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ANNIHILATOR CONDITIONS ON NEARRING OF SKEW POLYNOMIALS OVER A RING

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Abstract. Let R be a ring with unity. For a ring endomorphism α and an α-derivation δ, the system $R[x;\alpha,\delta]$ forms an abelian nearring under addition and substitution operations. In this paper we extend the study of annihilator conditions on nearring of polynomials to skew nearring $(R[x;\alpha,\delta],\ +,\ \circ)$, when R is an α-rigid ring. Also, we give a characterization of α-rigid rings. An example to show that "α-rigid condition on R" is not superfluous is given.

1. Introduction

Throughout this paper all rings are associative and all nearrings are left nearrings. We use R and N to denote a ring and a nearring respectively. Recall from [12] that a ring R is *Baer* if R has a unity and the right annihilator of every nonempty subset of R is generated, as a right ideal, by an idempotent. Kaplansky [12] shows that the definition of a Baer ring is left-right symmetric. For example, the class of Baer rings includes all right Notherian PP rings and all von Neumann regular rings. In 1974, Armendariz obtained the following result [2, Theorem B]: Let R be a reduced ring. Then $R[x]$ is a Baer ring if and only if R is a Baer ring. Recall a ring or a nearring is said to be *reduced* if it has no nonzero nilpotent element. A generalization of Armendariz's result for several types of polynomial extensions over Baer rings, are obtained by various authors, [9-10]. According to Krempa [14], an endomorphism α of a ring R is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. Note that any rigid endomorphisms of a ring is a monomorphism and α-rigid rings are reduced, by Hong et al. [9]. Properties of α-rigid rings had been studied in Krempa [14], Hong et al. [9] and Matczuk [16]. In [9] Hong et al. studied Ore extensions of Baer rings over α-rigid rings. Birkenmeier and Huang in [4], had defined the *Baer-type annihilator conditions* in the class of nearrings as follows (for a nonempty $S \subseteq N$, let $r_N(S) = \{a \in N \mid Sa = 0\}$ and $\ell_N(S) = \{a \in N \mid aS = 0\}$):

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(1) $N \in \mathcal{B}_{r1}$ if $r_N(S) = eN$ for some idempotent $e \in N$;
(2) $N \in \mathcal{B}_{r2}$ if $r_N(S) = r_N(e)$ for some idempotent $e \in N$;
(3) $N \in \mathcal{B}_{\ell 1}$ if $\ell_N(S) = Ne$ for some idempotent $e \in N$;
(4) $N \in \mathcal{B}_{\ell 2}$ if $\ell_N(S) = \ell_N(e)$

for some idempotent $e \in N$.

If N is a ring with unity then $N \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$ is equivalent to N being a Baer ring. When S is a singleton, the *Rickart-type annihilator conditions* on nearrings are also defined similarly except replacing $\mathcal{B}$ by $\mathcal{R}$. In [3, p. 28], the $\mathcal{R}_{r2}$ condition is considered for rings with involution. In [5-6] Birkenmeier and Huang, studied Baer-type annihilator conditions in the class of nearrings. In particular they studied Baer-type conditions on the nearring of polynomials $R[x]$ (with the operations of addition and substitution) and formal power series by obtaining the following results: Let R be a reduced ring. (1) If R is Baer, then $R_0[x]$ (resp. $R_0[[x]]$) satisfies all the Baer-type conditions. (2) If $R_0[x]$ (resp. $R_0[[x]]$) satisfies any one of the Baer-type conditions, then R is Baer.

Let α be an endomorphism of R and δ is an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. Since $R[x;\alpha,\delta]$ is an abelian nearring under addition and substitution, it is natural to investigate the nearring of skew polynomials ($R[x;\alpha,\delta]$,+,$\circ$) when R is Baer. We use $R[x;\alpha,\delta]$ to denote the left nearring of skew polynomials ($R[x;\alpha,\delta]$,+,$\circ$) with coefficients from R and $R_0[x;\alpha,\delta] = \{f \in R[x;\alpha,\delta] \mid f \text{ has zero constant term}\}$ the 0-symmetric subnearring of $R[x;\alpha,\delta]$. Let $(x)f = a_0 + a_1x$ and $(x)g = b_0 + b_1x + b_2x^2 \in R[x;\alpha,\delta]$. Through a simple calculation, we have $(x)f \circ (x)g = ((x)f)g = b_0 + b_1((x)f) + b_2((x)f)^2 = (b_0 + b_1a_0 + b_2a_0^2 + b_2a_1\delta(a_0)) + (b_1a_1 + b_2a_0a_1 + b_2a_1\alpha(a_0) + b_2a_1\delta(a_1))x + b_2a_1\alpha(a_1)x^2$.

