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WEAK AND STRONG CONVERGENCE IN THE HYPERSPACE $CC(X)$

Thakyin Hu and Jui-Chi Huang

Abstract. Let $CC(X)$ be the collection of all non-empty compact, convex subsets of a complex Banach space X endowed with the usual Hausdorff metric h. We shall define a natural weak topology $\mathcal{T}_w$ on $CC(X)$ and investigate properties of $\mathcal{T}_w$-convergent sequences. Our main result is a theorem which states that if A_n, $A \in CC(X)$ and A_n is $\mathcal{T}_w$-convergent to A, then there exists a sequence $\{B_n\}$ (each B_n is a finite convex combination of A_k's) such that B_n converges to A with respact to the Hausdorff metric h.

1. Introduction

Let X be a Banach space and $BCC(X)$ be the collection of all non-empty bounded, closed, convex subsets of X endowed with Hausdorff metric h. If $\dim(X) < \infty$ and $A_n \in BCC(X)$ is a bounded sequence (i.e. there exists $M < \infty$ such that $h(A_n, \{0\}) \leq M < \infty$ for all $n = 1, 2, \ldots$), Blaschke [2] proved that $\{A_n\}$ has a subsequence $\{A_{n_k}\}$ such that $\{A_{n_k}\}$ converges to some $A \in BCC(X)$. DeBlasi and Myjak [3] introduced the concept of weak convergence of a sequence in $BCC(X)$ and they proved an infinite dimensional version of Blaschke's theorem. Suppose X is regarded as a real Banach space, A_n and A are assumed to lie in finite dimensional subspaces of X and A_n converges weakly to A. DeBlasi and Myjak [3] also proved that there exists a sequence $\{Z_n\}$, where each Z_n is a finite convex combination of the A_n's such that Z_n converges to A strongly (i.e. $Z_n \xrightarrow{h} A$). This result is an extension of the classical Mazur's theorem. In this paper, we will define a certain weak topology $\mathcal{T}_w$ on the hyperspace $CC(X)$, which is the collection of all non-empty compact, convex subsets of X. We obtain some basic properties of the hyperspace $(CC(X), \mathcal{T}_w)$; moreover, we prove a $\mathcal{T}_w$-convergence theorem for a sequence $\{A_n\} \subset CC(X)$ which is more general than the above-mentioned result of DeBlasi and Myjak.

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2. Notations and Preliminaries

Let X be a Banach space, X^* its topological dual, and $\mathbb{Z}$ the complex plane. The closed unit balls of X and X^* are denoted by B and B^*, respectively; $BCC(X)$ is the collection of all non-empty bounded, closed convex subsets of X, $CC(X)$ is the collection of all all non-empty compact convex subsets of X, $CC(\mathbb{Z})$ is the collection of all non-empty compact convex subsets of the complex plane. For $A, B \in BCC(X)$, define $N(A;\varepsilon) = \{x \in X : \|x - a\| < \varepsilon \text{ for some } a \in A\}$ and $h(A, B) = \inf\{\varepsilon > 0 : A \subset N(B;\varepsilon) \text{ and } B \subset N(A;\varepsilon)\}$. Equivalently, $h(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$. In this paper, we are concerned only with the hyperspace $CC(X)$.

Lemma 1. *Let $A,\ B,\ C,\ D \in CC(X)$ and $\alpha \in \mathbb{Z}$. We have*

(a) $h(A, \{0\}) = \sup\{\|a\| : a \in A\}$;

(b) $h(A, B) \le h(A, C)$ *if* $B \subset C$;

(c) $h(\alpha A, \alpha B) = |\alpha| h(A, B)$;

(d) $h(A + C, B + D) \le h(A, B) + h(C, D)$.

The proofs follow immediately from the definition and shall be omitted. Observe that for $A, B \in CC(X)$ and $\lambda \in \mathbb{Z}$, we have $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ both belong to $CC(X)$ since addition and scalar multiplication are continuous. Also for $x^* \in X^*$ and $A \in CC(X)$, it follows from the continuity and linearity of x^* that $x^*(A)$ is a compact, convex subset of the complex plane $\mathbb{Z}$, i.e., $x^*(A) \in CC(\mathbb{Z})$.

