ij})$ for convenience. Because of the natural connection between the unit sphere and the Euclidean space, by solving a certain system of partial differetial equations, we can adapt the technique used by Levine and Weinberger [6] for the task of constructing trial functions. Based on the same scheme as Levine and Weinberger, we obtain the following main result.

**Theorem 1.1.** *Let $M$ be a compact domain in the unit $n$-sphere with smooth boundary. If the mean curvature $h$ of the boundary $\partial M$ is nonpositive, then*

$$\mu_k \leq \lambda_k \text{ for } k = 1, 2, \cdots .$$

*Moreover, if one of the equalities holds, then $\partial M$ is a minimal hypersurface.*

It is a simple fact that if a compact minimal hypersurface $\Sigma$ of the unit sphere is contained in a closed hemisphere then $\Sigma$ is an equator. Thus the previous theorem involves the case of domains which are not contained in any closed hemisphere. To make the second assertion more clear, if we further assume that $M$ is contained in a closed hemisphere, then one of the equality holds if and only if $M$ is the closed hemisphere.

For Euclidean domains the resulting inequalities are better if additional convex restrictions are imposed on the boundary [6]. For spherical domains little improvement seems possible. Indeed, if we assume that the second fundamental form of the boundary $\partial M$ is nonpositive, then one of the equalities of the previous theorem holds if and only if $M$ is a closed hemisphere. However, the following result shows that the resulting inequalities are better if we further assume that the second fundamental form of the boundary is nonpositive.

**Theorem 1.2.** *Let $M$ be a compact domain in the unit $n$-sphere with smooth boundary. If the second fundamental form is nonpositive, then*

$$\mu_{k+n_0} \leq \lambda_k \; for \; k = 1, 2, \cdots ,$$

*where* $n_0 = \left[\frac{n-1}{2}\right]$, *the greatest integer smaller than or equal to* $\frac{n-1}{2}$.

Finally, the paper closes with two examples on the Dirichlet-Neumann inequalities of the geodesic disks in the unit sphere. Because the geometric structure of the geodesic disks is simple, we have better inequalities, $\mu_{k+n-1} \le \lambda_k$, in both cases. It is a matter of interest to us that for every $n$, there are infinitely many eigenvalues so that the equalities of Theorems 1.1 and 1.2 hold when $M$ is a closed hemisphere.

## 2. Preliminaries

Let $M$ be a compact domain in the unit $n$-sphere $\mathbf{S}^n$ with smooth boundary $\partial M$. We choose a local field of orthonormal frames $e_1, e_2, \cdots, e_n$ in $M$ such that, restricted to $\partial M$, the vectors $e_1, e_2, \cdots, e_{n-1}$ are tangent to $\partial M$ and the remaining vector $e_n$ is the outward unit normal vector on $\partial M$. In what follows, it is convenient to agree to the following range of indices for differentiation:

$$1 \le i, j, k, \cdots \le n-1;\ 1 \le \alpha, \beta, \gamma, \cdots \le n.$$

With respect to the frame field of $M$ chosen above, let $\omega_1, \omega_2, \cdots, \omega_n$ be the corresponding field of dual coframes. Denote by $(h_{ij})$ the second fundamental form of the boundary $\partial M$ and by $h = \sum h_{ii}$ the mean curvature of the boundary $\partial M$. The hypersurface $\partial M$ is said to be minimal if its mean curvature $h$ vanishes identically. Here we agree that the second fundamental form $(h_{ij})$ of the boundary of a geodesic ball in the unit sphere with radius $r$ is $(-(\cot r)\delta_{ij})$ for convenience.