In this paper we show that R is α-rigid if and only if α is an injective endomorphism, R is reduced and if for polynomials $(x)f = a_0 + a_1x + \cdots + a_nx^n$, $(x)g = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha,\delta]$, $(x)f \circ (x)g = 0$ implies $b_ja_i = 0$ for each $1 \le i \le n, 1 \le j \le m$. Moreover if R is an α-rigid ring, then the nearring $R[x;\alpha,\delta]$ is reduced. Also, for an α-rigid ring R we show that: (1) If R is Baer, then $R_0[x;\alpha,\delta]$ satisfies all the Baer-type annihilator conditions. (2) If $R_0[x;\alpha,\delta]$ satisfies any one of the Baer-type annihilator conditions, then R is Baer. An example has presented to show that "α-rigid condition on R" is not superfluous.

Note that these results extend Hong et al.' results [9] on the skew polynomial rings over an α-rigid Baer ring to the Baer-type annihilator conditions in a nearring of skew polynomials.

3. Nearrings of Skew Polynomials

Definition 1.1. (Krempa [14]). Let α be an endomorphism of R. α is called a *rigid* endomorphism if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called to be *α-rigid* if there exists a rigid endomorphism α of R.

Clearly, any rigid endomorphism is a monomorphism. Note that α-rigid rings are reduced rings. In fact, if R is an α-rigid ring and $a^2 = 0$ for $a \in R$, then $a\alpha(a)\alpha(a\alpha(a)) = 0$. Thus $a\alpha(a) = 0$ and so $a = 0$. Therefore R is reduced.

Lemma 1.2. (Hong et al. [9]). *Let R be an α-rigid ring and $a, b \in R$. Then we have the following:*

(i) *If $ab = 0$ then $a\alpha^n(b) = \alpha^n(a)b = 0$ for each positive integer n.*

(ii) *If $a\alpha^k(b) = 0$ or $\alpha^k(a)b = 0$ for some positive integer k, then $ab = 0$.*

(iii) *If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for any positive integers m, n.*

(iv) *If $e^2 = e \in R$, then $\alpha(e) = e$ and $\delta(e) = 0$.*

A nearring N is said to have the *insertion of factors property* (IFP) if for all $a, b, n \in N$, $ab = 0$ implies $anb = 0$.

The following is a characterization of α-rigid rings:

Proposition 1.3. *Let δ be an α-derivation of a ring R. Then the following are equivalent:*

(1) *α is an injective endomorphism, R is reduced and if for each polynomials $(x)f = a_0 + a_1x + \cdots + a_nx^n$, $(x)g = b_0 + b_1x + \cdots + b_mx^m \in R[x; \alpha, \delta]$, $(x)f \circ (x)g = 0$ implies $b_ja_i = 0$ for each $1 \le i \le n, 1 \le j \le m$;*

(2) *α is an injective endomorphism, R is reduced and if for each polynomials $(x)f = a_1x + \cdots + a_nx^n$, $(x)g = b_1x + \cdots + b_mx^m \in R_0[x; \alpha, \delta]$, $(x)f \circ (x)g = 0$ implies $b_ja_i = 0$ for each $1 \le i \le n, 1 \le j \le m$.;*

(3) *R is α-rigid.*

Proof. (1)⇒(2) is clear.

(2)⇒(3) Let $a \in R$ with $a\alpha(a) = 0$. Then $\delta(a\alpha(a)) = \delta(a)\alpha(a) + \alpha(a)\delta(\alpha(a)) = 0$. Let $(x)f = \alpha(a)x$ and $g(x) = \delta(a)x + x^2 \in R_0[x; \alpha, \delta]$. Then $(x)f \circ (x)g = (\delta(a)\alpha(a) + \alpha(a)\delta(\alpha(a)))x + \alpha(a)\alpha^2(a)x^2 = 0$. Hence $\delta(a)\alpha(a) = \alpha(a) = 0$, by (2). Therefore $a = 0$, since α is an injective. Consequently R is α-rigid.