Lemma 2. (cf [4]). *Let $A,\ B,\ A_1,\ A_2, \ldots,\ A_n \in CC(X)$, $\alpha_1,\ \alpha_2, \ldots,\ \alpha_n \in [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$, and $x^* \in X^*$. Then we have*

(a) $\sum_{i=1}^n \alpha_i A_i \in CC(X)$;

(b) $A = B$ *if and only if* $x^*(A) = x^*(B)$ *for each* $x^* \in X^*$;

(c) $x^* : (CC(X), h) \to (CC(\mathbb{Z}), h)$ *is continuous. In fact,* $h(x^*(A), x^*(B)) \le \|x^*\| h(A, B)$.

The proof of (b) is a simple application of the Hahn-Banach theorem and the proof of (c) is a consequence of the fact that $|x^*(a) - x^*(b)| = |x^*(a - b)| \le \|x^*\| \|a - b\|$.

Recall now that the weak topology τ_w on X is defined to be the weakest topology on X which makes each $x^* : (X, \tau_w) \to (\mathbb{Z}, |\ |)$ continuous. Now that we have, by Lemma 2, that each $x^* : (CC(X), h) \to (CC(\mathbb{Z}), h)$ is continuous, we may define $\mathcal{T}_w$ to be the weakest topology on the hyperspace $CC(X)$ such that each $x^* : (CC(X), \mathcal{T}_w) \to (CC(\mathbb{Z}), h))$ is continuous. We shall use the notation $\mathcal{W}(A; x_1^*, \ldots, x_n^*; \varepsilon) = \{B \in CC(X) : h(x_i^*(B), x_i^*(A)) < \varepsilon$ for $i = 1, 2, \ldots, n\}$ to denote a $\mathcal{T}_w$-neighborhood of A in $CC(X)$. The Hausdorff metric topology and the $\mathcal{T}_w$-topology shall be called as the strong topology and the weak topology on $CC(X)$, respectively.

Thus we have the following

Definition 1. Let $A_\alpha,\ A \in CC(X)$ and $\mathcal{K} \subset CC(X)$.

(a) $A_\alpha \to A$ strongly if and only if $h(A_\alpha, A) \to 0$;

$A_\alpha \to A$ weakly if and only if $h(x^*(A_\alpha), x^*(A)) \to 0$ for each $x^* \in X^*$.

(b) $\mathcal{K}$ is strongly bounded if and only if there exists some $M > 0$ such that $\sup\{h(A, \{0\}) : A \in \mathcal{K}\} \leq M < \infty$;

$\mathcal{K}$ is weakly bounded if and only if there exists some $M_{x^*} > 0$ such that $\sup\{h(x^*(A), \{0\}) : A \in \mathcal{K}\} \leq M_{x^*} < \infty$ for each $x^* \in X^*$.

Suppose x^* is a complex-linear functional on X and u its real part, the equation $x^*(x) = u(x) - iu(ix) = u(x) + iu(-ix)$ tells us that each complex-linear functional is uniquely determined by its real part. we now define

Definition 2. Let $A \in CC(X)$ and $x^* \in X^*$. Define $\sigma_A(x^*) = s_A(Re\, x^*) = \max\{Re\, x^*(a) : a \in A\}$, where s_A is known as the support function of A.

Let $\mathbb{R}$ be the set of all real numbers. Then $CC(\mathbb{R})$ is simply the collection of all non-empty closed intervals and it is well-known that $h([a, b], [c, d]) = \max\{|c - a|, |d - b|\}$. The basic and important relationship between $\sigma_A(x^*)$ and $x^*(A)$ is now given by the next lemma.