Let $\Phi$ be a function defined on $M$. We restrict $\Phi$ to $\partial M$ and denote it by $\phi$. Then we have

$$\Phi_i|_{\partial M} = \phi_i,$$

$$\Phi_n|_{\partial M} = \frac{\partial \Phi}{\partial n},$$

where $\frac{\partial \Phi}{\partial n}$ denotes the directional derivative of $\Phi$ in the outward normal direction. Taking exterior differentiation of both $\Phi_i$ and $\phi_i$, we get

$$\Phi_{ij}|_{\partial M} = \phi_{ij} - h_{ij}\Phi_n \tag{2.1}$$

and

$$\Phi_{ni}|_{\partial M} = (\Phi_n)_i + \sum h_{ij}\phi_j. \tag{2.2}$$

Let $\mathbf{Y}$ be a vector field in $M$, $\mathbf{Y} = \sum Y^\alpha e_\alpha$. Take the exterior differentiation of $Y^\alpha$ and define $Y^\alpha_\beta$ by $dY^\alpha + \sum Y^\beta \omega_{\alpha\beta} = \sum Y^\alpha_\beta \omega_\beta$. Take the exterior differentiation of $Y^\alpha_\beta$ and define $Y^\alpha_{\beta\gamma}$ by $dY^\alpha_\beta + \sum Y^\gamma_\beta \omega_{\gamma\alpha} + \sum Y^\alpha_{\gamma\beta}\omega_{\gamma\beta} = \sum Y^\alpha_{\beta\gamma}\omega_\gamma$.

Suppose that $W$ is a vector field on $M$ having the following properties ($W$ will be constructed in the next section):

1. $W^\alpha_\beta + W^\beta_\alpha = 0$ for all $\alpha$, $\beta$,
2. $W^\alpha_{\beta\gamma} + W^\beta_{\alpha\gamma} = 0$ for all $\alpha$, $\beta$, $\gamma$,
3. $\sum W^\alpha_{\beta\beta} = -(n-1)W^\alpha$ for all $\alpha$.

Let $\lambda$ be a Dirichlet eigenvalue of $M$, $u$ an eigenfunction corresponding to $\lambda$, and $\varphi$ be the function given by $\varphi = \langle W, \nabla u\rangle$. Then we have

**Lemma 2.1.** $\Delta\varphi = -\lambda\varphi$.

*Proof.* From the above properties of $W$, we have

$$\begin{aligned}\Delta\varphi &= \sum W^\alpha_{\beta\beta}u_\alpha + 2\sum W^\alpha_\beta u_{\alpha\beta} + \sum W^\alpha u_{\alpha\beta\beta}\\ &= -\sum(n-1)W^\alpha u_\alpha + \sum W^\alpha((\Delta u)_\alpha + (n-1)u_\alpha)\\ &= \sum W^\alpha(\Delta u)_\alpha\\ &= -\lambda\varphi.\end{aligned}$$

Here we have used the Ricci identity $\sum u_{\beta\alpha\alpha} = (\Delta u)_\beta + (n-1)u_\beta$. ∎

Now we are in a position to construct our test function $F$. Let $u^1, u^2, \cdots, u^k$ be the first $k$ normalized Dirichlet eigenfunctions corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, respectively. Let $F$ be the function defined by $F = T + Z$, where $T = \langle W, \nabla u^k\rangle$, $Z = \sum_{j=1}^k c_j u^j$, and $c_1, c_2, \cdots, c_k$ are constants. Using Stokes's theorem and Lemma 2.1, we have

$$\int_M |\nabla F|^2 = \lambda_k \int_M F^2 - \int_M (\lambda_k Z^2 + Z\Delta Z) + \int_{\partial M} T\frac{\partial T}{\partial n}. \tag{2.3}$$

Since

$$-\int_M (\lambda_k Z^2 + Z\Delta Z) = -\lambda_k \sum c_i c_j \int_M u^i u^j + \sum \int_M \lambda_i u^i u^j,$$

the second term in the right-hand side of (2.3) is nonpositive,

$$-\int_M (\lambda_k Z^2 + Z\Delta Z) = \sum(\lambda_i - \lambda_k)c_i^2 \le 0, \tag{2.4}$$

and it is zero if and only if $\Delta Z + \lambda_k Z = 0$ in $M$.

