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CLASSIFICATION OF A FAMILY OF HAMILTONIAN-STATIONARY LAGRANGIAN SUBMANIFOLDS IN COMPLEX HYPERBOLIC 3-SPACE

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Abstract. A Lagrangian submanifold in a Kaehler manifold is said to be Hamiltonian-stationary (or simply H-stationary) if it is a critical point of the area functional restricted to (compactly supported) Hamiltonian variations. In an earlier paper [12], H-stationary Lagrangian submanifolds of constant curvature in the complex projective 3-space CP^3 with positive relative nullity are classified. In this paper we completely classify H-stationary Lagrangian submanifolds of constant curvature in the complex hyperbolic 3-space CH^3 with positive relative nullity. As an immediate by-product, several explicit new families of H-stationary Lagrangian submanifolds in CH^3 are obtained.

1. Introduction

Let $\tilde{M}^n(4c)$ denote a Kähler n-manifold of constant holomorphic sectional curvature $4c$. Let J and $\langle\ ,\ \rangle$ be the complex structure and the Kaehler metric $\langle\ ,\ \rangle$ on $\tilde{M}^n(4c)$. The Kaehler 2-form ω is defined by $\omega(\cdot\,,\cdot) = \langle J\cdot, \cdot\rangle$.

An immersion $\psi : M \to \tilde{M}^n(4c)$ of an n-manifold M into $\tilde{M}^n(4\tilde{c})$ is called *Lagrangian* if $\psi^*\omega = 0$ on M. A vector field X on $\tilde{M}^n(4c)$ is called Hamiltonian if $\mathcal{L}_X\omega = f\omega$ for some function $f \in C^\infty(\tilde{M}^n(4c))$, where $\mathcal{L}$ is the Lie derivative. Thus, there exists a smooth real-valued function φ on $\tilde{M}^n(4c)$ such that $X = J\tilde{\nabla}\varphi$, where $\tilde{\nabla}$ is the gradient in $\tilde{M}^n(4c)$. Since the diffeomorphisms of the flux ψ_t of X satisfy $\psi_t\omega = e^{h_t}\omega$, they transform Lagrangian submanifolds into Lagrangian submanifolds.

A normal vector field ξ to a Lagrangian immersion $\psi : M^n \to \tilde{M}^n(4c)$ is called Hamiltonian if $\xi = J\nabla f$, where f is a smooth function on M^n and ∇f is the gradient of f with respect to the induced metric.

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The notion of Hamiltonian-stationary (or H-stationary for brevity) Lagrangian submanifolds was introduced by Oh in 1990 (see [19]) as the critical points of the volume functional for all Hamiltonian isotropy of the Lagrangian submanifold. The Euler-Lagrange equation of this variational problem is

$$\delta\alpha_H = 0, \tag{1.1}$$

where H is the mean curvature vector of the submanifold, α_H is the Maslov form, and δ is the Hodge-dual of the exterior derivative d on M with respect to the induced metric. Clearly, Lagrangian submanifolds with parallel mean curvature vector are H-stationary. Among others, H-stationary Lagrangian submanifolds in complex space forms have been studied in [1-10, 12, 13, 16-19].

In an earlier paper [12], the author and Garay classify H-stationary Lagrangian submanifolds of constant curvature in CP^3 with positive relative nullity. In this paper, we completely classify H-stationary Lagrangian submanifolds of constant curvature in CH^3 with positive relative nullity. As an immediate by-product, several explicit new families of H-stationary Lagrangian submanifolds in CH^3 are obtained.

2. Preliminaries

2.1. Basic notation and formulas

Let $f : M \to \tilde{M}^n(4c)$ be a Lagrangian isometric immersion of a Riemannian n-manifold M into $\tilde{M}^n(4c)$. Denote by ∇ and $\tilde{\nabla}$ the Riemannian connections of M and $M^n(4c)$, respectively. Let D be the connection on the normal bundle of the submanifold.

The formulas of Gauss and Weingarten are given respectively by (cf. [6])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for tangent vector fields X, Y and normal vector field ξ. If we denote the Riemann curvature tensor of ∇ by R, then the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = {} & \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \\ & + c\{\langle X,W\rangle \langle Y,Z\rangle - \langle X,Z\rangle \langle Y,W\rangle\},\end{aligned} \tag{2.3}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.4}$$

where $(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

For a Lagrangian submanifold M we also have (cf. [14])

$$D_X JY = J\nabla_X Y, \tag{2.5}$$

$$\langle h(X,Y), JZ\rangle = \langle h(Y,Z), JX\rangle = \langle h(Z,X), JY\rangle\,. \tag{2.6}$$

At a given point $p \in M$, the *relative null space* $\mathcal{N}_p$ at p is the subspace of the tangent space T_pM defined by

$$\mathcal{N}_p = \{X \in T_pM : h(X,Y) = 0\ \forall Y \in T_pM\}.$$

The dimension ν_p of $\mathcal{N}_p$ is called the *relative nullity* at p. The submanifold is said to have positive relative nullity if ν_p is positive at each point $p \in M$.

2.2. Lagrangian and Legendrian submanifolds

If $\tilde{M}^n(4c)$ is a complete and simply-connected Kähler manifold of constant holomorphic sectional curvature $4c$ with $c < 0$, then $\tilde{M}^n(4c)$ is holomorphically isometric to the complex hyperbolic n-space $CH^n(4c)$.

Consider the complex number $(n+1)$-space $\mathbf{C}_1^{n+1}$ equipped with the pseudo-Euclidean metric:

$$g_0 = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n+1} dz_j d\bar{z}_j.$$

Put $H_1^{2n+1}(-1) = \left\{z \in \mathbf{C}_1^{n+1} : \langle z,z\rangle = -1\right\}$ and $H_1^1 = \{\lambda \in \mathbf{C} : \lambda\bar{\lambda} = 1\}$.

On $\mathbf{C}_1^{n+1}$ we consider the canonical complex structure J induced by $i = \sqrt{-1}$. On $H^{2n+1}(-1)$ we consider the canonical contact structure consisting of ϕ given by the projection of the complex structure J of $\mathbf{C}_1^{n+1}$ on the tangent bundle of $H_1^{2n+1}(-1)$ and the structure vector field $\xi = Jx$ with x being the position vector.

There exists an H_1^1-action on $H_1^{2n+1}(-1)$ given by $z \mapsto \lambda z$. At each point $z \in H_1^{2n+1}(-1)$, iz is tangent to the flow of the action. The orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2n+1}(-1)/\sim$ is the complex hyperbolic space $CH^n(-4)$ with constant holomorphic sectional curvature -4, whose complex structure is induced from the complex structure on $\mathbf{C}_1^{n+1}$ via Hopf's fibration: $\pi\colon\ H_1^{2n+1}(-1) \to CH^n(-4)$.

An isometric immersion $f:\ M \to H_1^{2n+1}(-1)$ is called *Legendrian* if ξ is normal to $f_*(TM)$ and $\langle \phi(f_*(TM)), f_*(TM)\rangle = 0$, where $\langle\ ,\ \rangle$ denotes the inner product on $\mathbf{C}_1^{n+1}$. The vectors of $H_1^{2n+1}(-1)$ normal to ξ at a point z define the horizontal subspace $\mathcal{H}_z$ of the Hopf fibration $\pi\colon\ H_1^{2n+1}(-1) \to CH^n(-4)$.

Let $\psi\colon\ M \to CH^n(-4)$ be a Lagrangian immersion. Then there is an isometric covering map $\tau\colon\ \hat{M} \to M$ and a Legendrian immersion $f\colon\ \hat{M} \to H_1^{2n+1}(-1)$

such that $\psi(\tau) = \pi(f)$. Hence, every Lagrangian immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a Legendrian immersion of the same Riemannian manifold (see [20] for details).

Conversely, suppose that $f\colon\ \hat{M} \to H_1^{2n+1}(-1)$ is a Legendrian immersion. Then $\psi = \pi(f)\colon\ M \to CH^n(-4)$ is again an isometric immersion, which is Lagrangian. Under this correspondence, the second fundamental forms h^f and h^ψ of f and ψ satisfy $\pi_* h^f = h^\psi$. Moreover, h^f is horizontal with respect to π. We shall denote h^f and h^ψ simply by h.