Lemma 3. *Let* $A,\ B \in CC(X)$, $x^* \in X^*$. *Then* $|\sigma_A(x^*) - \sigma_B(x^*)| \leq h(Re\, x^*(A), Re\, x^*(B)) \leq h(x^*(A), x^*(B)) \leq \|x^*\| h(A, B)$. *In particular* $|\sigma_A(x^*)| \leq h(Re\, x^*(A), \{0\}) \leq h(x^*(A), \{0\}) \leq \|x^*\| h(A, \{0\})$, *where the same symbol* $\{0\}$ *is used to denote the zero-element of the corresponding spaces, namely* $\mathbb{R}$, $\mathbb{Z}$ *and* X.

Proof. Let $Re\, x^*(A) = [a_1, a_2]$, $Re\, x^*(B) = [b_1, b_2] \in CC(\mathbb{R})$. Then $|\sigma_A(x^*) - \sigma_B(x^*)| = |s_A(Re\, x^*) - s_B(Re\, x^*)| = |b_2 - a_2| \leq \max\{|b_2 - a_2|, |b_1 - a_1|\} = h([a_1, a_2], [b_1, b_2]) = h(Re\, x^*(A), Re\, x^*(B)) \leq h(x^*(A), x^*(B)) \leq \|x^*\| h(A, B)$ since $Re : (CC(\mathbb{Z}), h) \to (CC(\mathbb{R}), h)$ is a nonexpansive mapping and the last inequality follows from Lemma 2.

It follows from Lemma 3 and Proposition 2.1 of [3] that we have

Lemma 4. *Let* $A,\ B \in CC(X)$, $\alpha, \beta \geq 0$. *Then*

(a) $\sigma_{\alpha A+\beta B} = \alpha\sigma_A + \beta\sigma_B$.

(b) $h(A, B) = \sup\{h(x^*(A), x^*(B)) : \|x^*\| \leq 1\} = \sup\{|\sigma_A(x^*) - \sigma_B(x^*)| : \|x^*\| \leq 1\}$.

3. Main Results

Firstly, we show that $\mathcal{T}_w$-convergence and weak convergence in the sense of DeBlasi and Myjak [3] are equivalent.

Theorem 1. *Let X be a complex Banach space, and $A,\ A_n \in CC(X)$. Then $x^*(A_n) \to x^*(A)$ for each $x^* \in X^*$ if and only if $\sigma_{A_n}(x^*) \to \sigma_A(x^*)$ for each $x^* \in X^*$.*

Proof. Suppose $x^*(A_n) \to x^*(A)$ for each $x^* \in X^*$. Then $Re\, x^*(A_n) \to Re\, x^*(A)$ since $Re : (CC(\mathbb{Z}), h) \to (CC(\mathbb{R}), h)$ is nonexpansive. Also $\max : (CC(\mathbb{R}), h) \to \mathbb{R}$ is nonexpansive implies that $\sigma_{A_n}(x^*) = s_{A_n}(Re\, x^*) = \max\{Re\, x^*(A_n)\} \to \max\{Re\, x^*(A)\} = \sigma_A(x^*)$ for each $x^* \in X^*$.