Now we turn our attention to the third term on the right-hand side of (2.3). This term plays a crucial role in our construction. For simplicity, we shall replace $u^k$ by $u$. Since $u$ is the $k$th Dirichlet eigenfunction, by (2.1) and (2.2), we get $u_{nn} = h\frac{\partial u}{\partial n}$ and $T|_{\partial M} = W^n u_n$. By the definition of $\frac{\partial T}{\partial n}$ and taking exterior differentiation of $T$, we have

$$\left.\frac{\partial T}{\partial n}\right|_{\partial M} = \sum W^i u_{in} + \sum W^n u_{nn}.$$

Substituting these into the integral yields

$$\int_{\partial M} T\frac{\partial T}{\partial n} = \int_{\partial M} \left[\frac{1}{2}W^i W^n (u_n^2)_i + h(W^n)^2 \left(\frac{\partial u}{\partial n}\right)^2\right].$$

By the Stokes's theorem, the first term on the right-hand side of the above equation can be transformed into

$$\int_{\partial M} \sum W^i W^n (u_n^2)_i = -\int_{\partial M} div(W^n W)\left(\frac{\partial u}{\partial n}\right)^2.$$

Using the properties of $W$, we have $d(W^i W^n) = \sum W^n W_i^j \omega_j - \sum W^n W^j \omega_{ji} + \sum W^i W_j^n \omega_j + \sum h_{ij}(W^n)^2 \omega_j - \sum h_{jl} W^i W^j \omega_l$ and

$$div(W^n W) = h(W^n)^2 - h_{ij} W^i W^j + W^i W_i^n.$$

Putting this transformation together with the integral, the third term is adapted for use in the next section

$$\text{(2.5)} \qquad \int_{\partial M} T\frac{\partial T}{\partial n} = \frac{1}{2}\int_{\partial M} (h(W^n)^2 + h_{ij} W^i W^j - W^i W_i^n)\left(\frac{\partial u}{\partial n}\right)^2.$$

## 3. Convex Conditions

In this section, we first construct a family of vector fields $W$ which satisfies the properties stated in Section 2. We then make some assumptions concerning the convexity of the boundary which imply that the integral in equation (2.5) is nonpositive.

Consider the vector field $W$,

$$W = (x, a)b - (x, b)a,$$

where $a$, $b$ are mutually orthogonal constant vectors in $\mathbf{R}^{n+1}$ and $(\cdot, \cdot)$ is the usual inner product on $\mathbf{R}^{n+1}$. Using the structure equations of $\mathbf{S}^n$, we have

**Lemma 3.1.**

1. $W^\alpha = (x,a)(e_\alpha,b) - (e_\alpha,a)(x,b)$ *for all* $\alpha$,
2. $W^\alpha_\beta = (e_\beta,a)(e_\alpha,b) - (e_\alpha,a)(e_\beta,b)$ *for all* $\alpha$, $\beta$,
3. $W^\alpha_{\beta\gamma} = W^\beta\delta_{\alpha\gamma} - W^\alpha\delta_{\beta\gamma}$ *for all* $\alpha$, $\beta$, $\gamma$.

It follows that $W$ satisfies the properties stated in Section 2. Substituting the identities of the above lemma into equation (2.5), we can rewrite equation (2.5) in the following form

$$\int_{\partial M} T\frac{\partial T}{\partial n} = \frac{1}{2}I(a,b), \tag{3.1}$$

where

$$I(a,b) = \int_{\partial M}\left[h(W^n)^2 + h_{ij}W^iW^j + |a|^2(x,b)(e_n,b) + |b|^2(x,a)(e_n,a)\right]\left(\frac{\partial u}{\partial n}\right)^2,$$

$W^n = (x,a)(e_n,b) - (e_n,a)(x,b)$ and $W^i = (x,a)(e_i,b) - (e_i,a)(x,b)$.

Now we want to find conditions under which the integral of (3.1) is nonpositive. We prove our first crucial lemma as follows.