Let $L : M \to H_1^{2n+1}(-1) \subset \mathbf{C}_1^{n+1}$ be an isometric immersion. Denote by $\hat{\nabla}$ and ∇ the Levi-Civita connections of $\mathbf{C}_1^{n+1}$ and M, respectively. Let h denote the second fundamental form of M in $H_1^{2n+1}(-1)$. Then we have

$$\hat{\nabla}_X Y = \nabla_X Y + h(X,Y) + \langle X, Y\rangle\, L \tag{2.7}$$

for vector fields X, Y tangent to M.

3. Twisted Product Decompositions and Adapted Immersions

We recall a very effective method introduced by Chen, Dillen, Verstraelen and Vrancken for constructing Lagrangian submanifolds of constant curvature c in $\tilde{M}^n(4c)$ (see [11] for details).

Definition 3.1. Let $(M_1, g_1), \ldots, (M_m, g_m)$ be Riemannian manifolds, f_i a positive function on $M_1 \times \cdots \times M_m$ and $\pi_i : M_1 \times \ldots \times M_m \to M_i$ the i-th canonical projection for $i = 1, \ldots, m$. Then the *twisted product*

$$_{f_1}M_1 \times \cdots \times_{f_m} M_m$$

of $(M_1, g_1), \ldots, (M_m, g_m)$ is the differentiable manifold $M_1 \times \ldots \times M_m$ with the twisted product metric g defined by

$$g(X,Y) = f_1^2 \cdot g_1(\pi_{1*}X, \pi_{1*}Y) + \cdots + f_m^2 \cdot g_m(\pi_{m*}X, \pi_{m*}Y) \tag{3.1}$$

for all vector fields X and Y of $M_1 \times \cdots \times M_m$.

Let $N^{n-\ell}(c)$ be an $(n-\ell)$-dimensional real space form of constant curvature c. For $0 < \ell < n-1$, consider the twisted product:

$$_{f_1}I_1 \times \cdots \times_{f_\ell} I_\ell \times_1 N^{n-\ell}(c) \tag{3.2}$$

with twisted product metric given by

$$g = f_1^2 dx_1^2 + \cdots + f_\ell^2 dx_\ell^2 + g_0, \tag{3.3}$$

where g_0 is the canonical metric of $N^{n-\ell}(c)$ and $I_1, \ldots, I_\ell$ are open intervals.

For $\ell = n-1$ (resp., $\ell = n$), consider the following twisted product instead.

$$ {}_{f_1}I_1 \times \cdots \times_{f_{n-1}} I_{n-1} \times_1 I_n \quad (\text{resp., } {}_{f_1}I_1 \times \cdots \times_{f_n} I_n). \tag{3.4} $$

If the twisted product given by (3.2) or (3.4) is a real space form $M^n(c)$ of constant curvature c, it is called a *twisted product decomposition* of $M^n(c)$. The functions $f_1, \ldots, f_\ell$ are called the *twistor functions*. For simplicity, we denote such a decomposition of $M^n(c)$ by $\mathcal{TP}^n_{f_1\cdots f_\ell}(c)$.

The coordinates $\{x_1, \ldots, x_n\}$ on $\mathcal{TP}^n_{f_1\cdots f_\ell}(c)$ are called *adapted coordinates* if

(i) $\partial/\partial x_j$ is tangent to I_j for $j = 1, \ldots, \ell$;

(ii) $\partial/\partial x_r$ is tangent to $N^{n-\ell}(c)$ for $r = \ell+1, \ldots, n$; and

(iii) the metric tensor of $\mathcal{TP}^n_{f_1\cdots f_\ell}(c)$ takes the form (3.3).

$$ \Phi(\mathcal{TP}) = f_1 dx_1 + \cdots + f_\ell dx_\ell, \tag{3.5} $$

which is called the *twistor form* of $\mathcal{TP}^n_{f_1\cdots f_\ell}(c)$. The twistor form $\Phi(\mathcal{TP})$ is said to be *twisted closed* if we have

$$ \sum_{i,j=1}^{\ell} \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_i = 0. \tag{3.6} $$

When $\ell = 1$, the twistor form $\Phi(\mathcal{TP})$ is automatically twisted closed.

The following useful theorem was proved in [11].

Theorem 3.1. *Let $\mathcal{TP}^n_{f_1\cdots f_\ell}(c)$, $1 \le \ell \le n$, be a twisted product decomposition of a simply-connected real space form $M^n(c)$. If the twistor form $\Phi(\mathcal{TP})$ of $\mathcal{TP}^n_{f_1\cdots f_\ell}(c)$ is twisted closed, then up to rigid motions of $\tilde M^n(4c)$ there is a unique Lagrangian isometric immersion:*

$$ L_{f_1\cdots f_\ell} : \mathcal{TP}^n_{f_1\cdots f_\ell}(c) \to \tilde M^n(4c), \tag{3.7} $$

whose second fundamental form satisfies

$$ \begin{aligned} & h\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j}\right) = J\frac{\partial}{\partial x_j}, \;\; j = 1, \ldots, \ell, \\ & h\left(\frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_t}\right) = 0, \; \textit{otherwise}, \end{aligned} \tag{3.8} $$

for any adapted coordinate system $\{x_1, \ldots, x_n\}$ on $\mathcal{TP}^n_{f_1\cdots f_\ell}(c)$.

Conversely, if $L : M^n(c) \to \tilde M^n(4c)$ is a non-totally geodesic Lagrangian immersion of a real space form $M^n(c)$ of constant curvature c into a complex space *form $\tilde{M}^n(4c)$, then $M^n(c)$ admits an appropriate twisted product decomposition with twisted closed twistor form. Moreover, the Lagrangian immersion L is given by the corresponding adapted Lagrangian immersion of the twisted product.*

For an adapted immersion $L_{f_1\cdots f_\ell} : \mathcal{T}P^n_{f_1\cdots f_\ell}(c) \to \tilde{M}^n(4c)$, Dong and Han [16] computed the H-stationary condition $\delta\alpha_H = 0$ in terms of the twistor functions $f_1, \ldots, f_\ell$ and obtained the following.

Proposition 3.1. *Let $L_{f_1\cdots f_\ell} : \mathcal{T}P^n_{f_1\cdots f_\ell}(c) \to \tilde{M}^n(4c)$ be an adapted Lagrangian immersion given in Theorem 3.1. Then $L_{f_1\cdots f_\ell}$ is H-stationary if and only if the twistor functions $f_1, \ldots, f_\ell$ satisfy*

$$\sum_{j=1}^{\ell} \frac{1}{f_j^3}\frac{\partial f_j}{\partial x_j} = \sum_{1\leq i\neq j\leq \ell} \frac{1}{f_i f_j^2}\frac{\partial f_i}{\partial x_j}. \tag{3.9}$$

Corollary 3.1. [16]. *Any adapted Lagrangian immersion $L_{ff} : \mathcal{T}P^n_{ff}(c) \to \tilde{M}^n(4c)$ (with $k = 2$ and $f_1 = f_2 = f$) is H-stationary.*

From Proposition 3.1 we also have the following.

Corollary 3.2. *If the twistor functions $f_1, \ldots, f_\ell$ of $\mathcal{T}P^n_{f_1\cdots f_\ell}(c)$ are independent of the adapted coordinates $x_1, \ldots, x_\ell$, then the adapted Lagrangian immersion $L_{f_1\cdots f_\ell}\, \mathcal{T}P^n_{f_1\cdots f_\ell}(c) \to \tilde{M}^n(4c)$ is H-stationary.*

Corollary 3.3. *An adapted Lagrangian immersion $L_{f_1} : \mathcal{T}P^n_{f_1}(c) \to \tilde{M}^n(4c)$ is H-stationary if and only if the twistor function f_1 is independent of the adapted coordinate x_1.*

Remark 3.1. Let $\mathcal{T}P^2_{fk}(-1)$ be a twisted product decomposition of a simply-connected surface of constant curvature -1. Then the metric tensor of $\mathcal{T}P^2_{fk}(-1)$ takes the form:

$$g = f^2(y,z)dy^2 + k^2(y,z)dz^2, \tag{3.10}$$

where $f(y,z)$ and $k(y,z)$ are positive functions satisfying

$$\left(\frac{f_z}{k}\right)_z + \left(\frac{k_y}{f}\right)_y = fk. \tag{3.11}$$

From (3.11), we know that f, k cannot be both constant.