(3)⇒(1) Clearly, R is reduced and α is an injective endomorphism. Let $(x)f$, $(x)g \in R[x; \alpha, \delta]$ such that $(x)f \circ (x)g = 0$. We proceed by induction on deg(f) + deg(g). It is clear for deg(f) + deg(g) =2. Now suppose that our claim is true for each $(x)f, (x)g \in R[x; \alpha, \delta]$, with deg($f$), deg($g$) ≥ 1, deg(f) + deg(g) $< k$. Let $(x)f = a_0 + a_1x \cdots + a_nx^n, (x)g = b_0 + b_1x + \cdots + b_mx^m \in R[x; \alpha, \delta]$ such that $n, m \ge 1$ and $m + n = k$. Then $\sum_{j=0}^{m} b_j((x)f)^j = 0$ and that

$b_m a_n \alpha^n(a_n)\cdots\alpha^{(m-1)n}(a_n) = 0$. Hence $b_m a_n = a_n b_m = 0$, by Lemma 1.2(ii). Thus $\sum_{j=0}^{m-1} a_n b_j((x)f)^j = 0$ and that $(x)f\circ(a_n b_0 + a_n b_1 x + \cdots + a_n b_{m-1}x^{m-1}) = 0$. By induction hypothesis, we have $a_n b_j a_n = 0$ for $1 \leq j \leq m-1$. Hence $a_n b_j = 0$ for $1 \leq j \leq m$, since R is reduced. Therefore, as R satisfies IFP property and by using Lemma 1.2, $(x)f\circ(x)g = (a_0 + a_1 x \cdots + a_{n-1}x^{n-1})\circ(b_0 + b_1 x + \cdots + b_m x^m) = 0$. Our assertion then follows from induction hypothesis.

The following example shows that there exists a non α-rigid ring R such that if $(x)f = a_1 x + \cdots + a_n x^n$, $(x)g = b_1 x + \cdots + b_m x^m \in R_0[x;\alpha,\delta]$ with $(x)f\circ(x)g = 0$, then $b_j(a_i x^i)^j = 0$ for each $1 \leq i \leq n, 1 \leq j \leq m$.

Example 1.4.

Let F be a filed and $R = \left\{\begin{pmatrix} a & r \\ 0 & a \end{pmatrix} \mid a, r \in F\right\}$. Then R is a commutative ring. Let u be a non-zero element of F. Let $\alpha : R \to R$ be an automorphism defiend by $\alpha\left(\begin{pmatrix} a & r \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & ru \\ 0 & a \end{pmatrix}$.

(I) R is not α-rigid:
Since R is not reduced, hence it is not α-rigid.

(II) Let $(x)f = A_1 x + \cdots + A_n x^n$ and $(x)g = B_1 x + \cdots + B_m x^m \in R_0[x;\alpha]$, where $A_i = \begin{pmatrix} a_i & r_i \\ 0 & a_i \end{pmatrix}$ and $B_j = \begin{pmatrix} b_j & s_j \\ 0 & b_j \end{pmatrix}$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Assume that $(x)f\circ(x)g = 0$ such that $A_n \neq 0$, $B_m \neq 0$. We claim that $B_j(A_i x^i)^j = 0$ for each $1 \leq i \leq n$, $1 \leq j \leq m$. Since

$$(\dagger)\ \ 0 = (x)f\circ(x)g = B_1(A_1 x + \cdots + A_n x^n) + \cdots + B_m(A_1 x + \cdots + A_n x^n)^m$$

we have $B_m(A_n x^n)^m = 0$ and that $B_m A_n \alpha^n(A_n)\cdots\alpha^{n(m-1)}(A_n) = 0$. Hence $b_m a_n^m = 0$ and that $b_m = 0$ or $a_n = 0$.