On the other hand, assume $\sigma_{A_n}(x^*) \to \sigma_A(x^*)$ for each $x^* \in X^*$. If there exists some x^* such that $x^*(A_n) \not\to x^*(A)$, then there exists $\varepsilon > 0$ and a subsequence $\{A_{n_k}\}$ such that $h(x^*(A_{n_k}), x^*(A)) \geq \varepsilon$ for all $k = 1, 2, \ldots$, which in turn implies that either (a) $x^*(A_{n_k}) \not\subset N(x^*(A); \varepsilon)$ or (b) $x^*(A) \not\subset N(x^*(A_{n_k}); \varepsilon)$. Assume (a) is true. Then there exists $a_{n_k} \in A_{n_k}$ such that $x^*(a_{n_k}) \notin N(x^*(A); \varepsilon)$, i.e., $|x^*(a_{n_k}) - x^*(a)| \geq \varepsilon$ for all $a \in A$. Hence $|x^*(a_{n_k}) - x^*(a)| = |[u(a_{n_k}) - iu(ia_{n_k})] - [u(a) - iu(ia)]| \geq \varepsilon$, which implies that $|u(a_{n_k}) - u(a)| \geq \frac{\varepsilon}{\sqrt{2}}$ or $|u(ia_{n_k}) - u(ia)| \geq \frac{\varepsilon}{\sqrt{2}}$ for each $a \in A$. If $u(a_{n_k}) - u(a) \geq \frac{\varepsilon}{\sqrt{2}}$, then $u(a_{n_k}) \geq \frac{\varepsilon}{\sqrt{2}} + u(a)$, and thus $\sigma_{A_n}(x^*) = s_{A_n}(u) \geq \frac{\varepsilon}{\sqrt{2}} + u(a)$ for all $a \in A$ and consequently $\sigma_{A_n}(x^*) \geq \frac{\varepsilon}{\sqrt{2}} + \sigma_A(x^*)$, which implies that $\sigma_{A_n}(x^*) \not\to \sigma_A(x^*)$. If $u(a) - u(a_{n_k}) \geq \frac{\varepsilon}{\sqrt{2}}$, we let $v(x) = -u(x)$, $y^* = v(x) - iv(ix)$, to get that $v(a_{n_k}) \geq v(a) + \frac{\varepsilon}{\sqrt{2}}$ and thus $\sigma_{A_n}(y^*) = s_{A_n}(v) \not\to s_A(v) = \sigma_A(y^*)$. That is a contradiction. By similar reasoning, all the remaining cases also yield a contradiction and the theorem is proved.

The next results are extensions to the hyperspace $CC(X)$ of their corresponding counterparts in the underlying space X.

Theorem 2. *If $\{A_n\}$ is a $\mathcal{T}_w$-Cauchy sequence in $CC(X)$, then there exists $M > 0$ such that $\sup_{n\geq 1}\{h(A_n, \{0\})\} \leq M$. Moreover, if $A_n \to A$ weakly, then $h(A, \{0\}) \leq \liminf_{n\to\infty} h(A_n, \{0\})$.*

Proof. For each $x^* \in X^*$, $\{x^*(A_n)\}$ is a Cauchy sequence in $CC(\mathbb{Z})$. Thus there exists $M_{x^*} > 0$ such that $h(x^*(A_n), \{0\}) \leq M_{x^*}$, for all $n = 1, 2, \ldots$. Let

$B = \cup_{n=1}^{\infty} A_n \subset X$. Then for each $b \in B = \cup A_n$, there exists some $a_k \in A_k$ such that $b = a_k$. Hence $|x^*(b)| = h(\{x^*(b)\}, \{0\}) = h(\{x^*(a_k)\}, \{0\}) \leq h(x^*(A_k), \{0\}) \leq M_{x^*}$. Thus $B \subset X$ is a family of linear functionals on X^* such that $\sup_{b \in B} |x^*(b)| \leq M_{x^*}$ and it follows from the uniform boundedness principle that $\sup\{\|b\| : b \in B\} \leq M < \infty$. Consequently, $h(B, \{0\}) = \sup\{\|b\| : b \in B\} \leq M$ which implies that $h(A_n, \{0\}) \leq h(B, \{0\}) \leq M$. That proves the first part of the theorem. Suppose now $A_n \xrightarrow{\mathcal{T}_w} A$, and if $\liminf_{n \to \infty} h(A_n, \{0\}) < \alpha < h(A, \{0\})$. Then there exists some subsequence $\{A_{n_k}\}$ with $h(A_{n_k}, \{0\}) < \alpha$. On the other hand, $h(A, \{0\}) > \alpha$ implies the existence of some $a \in A$ such that $\|a\| > \alpha$. By Hahn-Banach theorem, there exists some $x^* \in X^*$ with $\|x^*\| = 1$, and $|x^*(a)| = \|a\|$ and thus $h((x^*(A), \{0\}) \geq \|a\| > \alpha$. But, by Lemma 3, $h(x^*(A_{n_k}), \{0\}) \leq \|x^*\| h(A_{n_k}, \{0\}) \leq h(A_{n_k}, \{0\}) < \alpha$. It follows that $x^*(A_{n_k})$ does not converge to $x^*(A)$. That is a contradiction and the proof is complete. ■