The twistor form Φ of $\mathcal{T}P^2_{fk}(-1)$ is given by

$$\Phi = f^2(y,z)dy + k^2(y,z)dz,$$

which is twisted closed if and only if we have

$$ff_z = kk_y.$$

It follows from Proposition 3.1 that the adapted Lagrangian immersion

$$L : \mathcal{T}P^2_{fk}(-1) \to CH^2(-4)$$

is H-stationary if and only if we have

$$k^3 f_y + f^3 k_z = f^2 k f_z + f k^2 k_y. \tag{3.13}$$

If the twistor functions f and k of $\mathcal{T}P^2_{fk}(-1)$ are equal and satisfy (3.12), then f can be chosen to be one of the following functions (see [11]):

$$f = a \sec\left(\frac{a}{\sqrt{2}}(x+y)\right), \text{ or } f = a \operatorname{csch}\left(\frac{a}{\sqrt{2}}(x+y)\right), \text{ or } f = \frac{\sqrt{2}}{x+y}, \tag{3.14}$$

with $a > 0$. Their corresponding adapted Lagrangian surfaces in $CH^2(-4)$ were determined in [11]. It follows from Corollary 3.1 that such Lagrangian surfaces are H-stationary automatically. We call these H-stationary Lagrangian surfaces *H-stationary Lagrangian surfaces of type I*.

Remark 3.2. If the twistor functions f and k of $\mathcal{T}P^2_{f^2k^2}(-1)$ are unequal and if they satisfy (3.11), (3.12) and (3.13), then the corresponding adapted Lagrangian immersion in $CH^2(-4)$ is also H-stationary. Such H-stationary Lagrangian surfaces are called *H-stationary Lagrangian surfaces of type II*.

Recently, Chen, Garay and Zhou has constructed in [13] five distinct families of H-stationary Lagrangian surfaces of type II in $CH^2(-4)$.

4. H-Stationary Lagrangian Submanifolds in $CH^3(-4)$

The following result completely classifies H-stationary Lagrangian submanifolds of constant curvature in $CH^3(-4)$ with positive relative nullity.

Theorem 4.1. *There exist ten families of Hamiltonian-stationary Lagrangian submanifolds of constant curvature in $CH^3(-4)$ with positive relative nullity:*

(1) *A totally geodesic Lagrangian submanifold $L : H^3(-1) \to CH^3(-4)$;*

(2) *A Lagrangian submanifold defined by*

$$L(s,y,z) = \tfrac{1}{2(1-y^2-z^2)}\Big((2i+s)e^{\frac{i}{2}s}\big(2by+\sqrt{1+b^2}(1+y^2+z^2)\big),$$
$$se^{\frac{i}{2}s}\big(2by+\sqrt{1+b^2}(1+y^2+z^2)\big), 4\sqrt{1+b^2}y+2b(1+y^2+z^2), 4z\Big),\ b \in \mathbf{R}.$$

(3) *A Lagrangian submanifold defined by*

$$L(s,y,z)=\frac{1}{1-y^2-z^2}\Bigg(\frac{e^{\frac{i}{2}s}(by+a(1+y^2+z^2))\{2\delta\cosh\delta s-i\sinh\delta s\}}{\delta\sqrt{4a^2-b^2}}$$
$$\frac{e^{\frac{i}{2}s}(by+a(1+y^2+z^2))\sinh\delta s}{\delta},\frac{4ay-b(1+y^2+z^2)}{\sqrt{4a^2-b^2}},2z\Bigg),$$

where a,b,δ *are real numbers satisfying* $4a^2-b^2>1$ *and* $2\delta=\sqrt{4a^2-b^2-1}$.

(4) *A Lagrangian submanifold defined by*

$$L(s,y,z)=\frac{1}{1-y^2-z^2}\Bigg(\frac{e^{\frac{i}{2}s}(by+a(1+y^2+z^2))\{2\gamma\cos\gamma s-i\sin\gamma s\}}{\gamma\sqrt{4a^2-b^2}}$$
$$\frac{e^{\frac{i}{2}s}(by+a(1+y^2+z^2))\sin\gamma s}{\gamma},\frac{4ay+b(1+y^2+z^2)}{\sqrt{4a^2-b^2}},2z\Bigg),$$

where a,b,γ *are real numbers satisfying* $4a^2<1+b^2$, $2\gamma=\sqrt{1+b^2-4a^2}$ *and* $4a^2\neq b^2$.

(5) *A Lagrangian submanifold defined by*

$$L(s,y,z)=\Bigg(\frac{2y-a^2(1+is)((1+y)^2+z^2)}{\sqrt{a^2-1}(1-y^2-z^2)},\frac{2z}{1-y^2-z^2},$$
$$\frac{1+y^2+z^2+ia^2s((1+y)^2+z^2)}{\sqrt{a^2-1}(1-y^2-z^2)},\frac{ae^{is}((1+y)^2+z^2)}{1-y^2-z^2}\Bigg),\quad a^2\neq 0,1.$$

(6) *A Lagrangian submanifold defined by*

$$L(s,y,z)=\Bigg(\frac{is}{2}+\frac{3}{2}-i+\frac{2i-3-is+(2i-2-is)y}{1-y^2-z^2},\frac{2z}{1-y^2-z^2},$$
$$\frac{is}{2}-\frac{1}{2}-i+\frac{1+2i-is+(2+2i-is)y}{1-y^2-z^2},\frac{e^{is}((1+y)^2+z^2)}{1-y^2-z^2}\Bigg).$$

(7) *A Lagrangian submanifold defined by*

$$L(x,s,t)=\frac{\cosh x}{\sqrt{1-2b}}\Bigg(\sqrt{2b}\tan s-i,\sqrt{2b}e^{is/\sqrt{2b}}\sec s\cos\left(\frac{\sqrt{1-2b}}{\sqrt{2b}}t\right),$$
$$\sqrt{2b}e^{is/\sqrt{2b}}\sec s\sin\left(\frac{\sqrt{1-2b}}{\sqrt{2b}}t\right),\sqrt{1-2b}\tanh x\Bigg),\quad 0<2b<1.$$

(8) *A Lagrangian submanifold defined by*

$$L(x,s,t)=\frac{\cosh x}{\sqrt{1-2b}}\left(\sqrt{2b}e^{is/\sqrt{2b}}\sec s\cosh\left(\frac{\sqrt{2b-1}}{\sqrt{2b}}t\right),\sqrt{2b}\tan s-i,\right.$$
$$\left.\sqrt{2b}e^{is/\sqrt{2b}}\sec s\sinh\left(\frac{\sqrt{2b-1}}{\sqrt{2b}}t\right),\sqrt{2b-1}\tanh x\right),\quad 2b>1.$$

(9) *A Lagrangian submanifold defined by*

$$L(x,s,t)=\frac{\cosh x}{\sqrt{2}(1+e^{2is})}\Big(i+2e^{2is}(s+i+it^2),i+2e^{2is}(s+it^2),$$
$$\sqrt{2}(1+e^{2is})\tanh x,2\sqrt{2}e^{2is}t\Big).$$

(10) *A Lagrangian submanifold defined by*

$$L(x,y,z)=\big(\tilde P(y,z)\cosh x,\sinh x\big),$$

where $\tilde P$ is a horizontal lift of a type II Hamiltonian-stationary Lagrangian surface $L:\mathcal{T}P^n_{f^2k^2}(-1)\to CH^2(-4)$ via the Hopf fibration $\pi: H^5_1(-1)\to CH^2(-4)$.

Conversely, locally every Hamiltonian-stationary Lagrangian submanifold of constant curvature in $CH^3(-4)$ with positive relative nullity is congruent to an open portion of a Lagrangian submanifold from one of the above tex families.

Proof. By a straight-forward long computation, we can verify that the each map defined by one of the above tex families gives rise to an H-stationary Lagrangian submanifold of constant curvature in $CH^3(-4)$ with positive relative nullity.