**Lemma 3.2.** *If the mean curvature* $h$ *of* $\partial M$ *is nonpositive, then there exist mutually orthogonal unit constant vectors* $a_0$, $b_0$ *in* $\mathbf{R}^{n+1}$ *such that* $I(a_0,b_0) \le 0$. *Moreover, if the equality holds, then* $\partial M$ *is a minimal hypersurface.*

*Proof.* First of all, consider the following function $\Psi$ defined on $\mathbf{S}^n$:

$$\begin{aligned}\Psi(a) &= \underset{b,b\perp a}{trace}\, I(a,b)\\ &= \int_{\partial M}[h_{ij}(2(x,a)^2\delta_{ij} + (e_n,a)^2\delta_{ij} + (e_i,a)(e_j,a))\\ &\quad + (n-1)(x,a)(e_n,a)]\left(\frac{\partial u}{\partial n}\right)^2,\end{aligned}$$

where the trace is carried out over all $b$, $b\perp a$. Let $\Delta_a^{\mathbf{S}^n}$ be the Laplace-Beltrami operator of the unit sphere $\mathbf{S}^n$ with respect to $a$. Then we have

$$\frac{1}{2}\Delta_a^{\mathbf{S}^n}\Psi(a) = -(n+1)\Psi(a) + 4\int_{\partial M} h\left(\frac{\partial u}{\partial n}\right)^2.$$

Since $\mathbf{S}^n$ is compact, $\Psi$ attains its minimum, say, at $a_0 \in \mathbf{S}^n$. Then $\Delta_a^{\mathbf{S}^n}\Psi(a_0) \ge 0$, and we have

$$\Psi(a_0) \le \frac{4}{n+1}\int_{\partial M} h\left(\frac{\partial u}{\partial n}\right)^2,$$

which is nonpositive. Choose a constant unit vector $b_0$ in $\mathbf{R}^{n+1}$ which is orthogonal to $a_0$, and minimizes the quadratic form $I(a_0, \cdot)$ restricted to the orthogonal complement of $a_0$. Since $\Psi(a_0)$ is nonpositive, $I(a_0, b_0)$ is nonpositive. For our choice of $a_0$ and $b_0$, if the equality holds, then

$$I(a_0, b) = 0$$

for all $b \in \mathbf{S}^n$, $b \perp a_0$. In particular, we have $\Psi(a_0) = 0$, and $\int_{\partial M} h(\frac{\partial u}{\partial n})^2 = 0$. Since $\frac{\partial u}{\partial n}$ cannot vanish on any open subset of $\partial M$, $h$ must be identically zero on $\partial M$. ∎

The following lemma is necessary for the proof of Theorem 1.2 .

**Lemma 3.3.** *If the second fundamental form of $\partial M$ is nonpositive, then there exists a constant unit vector $a_0$, and a subspace $V_0$ of $\mathbf{R}^{n+1}$ of dimension $n_0 + 1$ which is orthogonal to $a_0$ such that $I(a_0, b) \leq 0$ for all $b \in V_0$, where $n_0$ is given as in Theorem* 1.2.

*Proof.* Define a quadratic form $J$ on $\mathbf{R}^{n+1}$ by

$$J(a) = \int_{\partial M} (x, a)(e_n, a) \left(\frac{\partial u}{\partial n}\right)^2 .$$

Let $a_0$ be a constant unit vector which minimizes $J$, $J(a_0) = \min_{a \in S^n} J(a)$. Since the trace of $J$ is zero, $J(a_0)$ is nonpositive. Let $b_1, b_2, \cdots, b_n$ be a family of unit eigenvectors of the quadratic form $I(a_0, \cdot)$ defined on the orthogonal complement of $a_0$. Rearranging the eigenvalues of $I(a_0, \cdot)$ if necessary, we may assume that

$$I(a_0, b_1) \leq I(a_0, b_2) \leq \cdots \leq I(a_0, b_n).$$

Since the second fundamental form of $\partial M$ is nonpositive, we have

$$\begin{aligned} & I(a_0, b_1) + I(a_0, b_2) + \cdots + I(a_0, b_{n_0}) - \Psi(a_0) \\ \geq \; & J(b_1) + J(b_2) + \cdots + J(b_{n_0}) + n_0 J(a_0) - (n-1)J(a_0) \\ \geq \; & (2n_0 - (n-1))J(a_0) \\ \geq \; & 0, \end{aligned}$$

where $\Psi(\cdot)$ was defined in the proof of Lemma 3.2. This implies that

$$I(a_0, b_{n_0+1}) + I(a_0, b_{n_0+2}) + \cdots + I(a_0, b_n) \leq 0.$$

In particular, we have $I(a_0, b_{n_0+1}) \leq 0$. Let $V_0$ be the linear subspace spanned by $b_1, b_2, \cdots b_{n_0+1}$. Then $V_0$ is just the desired subspace. ∎