Corollary 1. *Let $A_n \to A$ weakly in $CC(X)$. Then $\sigma_{A_n} : B^* \to \mathbb{R}$ is a sequence of functions such that $\sigma_{A_n}(x^*) \to \sigma_A(x^*)$ and $\|\sigma_{A_n}\| = \sup_{x^* \in B^*} |\sigma_{A_n}(x^*)| \leq M < \infty$ for all $n = 1, 2, \ldots$.*

Proof. $A_n \to A$ weakly implies that $\sigma_{A_n}(x^*) \to \sigma_A(x^*)$ for each $x^* \in X^*$ by Theorem 1. And $\|\sigma_{A_n}\| = \sup_{x^* \in B^*} |\sigma_{A_n}(x^*)| \leq \sup_{x^* \in B^*} \{h(x^*(A_n), \{0\}\} \leq h(A_n, \{0\}) \leq M \; \forall \; n = 1, 2, \ldots$.

Theorem 3. *Let X be a complex Banach space, B^* the closed unit ball of X^* and $A \in CC(X)$, $f_A(x^*) = x^*(A)$. Then $f_A : (B^*, \tau_w^*) \to (CC(\mathbb{Z}), h)$ is continuous.*

Proof. Let $x^* \in B^*$ and $\varepsilon > 0$ be given. A is compact and $A \subset \sup_{a \in A} N(a; \frac{\varepsilon}{3})$ implies the existence of a finite number of points $a_1, a_2, \ldots, a_n$ such that $A \subset \cup_{i=1}^{n} N(a_i; \frac{\varepsilon}{3})$. Let $U(x^*; a_1, a_2, \ldots, a_n; \frac{\varepsilon}{3})$ be a τ_w^*-neighborhood of x^* and $y^* \in U(x^*; a_1, \ldots, a_n; \frac{\varepsilon}{3})$. For each $a \in A$, there exists some a_k $(1 \leq k \leq n)$ such that $\|a - a_k\| < \frac{\varepsilon}{3}$, and $|y^*(a) - x^*(a)| \leq |y^*(a) - y^*(a_k)| + |y^*(a_k) - x^*(a_k)| + |x^*(a_k) - x^*(a)| \leq \|y^*\| \|a - a_k\| + \frac{\varepsilon}{3} + \|x^*\| \|a_k - a\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ since $\|y^*\|, \|x^*\| \leq 1$. Hence $y^*(A) \subset N(x^*(A); \varepsilon)$ and $x^*(A) \subset N(y^*(A); \varepsilon)$ which implies that $h(f_A(y^*), f_A(x^*)) = h(y^*(A), x^*(A)) < \varepsilon$ and the proof is complete. ■

Corollary 2. *The mapping $\sigma_A : (B^*, \tau_w^*) \to \mathbb{R}$ is continuous.*

Proof. Since $|\sigma_A(y^*) - \sigma_A(x^*)| = |\max\{Re\, y^*(A)\} - \max\{Re\, x^*(A)\}| \leq h(Re\, y^*(A), Re\, x^*(A)) \leq h\ (y^*(A), x^*(A)) = h(f_A(y^*), f_A(x^*))$.

Finally, we prove the following extension of Mazur's theorem. Our theorem extends DeBlasi and Myjak's result to compact convex sets of a complex Banach spaces instead of finite dimensional convex sets.