Conversely, let us assume that $L: M\to CH^3(-4)$ is an H-stationary Lagrangian isometric immersion with positive relative nullity from a Riemannian 3-manifold of constant curvature K into $CH^3(-4)$.

If the relative nullity is three everywhere, then M is totally geodesic, which gives case (1) of the theorem.

So, from now on, we assume that M is non-totally geodesic. It follows from the assumption of positive relative nullity that there exists a local unit vector field e_1 such that

$$h(e_1,X)=0,\quad \forall X\in TM. \tag{4.1}$$

Hence, by applying equation (2.3) of Gauss, we obtain $K=-1$. Thus, from (2.3) and (2.6), we find

$$[A_{JX},A_{JY}]=0$$

for $X, Y \in TM$. By using (2.6), (4.1), and $[A_{JX}, A_{JY}] = 0$, we know that at each point $p \in M$ there exist orthonormal vectors e_2, e_3 perpendicular to e_1 such that the second fundamental form takes the following form:

$$\begin{aligned} &h(e_1, e_1) = h(e_1, e_2) = h(e_1, e_3) = h(e_2, e_3) = 0, \\ &h(e_2, e_2) = \alpha J e_2, \quad h(e_3, e_3) = \varphi J e_3 \end{aligned} \tag{4.2}$$

for some functions α, φ. Since M is assumed to be non-totally geodesic, at least one of α, φ is nonzero.

Let $\omega^1, \omega^2, \omega^3$ be the dual 1-forms of e_1, e_2, e_3 and $(\omega_i^j), i, j = 1, 2, 3$, be the connection forms. Then, by applying (4.2) and Codazzi's equation, we have

$$\alpha\omega_2^1(e_1) = \alpha\omega_2^1(e_3) = \varphi\omega_3^1(e_1) = \varphi\omega_3^1(e_2) = \omega_3^2(e_1) = 0, \tag{4.3}$$

$$\alpha\omega_3^2(e_3) = \varphi\omega_2^3(e_2), \tag{4.4}$$

$$e_1\alpha = \alpha\omega_2^1(e_2),\ e_3\alpha = \alpha\omega_2^3(e_2),\ e_1\varphi = \varphi\omega_3^1(e_3),\ e_2\varphi = \varphi\omega_3^2(e_3). \tag{4.5}$$

It follows from (4.1) that the mean curvature vector satisfies

$$3H = \alpha J e_2 + \varphi J e_3.$$

So, the Maslov form, i.e., dual 1-form α_H of JH, is given by

$$\alpha_H = -\frac{1}{3}(\alpha\omega^2 + \varphi\omega^3). \tag{4.6}$$

After applying δ to (4.6) and using (4.3) and the structure equations, we see that the H-stationary condition (1.1) is equivalent to

$$e_2\alpha + e_3\varphi = \alpha\omega_3^2(e_3) + \varphi\omega_2^3(e_2). \tag{4.7}$$

Case (a). $\alpha = 0$. Because M is non-totally geodesic, we have $\varphi \neq 0$. It follows from (4.2)-(4.5) and (4.7) that

$$h(e_1, e_j) = h(e_2, e_2) = h(e_2, e_3) = 0,\ h(e_3, e_3) = \varphi J e_3;\ \ j = 1, 2, 3, \tag{4.8}$$

$$\omega_3^1(e_1) = \omega_3^1(e_2) = \omega_3^2(e_1) = \omega_3^2(e_2) = 0, \tag{4.9}$$

$$e_1\varphi = \varphi\omega_3^1(e_3), \quad e_2\varphi = \varphi\omega_3^2(e_3), \quad e_3\varphi = 0. \tag{4.10}$$

Consider the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ spanned by $\{e_1, e_2\}$ and $\{e_3\}$, respectively. Clearly, $\mathcal{D}^\perp$ is integrable, since it is of rank one. Also, it follows from (4.9) that the distribution $\mathcal{D}$ is integrable with totally geodesic leaves. Moreover, (4.8) implies that the leaves of $\mathcal{D}$ are totally geodesic in $CH^3(-4)$ as well.

Because $\mathcal{D}$ and $\mathcal{D}^\perp$ are both integrable, there exist local coordinates $\{s, y, z\}$ such that $\partial/\partial s$ spans $\mathcal{D}^\perp$ and $\{\partial/\partial y, \partial/\partial z\}$ spans $\mathcal{D}$ according to Frobenius' theorem. Since $d(\varphi\omega^3) = 0$, we may choose s in such way that $\partial/\partial s = \varphi^{-1}e_3$.

From $e_3\varphi = 0$, we have $\varphi = \varphi(y, z)$. With respect to $\{s, y, z\}$, (4.8) becomes

$$h\Big(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\Big) = J\frac{\partial}{\partial s},\ h\Big(\frac{\partial}{\partial s}, \frac{\partial}{\partial x_j}\Big) = h\Big(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big) = 0,\ j, k = 2, 3, \tag{4.11}$$

with $x_2 = y, x_3 = z$.

Let N be an integral submanifold of $\mathcal{D}$. Then N is a totally geodesic and totally real surface in $CH^3(-4)$. Thus, N is an open portion of a unit hyperbolic 2-plane $H^2(-1)$. Hence, M is isometric to an open portion of the warped product manifold $I \times H^2(-1)$ equipped with the warped product metric (see, for instance, [15]):

$$g = \phi^2 ds^2 + g_1,\ \ \phi = \frac{1}{\varphi(y, z)}, \tag{4.12}$$

where I is an open interval and g_1 can be chosen to be an isothermal metric on the hyperbolic plane $H^2(-1)$; namely,

$$g_1 = \frac{4(dy^2 + dz^2)}{(1 - y^2 - z^2)^2}. \tag{4.13}$$

From (4.12) we know that the Levi-Civita connection of M satisfies

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} = -\frac{1}{4}(1 - y^2 - z^2)^2\phi\Big\{\phi_y\frac{\partial}{\partial y} + \phi_z\frac{\partial}{\partial z}\Big\},\\
&\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial y} = \frac{\phi_y}{\phi}\frac{\partial}{\partial s},\ \ \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial z} = \frac{\phi_z}{\phi}\frac{\partial}{\partial s},\\
&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{2}{1 - y^2 - z^2}\left(y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}\right),\\
&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z} = \frac{2}{1 - y^2 - z^2}\left(z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right),\\
&\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z} = \frac{2}{1 - y^2 - z^2}\left(-y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right).
\end{aligned} \tag{4.14}$$

Since M is of constant curvature -1, by computing the curvature tensor R of M, we find

$$\begin{aligned}
&(1 - y^2 - z^2)\big\{(1 - y^2 - z^2)\phi_{yy} - 2y\phi_y + 2z\phi_z\big\} = 4\phi,\\
&(1 - y^2 - z^2)\big\{(1 - y^2 - z^2)\phi_{zz} + 2y\phi_y - 2z\phi_z\big\} = 4\phi,\\
&(1 - y^2 - z^2)\phi_{yz} = 2y\phi_z + 2z\phi_y.
\end{aligned} \tag{4.15}$$

After solving this PDE system (4.15), we obtain

$$\phi = \frac{a(1+y^2+z^2)+by+cz}{1-y^2-z^2}$$

for some $a, b, c \in \mathbf{R}$. Therefore, by applying a suitable rotation on the yz-plane, we may put

$$\phi = \frac{a(1+y^2+z^2)+by}{1-y^2-z^2}. \tag{4.16}$$

It follows (2.7), (4.11)-(4.14), and (4.16) that

$$\begin{aligned}
&L_{ss} = iL_s - \frac{\phi}{4}(1-y^2-z^2)^2(\phi_y L_y + \phi_z L_z) + \phi^2 L, \\
&L_{sy} = \frac{\phi_y}{\phi} L_s, \quad L_{sz} = \frac{\phi_z}{\phi} L_s, \\
&L_{yy} = \frac{2yL_y - 2zL_z}{1-y^2-z^2} + \frac{4L}{(1-y^2-z^2)^2}, \\
&L_{yz} = \frac{2zL_y + 2yL_z}{1-y^2-z^2}, \\
&L_{zz} = \frac{-2yL_y + 2zL_z}{1-y^2-z^2} + \frac{4L}{(1-y^2-z^2)^2}.
\end{aligned} \tag{4.17}$$