## 4. Proofs of the Main Theorems

Now we are in the position to prove Theorem 1.1. Using the constant vectors $a_0$ and $b_0$ given in Lemma 3.2, let $W$ be the vector field corresponding to the constant vector $b = tb_0$, $T = \langle W, \nabla u^k \rangle$, $Z = \sum_{j=1}^{k} c_j u^j$ and $F = T + Z$. Let $v_0, v_1, \cdots, v_{k-1}$ be the first $k$ Neumann eigenfunctions corresponding to the eigenvalues $\mu_0, \mu_1, \cdots, \mu_{k-1}$, respectively. Consider the following linear system of equations

$$\int_M Fv_0 = \int_M Fv_1 = \cdots = \int_M Fv_{k-1} = 0. \tag{4.1}$$

The system has $k$ linear equations with $k+1$ variables $t, c_1, \cdots, c_k$. There exist $k+1$ constants $t_0, c_1^0, \cdots, c_k^0$, not all of which are zero, satisfying (4.1). When this is done, by (2.3) and (2.4), we have

$$\int_M |\nabla F|^2 \leq \lambda_k \int_M F^2 + \int_{\partial M} T \frac{\partial T}{\partial n}. \tag{4.2}$$

Lemma 3.2 implies that

$$\int_M |\nabla F|^2 \leq \lambda_k \int_M F^2.$$

If $F$ does not vanish identically on $M$, we then have

$$\mu_k \leq \frac{\int_M |\nabla F|^2}{\int_M F^2} \leq \lambda_k, \tag{4.3}$$

that is, the $k$th nonzero Neumann eigenvalue is less than or equal to the $k$th Dirichlet eigenvalue.

Suppose that $F$ vanishes everywhere on $M$. We observe that $t_0$ is not zero. Indeed, if $t_0 = 0$, since $0 = T = -Z$ on $M$, $Z$ vanishes identically on $M$. This means that all $c$'s are zero, a contradiction.

Suppose that $F$ vanishes everywhere on $M$. Then $T = -Z$ on $M$, and hence $T = 0$ on $\partial M$. Because of the definition of $T$ and $u_k = 0$ on $\partial M$, we have

$$(x, a_0)(e_n, b_0) - (x, b_0)(e_n, a_0) = 0 \text{ on } \partial M,$$

since $t_0$ is not zero. For our choice of $b_0$, we then have

$$\int_{\partial M} T_b \frac{\partial T_b}{\partial n} = 0$$

for all $b \in \mathbf{R}^{n+1}$, $b \perp a_0$, where $T_b$ is the function corresponding to the vector field $W_b = (x, a_0)b - (x, b)a_0$. If all the corresponding functions $F_b = T_b + Z$ vanish identically on $M$, for all $b \in \mathbf{S}^n$, $b \perp a_0$, then we have

$$(x, a_0)(e_n, b) - (x, b)(e_n, a_0) = 0 \quad \text{on} \quad \partial M \tag{4.4}$$

by the same argument as above. Multiplying equation (4.4) by $(e_n, b)$ and taking the trace of the resulting equation with respect to $b$, $b \perp a_0$, we have $(x, a_0) = 0$ on $\partial M$, and conclude that $M$ is a closed hemishpere. The spectrum of the Dirichlet and Neumann eigenvalue problems for the closed hemisphere are well-known (see Example 1 in the next section). In this case, the assertion follows. Thus we may assume that there is a resulting function $F_b$ not identically zero for some $b \in \mathbf{S}^n$, $b \perp a_0$. Consequently we have the inequalities (4.3) for $k = 1, 2, \cdots$.