(1) Suppose $b_m \neq 0$ and $a_n = 0$. Since $A_n \neq 0$ so $r_n \neq 0$. Multiplying A_n to Eq. ($\dagger$) from the left-hand side, we have $A_n B_1(A_1 x + \cdots + A_{n-1}x^{n-1}) + \cdots + A_n B_m(A_1 x + \cdots + A_{n-1}x^{n-1})^m = 0$, since $A_n\alpha^k(A_n) = 0$ for each $i \geq 0$ and R is commutative. Then $A_n B_m(A_{n-1}x^{n-1})^m = 0$. Thus $r_n b_m a_{n-1}^m = 0$ and that $a_{n-1} = 0$. Therefore $A_n B_1(A_1 x + \cdots + A_{n-2}x^{n-2}) + \cdots + A_n B_m(A_1 x + \cdots + A_{n-2}x^{n-2})^m = 0$, since $A_n\alpha^k(A_{n-1}) = 0$ for each $i \geq 0$ and R is commutative. Continuing this process, we have $a_1 = \cdots = a_n = 0$. Thus $B_j(A_i x^i)^j = 0$ for each $2 \leq j \leq m$ and $1 \leq i \leq n$, since $A_i\alpha^i(A_i) = 0$ for each $i \geq 1$. Hence $0 = (x)f\circ(x)g = B_1(A_1 x + \cdots + A_n x^n)$ and that $B_1(A_i x^i) = 0$ for each $1 \leq i \leq n$. Consequently in this case, $B_j(A_i x^i)^j = 0$ for $1 \leq i \leq n$, $1 \leq j \leq m$.

(2) Suppose that $b_m = 0$ and $a_n \neq 0$. Since $B_m A_n \alpha^n(A_n) \cdots \alpha^{n(m-1)}(A_n) = 0$, we have $s_m a_n^m = 0$ and that $s_m = 0$. Hence $B_m = 0$, a contradiction.

(3) Suppose that $b_m = 0$ and $a_n = 0$. We claim that $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_m = 0$. Assume, to the contrary, that there exists n_1, m_1 such that $a_{n_1+1} = \cdots = a_n = 0$, $b_{m_1+1} = \cdots = b_m = 0$, $a_{n_1} \neq 0$ and $b_{m_1} \neq 0$. Then $A_n B_1(A_1x + \cdots + A_{n_1}x^{n_1}) + \cdots + A_n B_{m_1}(A_1x + \cdots + A_{n_1}x^{n_1})^{m_1} = 0$. Thus $A_n B_{m_1}(A_{n_1}x^{n_1})^{m_1} = 0$ and that $r_n b_{m_1} a_{n_1}^{m_1} = 0$. Hence $r_n = 0$, a contradiction. If $a_1 = \cdots = a_n = 0$, then by using the case (i), $B_j(A_i x^i)^j = 0$ for each $1 \leq i \leq n$, $1 \leq j \leq m$. If $b_1 = \cdots = b_m = 0$, then $0 = (x)f \circ (x)g = B_1(A_1x + \cdots + A_{n-1}x^{n-1}) + \cdots + B_m(A_1x + \cdots + A_{n-1}x^{n-1})^m$ and that $B_m(A_{n-1}x^{n-1})^m = 0$. Hence $a_{n-1} = 0$. Continuing this process, we can prove $a_1 = \cdots = a_{n-1} = 0$. Hence $B_j(A_i x^i)^j = 0$ for each $1 \leq i \leq n$, $1 \leq j \leq m$.

Lemma 1.5. *Let δ be an α-derivation of ring R and $R[x;\alpha,\delta]$ the nearring of skew polynomials over R. Let R be an α-rigid ring. Then:*

(1) *If $(x)E \in R[x;\alpha,\delta]$ is an idempotent, then $(x)E = e_1x + e_0$, where e_1 is an idempotent in R with $e_1e_0 = 0$.*

(2) *$R[x;\alpha,\delta]$ is reduced.*

Proof.

(1) Let $(x)E = e_0 + \cdots + e_nx^n$ be an idempotent. Since $(x)E \circ (x)E = (x)E$, we have $(x)E \circ ((x)E - x) = 0$, and that $(e_0 + \cdots + e_nx^n) \circ ((e_0 + (e_1 - 1)x + \cdots + e_nx^n) = 0$. Then $e_i^2 = 0$ for all $i \geq 2$, by Proposition 1.3. Hence $e_i = 0$ for all $i \geq 2$, since R is reduced. Thus we have $e_0 + e_1(e_0 + e_1x) = e_0 + e_1x$ and that $e_1e_0 = 0$, $e_1^2 = e_1$.