Case (a.i). $4a^2 \neq b^2$. In this case, after solving system (4.17), we obtain

$$L(s,y,z) = \phi H(s) - \frac{c_1 y + c_2 z + c_3(1+y^2+z^2)}{1-y^2-z^2} - \frac{4ac_3 - bc_1}{4a^2-b^2}\phi \tag{4.18}$$

for some vectors $c_1, c_2, c_3 \in \mathbf{C}^4_1$, where $H(s)$ is a $\mathbf{C}^4_1$-valued function satisfying

$$4H''(s) - 4iH'(s) - (4a^2-b^2)H(s) = 0. \tag{4.19}$$

Case (a.i.1). $4a^2 - b^2 = 1$. In this case, (4.16) reduces to

$$\phi = \frac{2by + \sqrt{1+b^2}(1+y^2+z^2)}{2(1-y^2-z^2)}. \tag{4.20}$$

By solving (4.19) and by applying (4.18) we find

$$\begin{aligned}
L(s,y,z) = \phi e^{\frac{i}{2}s}(c_4 + c_5 s) + \tfrac{c_1 y + c_2 z + c_3(1+y^2+z^2)}{1-y^2-z^2} \\
+(2\sqrt{1+b^2}c_3 - bc_1)\phi
\end{aligned} \tag{4.21}$$

for some vectors $c_1, \ldots, c_5 \in \mathbf{C}^4_1$. Hence, after choosing suitable initial conditions, we obtain case (2) of the theorem.

Case (a.i.2). $4a^2 - b^2 > 1$. After solving (4.19) we obtain from (4.18) that

$$\begin{aligned} L(s,y,z) = {} & \phi e^{\frac{i}{2}s}\left(c_4 \cosh\delta s + c_5 \sinh\delta s\right) - \frac{4ac_3 - bc_1}{4a^2 - b^2}\phi \\ & + \frac{c_1 y + c_2 z + c_3(1+y^2+z^2)}{1-y^2-z^2} \end{aligned} \tag{4.22}$$

for some vectors $c_4, c_5 \in \mathbf{C}^4_1$, where ϕ is given by (4.16) and δ is given by

$$2\delta = \sqrt{4a^2 - b^2 - 1}.$$

Hence, after choosing suitable initial conditions, we obtain case (3) of the theorem.

Case (a.i.3). $4a^2 - b^2 < 1$. After solving (4.19) we obtain from (4.18) that

$$\begin{aligned} L(s,y,z) = {} & \phi e^{\frac{i}{2}x}\left(c_4 \cos\gamma s + c_5 \sin\gamma s\right) + \tfrac{4ac_3 - bc_1}{4a^2 - b^2}\phi \\ & - \frac{c_1 y + c_2 z + c_3(1+y^2+z^2)}{1-y^2-z^2} \end{aligned} \tag{4.23}$$

for some vectors $c_4, c_5 \in \mathbf{C}^4_1$, where $\gamma = \frac{1}{2}\sqrt{1 + b^2 - 4a^2}$. Hence, after choosing suitable initial conditions, we obtain case (4).

Case (a.ii.) $4a^2 = b^2$. In this case, without loss of generality, we may put

$$\phi = \frac{2ay + a(1+y^2+z^2)}{1-y^2-z^2}. \tag{4.24}$$

Hence, (4.17) reduces to

$$\begin{aligned} L_{ss} &= iL_s - \frac{a^2((1+y)^2+z^2)}{2(1-y^2-z^2)} \\ &\quad \times\{((1+y)^2 - z^2)L_y + 2(1+y)zL_z\} + \phi^2 L, \end{aligned}$$

$$\begin{aligned} L_{sy} &= \frac{2((1+y)^2 - z^2)}{(1-y^2-z^2)((1+y)^2+z^2)} L_s, \\ L_{sz} &= \frac{4(1+y)z}{(1-y^2-z^2)((1+y)^2+z^2)} L_s, \\ L_{yy} &= \frac{2yL_y - 2zL_z}{1-y^2-z^2} + \frac{4L}{(1-y^2-z^2)^2}, \\ L_{yz} &= \frac{2zL_y + 2yL_z}{1-y^2-z^2}, \\ L_{zz} &= \frac{-2yL_y + 2zL_z}{1-y^2-z^2} + \frac{4L}{(1-y^2-z^2)^2}. \end{aligned} \tag{4.25}$$

Solving this system yields

$$L(s,y,z)=\frac{(ia^2c_1s-c_2e^{is}+c_3)((1+y)^2+z^2)}{1-y^2-z^2}+\frac{c_1(1+y^2+z^2)-c_4z}{1-y^2-z^2} \tag{4.26}$$

for some vectors $c_1, c_2, c_3, c_4 \in \mathbf{C}^4_1$.

Case (a.ii.1). $a^2 \neq 1$. After choosing suitable initial conditions we obtain case (5) of the theorem.

Case (a.ii.2). $a^2 = 1$. In this case, we obtain case (6) of the theorem.

Notice that (4.13) and Corollary 3.3 imply that every Lagrangian submanifold obtained from cases (2)-(6) of the theorem is H-stationary.

Case (b). $\varphi = \alpha \neq 0$. In this case, (4.2)-(4.5) and (4.7) imply that

$$\begin{aligned}&h(e_1,e_1)=h(e_1,e_2)=h(e_1,e_3)=h(e_2,e_3)=0,\\&h(e_2,e_2)=\alpha Je_2,\ h(e_3,e_3)=\alpha Je_3\end{aligned} \tag{4.27}$$

and

$$\omega_2^1(e_1)=\omega_3^1(e_1)=\omega_3^1(e_2)=\omega_2^1(e_3)=\omega_3^2(e_1)=0,\ \omega_3^2(e_3)=\omega_2^3(e_2), \tag{4.28}$$

$$e_1\alpha=\alpha\omega_2^1(e_2)=\alpha\omega_3^1(e_3),\ \ e_2\alpha=e_3\alpha=\alpha\omega_3^2(e_3). \tag{4.29}$$

From (4.28) and (4.29) we get

$$[e_1,\alpha^{-1}e_2]=[e_1,\alpha^{-1}e_3]=[\alpha^{-1}e_2,\alpha^{-1}e_3]=0.$$

Thus, there exists a coordinate system $\{x, y, z\}$ such that $e_1 = \partial/\partial x, e_2 = \alpha\partial/\partial y$ and $e_3 = \alpha\partial/\partial z$. So, the metric tensor is given by

$$g=dx^2+\frac{dy^2+dz^2}{\alpha^2}. \tag{4.30}$$

Thus, the Levi-Civita connection satisfies

$$\begin{aligned}&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=0,\ \ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=-\frac{\alpha_x}{\alpha}\frac{\partial}{\partial y},\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial z}=-\frac{\alpha_x}{\alpha}\frac{\partial}{\partial z},\\&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=\frac{\alpha_x}{\alpha^3}\frac{\partial}{\partial x}-\frac{\alpha_y}{\alpha}\frac{\partial}{\partial y}+\frac{\alpha_z}{\alpha}\frac{\partial}{\partial z},\\&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial z}=-\frac{\alpha_z}{\alpha}\frac{\partial}{\partial y}-\frac{\alpha_y}{\alpha}\frac{\partial}{\partial z},\\&\nabla_{\frac{\partial}{\partial z}}\frac{\partial}{\partial z}=\frac{\alpha_x}{\alpha^3}\frac{\partial}{\partial x}+\frac{\alpha_y}{\alpha}\frac{\partial}{\partial y}-\frac{\alpha_z}{\alpha}\frac{\partial}{\partial z}.\end{aligned} \tag{4.31}$$

From (2.7), (4.27), (4.30) and (4.31), we obtain

$$
\begin{aligned}
&L_{xx} = L,\\
&L_{xy} = -(\ln\alpha)_x L_y,\\
&L_{xz} = -(\ln\alpha)_x L_z,\\
&L_{yy} = \frac{\alpha_x}{\alpha^3}L_x + (i-(\ln\alpha)_y)L_y + \frac{\alpha_z}{\alpha}L_z + \frac{L}{\alpha^2},\\
&L_{yz} = -(\ln\alpha)_z L_y - (\ln\alpha)_y L_z,\\
&L_{zz} = \frac{\alpha_x}{\alpha^3}L_x + \frac{\alpha_y}{\alpha}L_y + (i-(\ln\alpha)_z)L_z + \frac{L}{\alpha^2}.
\end{aligned}
\tag{4.32}
$$