Theorem 4. *Let X be a complex Banach space, $A_n, A \in CC(X)$ and $A_n \to A$ weakly. Then there exists a sequence $\{Z_n\}$ of finite convex combinations of A_n's such that $Z_n \to A$ strongly.*

Proof. Let $A_n \to A$ weakly and σ_{A_n}, σ_A are their corresponding support functionals. It follows from that $\sigma_{A_n}, \sigma_A : (B^*, \tau_w^*) \to \mathbb{R}$ are continuous and hence $\sigma_{A_n}, \sigma_A \in C((B^*, \tau_w^*); \mathbb{R})$, the Banach space of all real-valued, continuous, functions on (B^*, τ_w^*) with supremum norm. If μ is any complex measure on (B^*, τ_w^*), Lebesgue's dominated convergence theorem implies that $\int \sigma_{A_n} d\mu \to \int \sigma_A d\mu$. Since the Riesz representation theorem identifies the dual of $C(B^*, \tau_w^*)$ with the space of all complex regular Borel measures on B^*, we have $\sigma_n = \sigma_{A_n} \to \sigma = \sigma_A$ weakly (for notational simplicity we set $\sigma_n = \sigma_{A_n}$, $\sigma = \sigma_A$). Thus it follows from Mazur's theorem that there exists a sequence of finite convex combinations of σ_n's, say $\sum_{i=1}^{k_n} t_{n(i)}\sigma_{n(i)}$, where $t_{n(i)} \geq 0$ and $\sum_{i=1}^{k_n} t_{n(i)} = 1$ such that $\| \sum_{i=1}^{k_n} t_{n(i)}\sigma_{n(i)} - \sigma\| \to 0$ as $n \to +\infty$. Suppose $\varepsilon > 0$ is given, choose n_0 such that $n \geq n_0$ implies $\| \sum_{i=1}^{k_n} t_{n(i)}\sigma_{n(i)} - \sigma\| < \varepsilon$. It follows now from Lemma 4 that $\sum_{i=1}^{k_n} t_{n(i)}\sigma_{n(i)} = \sum_{i=1}^{k_n} t_{n(i)}\sigma_{A_{n(i)}} = \sigma_{(\sum_{i=1}^{k_n} t_{n(i)}A_{n(i)})}$. Let $Z_n = \sum_{i=1}^{k_n} t_{n(i)}A_{n(i)}$. Then $Z_n \in \operatorname{Conv}\{A_1, A_2, \ldots\}$, and $h(Z_n, A) = \| \sum_{i=1}^{k_n} t_{n(i)}\sigma_{n(i)} - \sigma\|$. Thus for $n \geq n_0$ we have $h(Z_n, A) < \varepsilon$. That is, $Z_n \to A$ strongly as $n \to \infty$ and the proof is complete. ∎

That A and A_n's must all be compact in Theroem 4 is essential can be illustrated by the next example which is due to DeBlasi and Myjak [3].

Example. Let $l_2 = \{x = (x_n) \mid \|x\| = (\sum |x_n|^2)^{\frac{1}{2}} < \infty\}$, $A_n = \{x = (x_1, x_2, \ldots, x_n, 0, \ldots) \mid \|x\| \leq 1\}$, $B_1 = \{x \mid \|x\| \leq 1\}$. Then A_n is $\mathcal{T}_w$-convergent to B_1, but $h(A_n, B_1) = 1 \; \forall\, n = 1, 2, \ldots$. Note that B_1 is not compact.

Remarks. The notion of weak convergence of bounded closed convex sets has been studied by many mathematicans ([1, 3, 7-9]), and this paper is inspired by their work. However, our approach is different that enables us to define the weak topology $\mathcal{T}_w$ instead of the notion of weak convergence. Consequently, validity of fixed point theorems for mappings defined on $\mathcal{T}_w$-compact sets $\mathcal{K} \subset CC(X)$ may be further investigated ([5, 6]). In fact, it has been shown [4] that the classical Markov-Kakutani theorem can be extended to the hyperspace $(CC(X), \mathcal{T}_w)$. We are influenced and indebted to DeBlasi and Myjak [3] for the completion of this paper

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Thakyin Hu
Department of Mathematics,
Tamkang University,
Tamsui 25137, Taipei,
Taiwan, R.O.C.

Jui-Chi Huang
Center for General Education,
Northern Taiwan Institute of Science and Technology,
No. 2, Xue Yuan Road,
Peito, Taipei 112,
Taiwan, R.O.C.
E-mail: juichi@ntist.edu.tw