To see what happens as $\mu_k = \lambda_k$, we first observe that if $F$ does not vanish identically on $M$ then $t_0$ is not zero. Indeed, if $t_0 = 0$, then $F = Z$. From (2.4), $Z$, and therefore $F$, is a Dirichlet eigenfunction with eigenvalue $\lambda_k$. On the other hand, since $F$ satisfies conditions of the variational characterization of $\mu_k$, $F$ is also a Neumann eigenfunction with eigenvalue $\mu_k$, a contradiction.

Finally, if $\mu_k = \lambda_k$, since $t_0$ is not zero, then in either case of $F$, the testing function corresponding to the vectors $a_0$ and $b_0$, vanishing or otherwise, we have

$$I(a_0, b_0) = 0.$$

It follows from Lemma 3.2 that $\partial M$ is a minimal hypersurface. This completes the proof of Theorem 1.1. ■

The proof of Theorem 1.2 is essentially a slight modification of that of Theorem 1.1 except that we choose new $a_0$ from Lemma 3.3. We include the proof here for completeness. Using the constant vector $a_0$ and the linear subspace $V_0$ given in Lemma 3.3, let $W$ be the vector field related to $b \in V_0$, $b = \sum_{i=1}^{n_0+1} t_i b_i$, $T = \langle W, \nabla u^k \rangle$, $Z = \sum_{j=1}^{k} c_j u^j$ and $F = T + Z$. Then there exist $k + n_0 + 1$ constants $t_1^0, \cdots, t_{n_0+1}^0; c_1^0, \cdots, c_k^0$, not all of which are zero, satisfying the following linear system of equations

$$\int_M F v_0 = \int_M F v_1 = \cdots = \int_M F v_{k+n_0-1} = 0, \tag{4.5}$$

where $v_0, v_1, \cdots, v_{k+n_0-1}$ are the first $k + n_0$ Neumann eigenfunctions corresponding to the eigenvalues $\mu_0, \mu_1, \cdots, \mu_{k+n_0-1}$, respectively. Pick $b_0 = \sum_{i=1}^{n_0+1} t_i^0 b_i$. It follows from (2.3) and (2.4) that

$$\int_M |\nabla F|^2 \leq \lambda_k \int_M F^2 + \int_{\partial M} T \frac{\partial T}{\partial n}.$$

Lemma 3.3 implies that

$$\int_M |\nabla F|^2 \le \lambda_k \int_M F^2.$$

If $F$ does not vanish identically on $M$, we then have

$$\mu_{k+n_0} \le \frac{\int_M |\nabla F|^2}{\int_M F^2} \le \lambda_k, \tag{4.6}$$

that is, the $(k+n_0)$th nonzero Neumann eigenvalue is less than or equal to the $k$th Dirichlet eigenvalue.

Suppose that $F$ vanishes everywhere on $M$. We observe that $b_0$ is not zero. Indeed, if $b_0 = 0$, since $T = -Z$ on $M$, $Z$ vanishes identically on $M$. This means that all $t$'s and $c$'s are zero, and we get a contradiction. Furthermore, if $F$ vanishes everywhere on $M$, then $T = -Z$ on $M$, and hence $T = 0$ on $\partial M$. Because of the definition of $T$ and $u_k = 0$ on $\partial M$, we have

$$(x, a_0)(e_n, b_0) - (x, b_0)(e_n, a_0) = 0 \text{ on } \partial M.$$

This means that $(x, b_0) = s(x, a_0)$ and $(e_n, b_0) = s(e_n, a_0)$ for some function $s$ defined on $\partial M$. We then have

$$\begin{aligned} 0 &= \int_{\partial M} T\frac{\partial T}{\partial n} \\ &= \frac{1}{2} I(a_0, b_0) \\ &\le \frac{1}{2}\int_{\partial M} (s^2 + |b_0|^2)(x, a_0)(e_n, a_0)\left(\frac{\partial u}{\partial n}\right)^2 \\ &\le 0. \end{aligned}$$