The compatibility condition of this system is given by

$$\alpha\alpha_{xx} - 2\alpha_x^2 + \alpha^2 = 0,\ \ \alpha_y = \alpha_z,\ \ \alpha_{xy} = \alpha_x\alpha_y, \tag{4.33}$$

$$2\alpha^3\alpha_{yy} + \alpha^2 = \alpha_x^2 + 2\alpha^2\alpha_y^2. \tag{4.34}$$

Solving the first two equations in (4.33) gives

$$\alpha = \frac{\operatorname{sech}(x+u(w))}{f(w)},\ \ w = y+z \tag{4.35}$$

for some functions $f(w), u(w)$. Substituting (4.35) into (4.34) gives

$$2ff'' - 2f'^2 - f^4 + (\tanh(x+u)u'' + \operatorname{sech}^2(x+u)u'^2)f^2 = 0,$$

which implies that $u' = 0$ and $2ff'' - 2f'^2 + f^4 = 0$. Thus, u is a constant. Hence, by applying a suitable translation in x, we obtain

$$\alpha = \frac{\operatorname{sech} x}{f(y+z)},\ \ \ 2ff'' - 2f'^2 - f^4 = 0. \tag{4.36}$$

After solving the second order differential equation in (4.36) and applying a suitable translation in y, z, we have

$$f = \sqrt{b}\sec\left(\sqrt{\tfrac{b}{2}}(y+z)\right),\ \ \alpha = \frac{1}{\sqrt{b}}\operatorname{sech} x\cos\left(\sqrt{\tfrac{b}{2}}(y+z)\right)$$

for some positive number b. Thus, if we put $x_2 = \sqrt{b/2}y$ and $x_3 = \sqrt{b/2}z$, then we obtain

$$f = \sqrt{2}\sec(x_2+x_3),\ \ \alpha = \frac{1}{\sqrt{2}}\operatorname{sech} x\cos(x_2+x_3). \tag{4.37}$$

Substituting this into (4.32) we obtain

$$
\begin{aligned}
&L_{xx}=L,\ \ L_{xx_2}=\tanh x L_{x_2},\ L_{xx_3}=\tanh x L_{x_3},\\
&L_{x_2x_2}=-\sinh 2x\sec^2(x_2+x_3)L_x+\left(\frac{i\sqrt{2}}{\sqrt{b}}+\tan(x_2+x_3)\right)L_{x_2}\\
&\qquad\qquad -\tan(x_2+x_3)L_{x_3}-2\cosh^2 x\sec^2(x_2+x_3)L,\\
&L_{x_2x_3}=\tan(x_2+x_3)(L_y+L_z),\\
&L_{x_3x_3}=-\sinh 2x\sec^2(x_2+x_3)L_x-\tan(x_2+x_3)L_{x_2}\\
&\qquad\qquad +\left(\frac{i\sqrt{2}}{\sqrt{b}}+\tan(x_2+x_3)\right)L_{x_3}+2\cosh^2 x\sec^2(x_2+x_3)L.
\end{aligned} \tag{4.38}
$$

To solving this system we make the following change of variables:

$$s=x_2+x_3,\ \ t=x_2-x_3.$$

Then we get from (4.38) that

$$
\begin{aligned}
&L_{xx}=L,\\
&L_{xs}=\tanh x L_s,\\
&L_{xt}=\tanh x L_t,\\
&L_{ss}=-\frac{1}{2}\sinh 2x\sec^2 sL_x+\left(\frac{i}{\sqrt{2b}}+\tan s\right)L_s+\cosh^2 x\sec^2 sL,\\
&L_{st}=\left(\frac{i}{\sqrt{2b}}+\tan s\right)L_t,\\
&L_{tt}=-\frac{1}{2}\sinh 2x\sec^2 sL_x+\left(\frac{i}{\sqrt{2b}}-\tan s\right)L_s+\cosh^2 x\sec^2 sL.
\end{aligned} \tag{4.39}
$$

Case (b.i). $2b<1$. After solving system (4.39) we find

$$
\begin{aligned}
L(x,s,t)=\ &c_1\sinh x+c_2(\sqrt{2b}\tan s-i)\cosh x\\
&+\left\{c_3\cos\left(\frac{\sqrt{1-2b}}{\sqrt{2b}}t\right)+c_4\sin\left(\frac{\sqrt{1-2b}}{\sqrt{2b}}t\right)\right\}e^{is/\sqrt{2b}}\sec\ s\cosh x.
\end{aligned}
$$

Hence, by choosing suitable initial conditions, we obtain case (7) of the theorem.

Case (b.ii). $2b>1$. In this case, the solution of system (4.39) is given by

$$
\begin{aligned}
L(x,s,t)=\ &c_1\sinh x+c_2(\sqrt{2b}\tan s-i)\cosh x\\
&+\left\{c_3\cosh\left(\frac{\sqrt{2b-1}}{\sqrt{2b}}t\right)+c_4\sinh\left(\frac{\sqrt{2b-1}}{\sqrt{2b}}t\right)\right\}e^{is/\sqrt{2b}}\sec\ s\cosh x.
\end{aligned}
$$

Hence, after choosing suitable initial conditions, we obtain case (8) of the theorem.

Case (b.iii). $2b = 1$. Solving system (4.39) yields

$$L(x, s, t) = c_1 \sinh x + \frac{c_2(i + 2e^{2is}(s + it^2)) + e^{2is}(c_3 t + 2c_4)}{1 + e^{2is}} \cosh x.$$

Hence, after choosing suitable initial conditions, we may obtain case (9) of the theorem.

Notice that the Lagrangian submanifolds given in cases (7), (8) and (9) of the theorem are H-stationary according to (4.30) and Corollary 3.1.

Case (c). $\varphi \neq \alpha$ *and* $\alpha \neq 0$. If $\varphi = 0$, this reduces to Case (a) after interchanging e_2 and e_3. Thus, without loss of generality, we may assume that $\varphi \neq 0$. Hence, (4.2)-(4.5) and (4.7) reduce to

$$\begin{aligned} &h(e_1, e_1) = h(e_1, e_2) = h(e_1, e_3) = h(e_2, e_3) = 0, \\ &h(e_2, e_2) = \alpha J e_2,\ h(e_3, e_3) = \varphi J e_3 \end{aligned} \tag{4.40}$$

$$\omega_2^1(e_1) = \omega_3^1(e_1) = \omega_3^1(e_2) = \omega_2^1(e_3) = \omega_3^2(e_1) = 0, \tag{4.41}$$

$$\alpha\omega_3^2(e_3) = \varphi\omega_2^3(e_2), \tag{4.42}$$

$$e_1\alpha = \alpha\omega_2^1(e_2),\ e_3\alpha = -\alpha\omega_3^2(e_2),\ e_1\varphi = \varphi\omega_3^1(e_3),\ e_2\varphi = \varphi\omega_3^2(e_3), \tag{4.43}$$

$$e_2\alpha + e_3\varphi = 2\alpha\omega_3^2(e_3). \tag{4.44}$$

From (4.41) and (4.43) we get

$$[e_1, \alpha^{-1}e_2] = [e_1, \varphi^{-1}e_3] = [\alpha^{-1}e_2, \varphi^{-1}e_3] = 0.$$