(2) Let $(x)f = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$ such that $(x)f \circ (x)f = 0$. Then $a_i^2 = 0$ for each $1 \leq i \leq n$, by Proposition 1.3. Hence $a_i = 0$ for each $1 \leq i \leq n$, since R is reduced. Therefore $(x)f = 0$.

Proposition 1.6. *Let R be an α-rigid ring. If $R[x;\alpha,\delta] \in \mathcal{B}_{r2}$, then R is a Baer ring.*

Proof. Let S be a nonempty subset of R and $S_x = \{sx | s \in S\} \subseteq R[x;\alpha,\delta]$. Since $R[x;\alpha,\delta] \in \mathcal{B}_{r2}$ and R is α-rigid, there exists an idempotent $(x)E = e_1x + e_0 \in R[x;\alpha,\delta]$ such that $r(S_x) = r((x)E)$, by Lemma 1.5. We claim that $\ell_R(S) = \ell_R(e_1)$. Let $a \in \ell_R(e_1)$. Then $(e_1x + e_0) \circ (ax - ae_0) = a(e_1x + e_0) - ae_0 = 0$. Hence $ax - ae_0 \in r((x)E) = r(S_x)$. Therefore $sx \circ (ax - ae_0) = 0$ and so $as = ae_0 = 0$, for each $s \in S$. Hence $a \in \ell_R(S)$ and $\ell_R(e_1) \subseteq \ell_R(S)$. Now let

$a \in \ell_R(S)$. Then $sx \circ ax = asx = 0$. Thus $ax \in r(S_x) = r((x)E)$. Therefore $0 = (x)E \circ ax = a(e_1x + e_0)$ and thus $ae_1 = ae_0 = 0$. Hence $a \in \ell_R(e_1)$ and $\ell_R(S) \subseteq \ell_R(e_1)$. Therefore $\ell_R(S) = \ell_R(e_1)$ and $R \in \mathcal{B}_{\ell 2}$. By [4, Lemma 2.3], we have $R \in \mathcal{B}_{r2}$. From [4, Proposition 1.4(1)], R has a unity. Therefore R is a Baer ring.

Converse of Proposition 1.6 is not true in general. The following example [4, Example 3.5], shows that there exists a finite reduced commutative Baer ring R such that $R[x] \notin \mathcal{B}_{r2}$.

Example 1.7. Let $R = \mathbb{Z}_6$ and $S = \{2x+2, 2x+5\}$. From Lemma 1.5, all idempotents in $\mathbb{Z}_6[x]$ are $\{0, 1, 2, 3, 4, 5, x, 3x, 3x+2, 3x+4, 4x, 4x+3\}$. Note that $x - c \in r(c)$ and $x - c \notin r(S)$ for all constant idempotents $c \in \mathbb{Z}_6[x]$. Also, by Proposition 1.3, the possible idempotents $(x)E \in \mathbb{Z}_6[x]$ such that $r(S) = r((x)E)$ are either $4x$ or $4x+3$. Observe that $3x \in r(4x)$ but $3x \notin r(S)$, and also $3x^3 + 3 \in r(4x+3)$ but $3x^3 + 3 \notin r(S)$. Therefore, there is no idempotent $(x)E \in \mathbb{Z}_6[x]$ such that $r(S) = r((x)E)$. Consequently, $\mathbb{Z}_6[x] \notin \mathcal{B}_{r2}$.

In the following result we show a weak form of the $\mathcal{B}_{r2}$ condition when considering a converse to Proposition 1.6.

If $(x)f = \sum_{i=0}^{n} a_i x^i \in R[x;\alpha,\delta]$, let $S_f^* = \{a_1, a_2, \cdots, a_n\}$.