Since $b_0 \ne 0$, this gives

$$\int_{\partial M} (x, a_0)(e_n, a_0)\left(\frac{\partial u}{\partial n}\right)^2 = 0.$$

For our choice of $a_0$, we have

$$\int_{\partial M} (x, a)(e_n, a)\left(\frac{\partial u}{\partial n}\right)^2 = 0$$

for all $a \in \mathbf{R}^{n+1}$. For the second fundamental form of $\partial M$ being nonpositive, we get

$$\int_{\partial M} T_a \frac{\partial T_a}{\partial n} \le 0,$$

where $T_a$ is the function corresponding to the vector field $W_a = (x, a_0)a - (x, a)a_0$. If the corresponding functions $F = T_a + Z$ vanish identically on $M$ for all $a \in \mathbf{S}^n$, $a \perp a_0$, then we have

$$(x, a_0)(e_n, a) - (x, a)(e_n, a_0) = 0 \text{ on } \partial M \tag{4.7}$$

by the same argument as above. Multiplying equation (4.7) by $(e_n, a)$ and taking the trace of the resulting equation with respect to $a$, $a \perp a_0$, we have $(x, a_0) = 0$ on $\partial M$ and $M$ is a closed hemishpere. Since the assertion holds for the spectrum of the closed hemisphere (see Example 1 in the next section), we may assume that there is a resulting function $F$ not identically zero for some $a$, $a \perp a_0$. Consequently we have the desired inequality (4.6). This completes the proof of Theorem 1.2. ∎

**Example 5.1.** (The closed hemisphere) The spectrum of the Dirichlet and Neumann eigenvalue problems for the closed hemisphere of the unit $n$-sphere are well-known (see [2]). The Dirichlet eigenvalues are given by $m(n+m-1)$ and the Neumann eigenvalues are given by $(m-1)(n+m-2)$, both of which have the same multiplicity $\binom{n+m-2}{n-1}$ for $m = 1, 2, \cdots$. It then follows that the equality of Theorem 1.1, $\mu_k = \lambda_k$, holds for all $k = \binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{n+l-2}{n-1}$, $l = 1, 2, \cdots$. There are also infinitely many eigenvalues so that the equality of Theorem 1.2 holds. In fact, $\mu_{k+[\frac{n-1}{2}]} = \lambda_k$ for $k = 1$ or $\binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{n+l-1}{n-1} - [\frac{n-1}{2}] \leq k \leq \binom{n-1}{n-1} + \binom{n}{n-1} + \cdots + \binom{n+l-1}{n-1}$ for $l = 1, 2, \cdots$. Furthermore, the best inequalities in this case can be written as $\mu_{k+n-1} \leq \lambda_k$ for all $k = 1, 2, \cdots$.

**Example 5.2.** (The closed geodesic ball with radius $r < \frac{\pi}{2}$) Let $B(r)$ be a geodesic ball in the unit $n$-sphere with radius $r$, $0 < r < \frac{\pi}{2}$. Assume that the center of $B(r)$ is at the north pole. Let $E_1, E_2, \cdots, E_{n+1}$ be the standard basis for $\mathbf{R}^{n+1}$. Then the outer normal $e_n$ of the boundary of $B(r)$ is given by $e_n \sin r = x \cos r - E_{n+1}$, where $x$ is the position vector. The second fundamental form $(h_{ij})$ and the mean curvature $h$ of the boundary of $B(r)$ are $(-(\cot r)\delta_{ij})$ and $-(n-1)\cot r$, respectively. Choose $E_{n+1}$ as the constant unit vector $a$ in the bilinear form $I(a, b)$. Then the integrand of the bilinear form $I(a, b)$ is negative since

$$\begin{aligned} &h(W^n)^2 + h_{ij}W^iW^j + (x, b)(e_n, b) + (x, a)(e_n, a) \\ &= -\cot r(1 + (n-2)\csc^2 r(x, b)^2) \\ &< 0 \end{aligned}$$

for all unit vectors $b$, $b \perp a$. Using the same technique as the one in the proof of the previous theorems, we have $\mu_{k+n-1} < \lambda_k$ for $k = 1, 2, \ldots$.

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