Thus, there exists a coordinate system $\{x, y, z\}$ such that $e_1 = \partial/\partial x, e_2 = \alpha\partial/\partial y$ and $e_3 = \varphi\partial/\partial z$. So, the metric tensor is given by

$$g = dx^2 + \frac{dy^2}{\alpha^2} + \frac{dz^2}{\varphi^2}. \tag{4.45}$$

The Levi-Civita connection of (4.45) satisfies

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0,\ \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = -\frac{\alpha_x}{\alpha}\frac{\partial}{\partial y},\ \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial z} = -\frac{\varphi_x}{\varphi}\frac{\partial}{\partial z}, \\ &\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{\alpha_x}{\alpha^3}\frac{\partial}{\partial x} - \frac{\alpha_y}{\alpha}\frac{\partial}{\partial y} + \frac{\alpha_z\varphi^2}{\alpha^3}\frac{\partial}{\partial z}, \\ &\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial z} = -\frac{\alpha_z}{\alpha}\frac{\partial}{\partial y} - \frac{\varphi_y}{\varphi}\frac{\partial}{\partial z}, \\ &\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \frac{\varphi_x}{\varphi^3}\frac{\partial}{\partial x} + \frac{\alpha^2\varphi_y}{\varphi^3}\frac{\partial}{\partial y} - \frac{\varphi_z}{\varphi}\frac{\partial}{\partial z}. \end{aligned} \tag{4.46}$$

From (2.7), (4.40), (4.45) and (4.46) we obtain

$$(4.47)\qquad \begin{aligned} &L_{xx}=L,\ L_{xy}=-(\ln\alpha)_xL_y,\ L_{xz}=-(\ln\varphi)_xL_z,\\ &L_{yy}=\frac{\alpha_x}{\alpha^3}L_x+(i-(\ln\alpha)_y)L_y+\frac{\alpha_z\varphi^2}{\alpha^3}L_z+\frac{L}{\alpha^2},\\ &L_{yz}=-(\ln\alpha)_zL_y-(\ln\varphi)_yL_z,\\ &L_{zz}=\frac{\varphi_x}{\varphi^3}L_x+\frac{\alpha^2\varphi_y}{\varphi^3}L_y+(i-(\ln\varphi)_z)L_z+\frac{L}{\varphi^2}. \end{aligned}$$

The compatibility condition of this system is given by

$$(4.48)\qquad \alpha\alpha_{xx}-2\alpha_x^2+\alpha^2=0,\ \ \varphi\varphi_{xx}-2\varphi_x^2+\varphi^2=0,$$

$$(4.49)\qquad \varphi(2\alpha_x\alpha_z-\alpha\alpha_{xz})=\alpha\alpha_z\varphi_x,$$

$$(4.50)\qquad \alpha(2\varphi_x\varphi_y-\varphi\varphi_{xy})=\varphi\varphi_y\alpha_x,$$

$$(4.51)\qquad \alpha^3\varphi_y=\varphi^3\alpha_z,$$

$$(4.52)\qquad \begin{aligned} &\alpha\varphi(\varphi^3\alpha_{zz}+\alpha^3\varphi_{yy}+\alpha^2\alpha_y\varphi_y+\varphi^2\varphi_z\alpha_z)+\alpha^2\varphi^2\\ &\quad=\alpha\alpha_x\varphi\varphi_x+2\varphi^4\alpha_z^2+2\alpha^4\varphi_y^2. \end{aligned}$$

Solving (4.48) gives

$$(4.53)\qquad \alpha=\frac{\operatorname{sech}(x+u(y,z))}{f(y,z)},\ \ \varphi=\frac{\operatorname{sech}(x+v(y,z))}{k(y,z)}$$

for some functions f,k,u,v with $f,k\neq 0$. Substituting (4.53) into (4.49)-(4.51) gives

$$(4.54)\qquad \sinh(u-v)f_z=-fu_z\cosh(u-v),$$

$$(4.55)\qquad \sinh(u-v)k_y=kv_y\cosh(u-v),$$

$$(4.56)\qquad kk_y=ff_z.$$

Case (c.i). u,v,f,k *are constants*. By applying a suitable translation in x, we may assume that

$$(4.57)\qquad \alpha=\frac{\operatorname{sech}x}{a},\ \ \varphi=\frac{\operatorname{sech}(x+c)}{b},\ \ a,b\neq 0.$$

Substituting (4.57) into (4.52) gives $\cosh c = 0$, which is impossible.

Case (c.ii). *At least one of* u, v, f, k *is non-constant.*
We divide this into three cases.

Case (c.ii.1). $u = v$. In this case, (4.54) and (4.55) imply $u_z = v_y = 0$. Thus, $u = v$ is constant. Hence, at least one of f, k is non-constant. Therefore, after applying a suitable translation in x, we may put

$$g = dx^2 + \cosh^2 x\{f^2(y,z)dy^2 + k^2(y,z)dz^2\},$$
$$\alpha = \frac{\operatorname{sech} x}{f(y,z)}, \quad \varphi = \frac{\operatorname{sech} x}{k(y,z)}. \tag{4.58}$$

Clearly, it follows from the assumption $\alpha \neq \varphi$ that $f \neq k$.
From (4.47) and (4.58) we obtain

$$\begin{aligned}
&L_{xx} = L,\\
&L_{xy} = \tanh x L_y,\\
&L_{xz} = \tanh x L_z,\\
&L_{yy} = -f^2 \sinh x \cosh x L_x + (i + (\ln f)_y)L_y - \frac{f f_z}{k^2} L_z + f^2 \cosh^2 x L,\\
&L_{yz} = (\ln f)_z L_y + (\ln k)_y L_z,\\
&L_{zz} = -k^2 \sinh x \cosh x L_x - \tfrac{k k_y}{f^2} L_y + (i + (\ln k)_z)L_z + k^2 \cosh^2 x L.
\end{aligned} \tag{4.59}$$

The compatibility condition of system (4.59) is given by

$$k k_y = f f_z, \tag{4.60}$$

$$f^2(k f_{zz} - f_z k_z) + k^2(f k_{yy} - f_y k_y) - f^3 k^3 = 0. \tag{4.61}$$

Moreover, since the Lagrangian submanifold is H-stationary, it follows from (4.58) and Proposition 3.1 that

$$k^3 f_y + f^3 k_z = f^2 k f_z + f k^2 k_y. \tag{4.62}$$

After solving the first three equations in (4.59) we have

$$L(x,y,z) = c_1 \sinh x + P(y,z)\cosh x \tag{4.63}$$

for some vector $c_1 \in \mathbf{C}^4_1$ and $\mathbf{C}^4_1$-valued function $P(y,z)$. Since $\langle L, L\rangle = -1$, we get

$$\langle c_1, c_1\rangle = -\langle P, P\rangle = 1, \quad \langle c_1, P\rangle = 0.$$

Thus, $P(y,z)$ lies in the unit anti-de Sitter space $H^7_1(-1) \subset \mathbf{C}^4_1$ and c_1 is a unit space-like vector satisfying $\langle c_1, P\rangle = 0$.
Moreover, it follows from (4.58), (4.63) and the Lagrangian condition that

(a) the induced metric of the surface $P(y,z)$ is given by

$$g_1 = f^2(y,z)^2 dy^2 + k^2(y,z)dz^2;$$

(b) c_1 is perpendicular to P_y, P_z, iP_y, iP_z; and
(c) $\langle P_y, iP_z\rangle = 0$.

Condition (b) implies that $\langle P, ic_1\rangle$ is constant, say b. Therefore, by choosing a suitable coordinate system on $\mathbf{C}^4_1$ with $c_1 = (0,0,0,1)$ and $P = (P_1, P_2, P_3, ib)$, we have

$$L(x,y,z) = \big(\tilde{P}(y,z)\cosh x, \sinh x + ib\cosh x\big) \tag{4.64}$$

with $\tilde{P}(y,z) = (P_1(y,z), P_2(y,z), P_3(y,z))$.

Now, by substituting (4.64) into the fourth equation in (4.59) we find $b = 0$. Hence, (4.64) reduces to

$$L(x,y,z) = \big(\tilde{P}(y,z)\cosh x, \sinh x\big). \tag{4.65}$$

It then follows from (4.58), (4.59) and (4.65) that $\tilde{P}(y,z)$ is a Legendrian surface in $H^5_1(-1) \subset \mathbf{C}^3_1$ whose metric tensor is also given by g_1. This Legendrian surface gives rise to a Lagrangian surface $\hat{M}$ in $CH^2(-4)$.

It follows from (4.58) and (4.65) that the second fundamental form $\hat{h}$ of this Lagrangian surface $\hat{M}$ in $CH^2(-4)$ satisfies

$$\hat{h}\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big) = J\frac{\partial}{\partial y},\ \ \hat{h}\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\Big) = 0,\ \ \hat{h}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big) = J\frac{\partial}{\partial z}.$$

Therefore, $\tilde{P}(y,z)$ gives rise to an H-stationary Lagrangian surface of type II in $CH^2(-4)$. Consequently, we obtain case (10) of the theorem.

Case (c.ii.2). *$u - v = c$ is a nonzero constant.* We have $u_y = v_y, u_z = v_z$ from $u - v = c$. So, it follows from (4.54) and (4.55) that

$$(\ln f)_z = -u_z \coth c,\ \ (\ln k)_y = u_y \coth c,\ \ f_z k u_y + k_y f u_z = 0. \tag{4.66}$$

Solving the first equation equations in (4.66) yields

$$f = \frac{e^{-bu}}{\phi(y)},\ \ k = \frac{e^{bu}}{\eta(z)},\ \ b = \coth c \tag{4.67}$$

for some positive function $\phi(y)$ and $\eta(z)$. Hence, the metric tensor in (4.53) becomes

$$g = dx^2 + \cosh^2(x+u)e^{2bu}\phi^2(y)dy^2 + \cosh^2(x+u-c)e^{-2bu}\eta^2(z)dz^2.$$

It is straight-forward to verify that the sectional curvature of the plane section spanned by $\partial/\partial y$ and $\partial/\partial z$ is not equal to -1, which is a contradiction. Hence, this case is impossible.

Case (c.ii.3). $u-v$ *is non-constant*. From (4.45), (4.47) and (4.53) we obtain

$$g = dx^2 + f^2(y,z)\cosh^2(x+u)dy^2 + k^2(y,z)\cosh^2(x+v)dz^2 \tag{4.68}$$

and

$$\begin{aligned}
&L_{xx} = L, \quad L_{xy} = \tanh(x+u)L_y, \quad L_{xz} = \tanh(x+v)L_z,\\
&L_{yy} = -\frac{f^2}{2}\sinh(2x+2u)L_x + \Big(i + \frac{f_y}{f} + u_y\tanh(x+u)\Big)L_y\\
&\qquad -\frac{f\cosh(x+u)}{k^2\cosh^2(x+v)}\{f\cosh(x+u)\}_z L_z + f^2\cosh^2(x+u)L,\\
&L_{yz} = \left(\frac{f_z}{f} + u_z\tanh(x+u)\right)L_y + \left(\frac{k_y}{k} + v_y\tanh(x+v)\right)L_z,\\
&L_{zz} = -\frac{k^2\sinh(2x+2v)}{2}L_x + \Big(i + \frac{k_z}{k} + v_z\tanh(x+v)\Big)L_z\\
&\qquad -\frac{k\cosh(x+v)}{f^2\cosh^2(x+u)}\{k\cosh(x+v)\}_y L_y + k^2\cosh^2(x+v)L.
\end{aligned} \tag{4.69}$$

From the compatibility condition of this system we have

$$f_z = -fu_z\coth(u-v), \tag{4.70}$$

$$k_y = kv_y\coth(u-v), \tag{4.71}$$

$$f^2u_z + k^2v_y = 0. \tag{4.72}$$

It follows from (4.70)-(4.72) that

$$ff_z = kk_y. \tag{4.73}$$

Since the Lagrangian submanifold is H-stationary, Proposition 3.1 and (4.68) imply that

$$\begin{aligned}
&f^3k_z\cosh^3(x+u)\cosh(x+v)\\
&+f^3k\cosh^3(x+u)(u_z\cosh(x+v)\coth(u-v) + v_z\sinh(x+v))\\
&-f^3ku_z\sinh(x+u)\cosh^2(x+u)\cosh(x+v)\\
&-fk^3v_y\cosh^2(x+u)\cosh^2(x+v)\,\mathrm{csch}\,(u-v)\\
&+k^3\cosh^3(x+v)\{f_y\cosh(x+u) + fu_y\sinh(x+u)\} = 0.
\end{aligned} \tag{4.74}$$

After replacing $\cosh(x+u), \sinh(x+u), \cosh(x+v), \sinh(x+v)$ in (4.74) using

$$\cosh(x+\gamma) = \cosh x \cosh\gamma - \sinh x \sinh\gamma,$$
$$\sinh(x+\gamma) = \sinh x \cosh\gamma + \cosh x \sinh\gamma,$$

and applying (4.70)-(4.72) and the following identities:

$$\begin{aligned}
\sinh^4 x &= \frac{3}{8} - \frac{\cosh 2x}{2} + \frac{\cosh 4x}{8},\\
\cosh^4 x &= \frac{3}{8} + \frac{\cosh 2x}{2} + \frac{\cosh 4x}{8},\\
\sinh^3 x \cosh x &= -\frac{\sinh 2x}{4} + \frac{\sinh 4x}{8},\\
\sinh x \cosh^3 x &= \frac{\sinh 2x}{4} + \frac{\sinh 4x}{8},\\
\sinh^2 x \cosh^2 x &= \frac{\cosh 4x - 1}{8},
\end{aligned}$$

we obtain from the coefficients of $\cosh 4x$ in (4.74) that

$$\begin{aligned}
& f^3\big\{(\sinh 4u - \sinh(2u+2v))k_z \\
& +k[4u_z \cosh(2(u+v)) + v_z(\cosh 4u - \cosh(2u+2v)]\big\} \\
& +k^3 f_y(\sinh(2u+2v) - \sinh 4v) - fk^3 u_y(\cosh 4v \\
& -\cosh(2u+2v)) = 0.
\end{aligned} \tag{4.75}$$

Similarly, from the coefficients of $\sinh 4x$ we get

$$\begin{aligned}
& f^3\big\{(\cosh 4u - \cosh(2u+2v))k_z \\
& +4ku_z \sinh(2u+2v) + kv_z(\sinh 4u - \sinh(2u+2v))\big\} \\
& +k^3 f_y(\cosh(2u+2v) - \cosh 4v) - fk^3 u_y(\sinh 4v \\
& -\sinh(2u+2v)) = 0.
\end{aligned} \tag{4.76}$$

From the coefficients of $\cosh 2x$ we get

$$\begin{aligned}
& f^3\big\{(2\sinh 2u + \sinh(4u-2v) - 3\sinh 2v)k_z \\
& +8ku_z(\cosh 2u + \cosh 2v) - kv_z\big(\cosh(4u-2v) \\
& -4\cosh 2u + 3\sinh 2v\big)\big\} \\
& +k^3 f_y(3\sinh 2u + \sinh(2u-2v) - 2\sinh 2v) \\
& +fk^3 u_y(3\cosh 2u + \cosh(2u-2v) - 4\cosh 2v) = 0.
\end{aligned} \tag{4.77}$$

From the coefficients of $\sinh 2x$ we get

$$
\begin{aligned}
& f^3\{(2\cosh 2u+\cosh(4u-2v)-3\cosh 2v)k_z \\
& +8ku_z(\sinh 2u+\sinh 2v)-kv_z\big(\sinh(4u-2v) \\
& -4\sinh 2u+3\cosh 2v)\big)\big\} \\
& +k^3f_y(3\cosh 2u-\cosh(2u-2v)-2\cosh 2v) \\
& +fk^3u_y(3\sinh 2u-\sinh(2u-2v)-4\sinh 2v)=0.
\end{aligned} \tag{4.78}
$$

Also, from the coefficients which do not involve $\cosh 4x, \sinh 4x, \cosh 2x$ or $\sinh 2x$, we find

$$
\begin{aligned}
& f^3\{3\sinh(2u-2v)k_z+4ku_z(2+\cosh(2u-2v))-6kv_z\sinh^2(u-v)\big\} \\
& +3k^3f_y\sinh(2u-2v)-2\cosh 2v)+6fk^3u_y\sinh(u-v)=0.
\end{aligned} \tag{4.79}
$$

Solving system (4.75)-(4.79) for f_y, u_y, k_z, u_z, v_z gives $f_y=u_y=k_z=u_z=v_z=0$. Combining these with conditions (4.70)-(4.73) shows that f, k, u, v are constant. This is a contradiction. Consequently, this case is impossible. ■

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