Theorem 1.8. *Let R be an α-rigid ring. If $R \in \mathcal{B}_{\ell 2} \cup \mathcal{B}_{r2}$, then $R[x;\alpha,\delta] \in \mathcal{R}_{r2}$.*

Proof. By [4, Lemma 2.3(3)] it is suffices to assume $R \in \mathcal{B}_{\ell 2}$. Let $(x)f = \sum_{i=0}^{m} a_i x^i \in R[x;\alpha,\delta]$. Then $\ell_R(S_f^*) = \ell_R(e_1)$ for some idempotent $e_1 \in R$, since $R \in \mathcal{B}_{\ell 2}$. Let $(x)E = e_1x + e_0$ where $e_0 = -e_1a_0 + a_0$. Clearly, $(x)E$ is an idempotent in $R[x;\alpha,\delta]$. We show that $r((x)f) = r((x)E)$. Let $(x)g = \sum_{j=0}^{n} b_j x^j \in r((x)f)$. Then $b_j \in \ell_R(S_f^*) = \ell_R(e_1)$, for all $1 \le j \le n$ and $b_0 + b_1a_0 + \cdots + b_na_0^n = 0$, by Proposition 1.3. By Lemma 1.2, $\alpha(e_1) = e_1$ and $\delta(e_1) = 0$, hence by a simple calculation one can show that $((x)E)^k = e_1x^k + e_0^k$. Thus $(x)E \circ (x)g = \sum_{j=0}^{n} b_j((x)E)^j = \sum_{j=1}^{n} b_j(e_1x^j + e_0^j) + b_0 = \sum_{j=1}^{n} b_je_1x^j + \sum_{j=1}^{n} b_je_0^j + b_0 = 0 + b_na_0^n + \cdots + b_1a_0 + b_0 = 0$. Therefore $(x)g \in r((x)E)$ and $r((x)f) \subseteq r((x)E)$. Now, let $(x)g = \sum_{j=0}^{n} b_j \in r((x)E)$. Then $b_j \in \ell_R(e_1) = \ell_R(S_f^*)$ for all $1 \le j \le n$ and $b_0 + b_1e_0 + \cdots + b_ne_0^n = 0$, by Proposition 1.3. This implies $b_0 + b_1a_0 + \cdots + b_na_0^n = 0$ and thus $(x)g \in r((x)f)$, since $e_0^t = -e_1a_0^t + a_0^t$ for all $t \ge 1$. Therefore $r((x)f) = r((x)E)$. Consequently, $R[x;\alpha,\delta] \in \mathcal{R}_{r2}$.

We now turn to the problem of extending Baer-type annihilator conditions from R to $R_0[x;\alpha,\delta]$.

Proposition 1.9. *Let R be an α-rigid ring. Then:*

(1) $R \in \mathcal{B}_{r1}$ *if and only if* $R_0[x;\alpha,\delta] \in \mathcal{B}_{\ell 1}$.

(2) $R \in \mathcal{B}_{r2}$ *if and only if* $R_0[x;\alpha,\delta] \in \mathcal{B}_{\ell 2}$.

Proof.

(1) Assume $R \in \mathcal{B}_{r1}$. Let S be a nonempty subset of $R_0[x;\alpha,\delta]$. Then $T = \cup_{f\in S} S_f^*$ is a nonempty subset of R. Hence $r_R(T) = eR$ for some idempotent $e \in R$, since $R \in \mathcal{B}_{r1}$. We show that $\ell(S) = R_0[x;\alpha,\delta] \circ (ex) = e \cdot R_0[x;\alpha,\delta]$. Let $(x)f = \sum_{i=1}^m a_i x^i \in S$. Since $\alpha(e) = e$ and $\delta(e) = 0$, we have $(ex) \circ (x)f = \sum_{i=1}^m a_i (ex)^i = \sum_{i=1}^m a_i e x^i = 0$. Thus $ex \in \ell(S)$ and hence $e \cdot R_0[x;\alpha,\delta] \subseteq \ell(S)$. Now, let $(x)h = \sum_{k=1}^n c_k x^k \in \ell(S)$. Then $c_k \in r_R(T)$ for all $1 \le k \le n$, by Proposition 1.3. Therefore $c_k = ec_k$ for all $1 \le k \le n$. Hence $(x)h = e\sum_{k=1}^n c_k x^k \in eR_0[x;\alpha,\delta]$ and so $\ell(S) = R_0[x;\alpha,\delta] \circ (ex)$. Thus $R_0[x;\alpha,\delta] \in \mathcal{B}_{\ell 1}$.

Now, assume $R_0[x;\alpha,\delta] \in \mathcal{B}_{\ell 1}$. Let S be a nonempty subset of R, and define $S_x = \{sx | s \in S\}$ a subset of $R_0[x;\alpha,\delta]$. Then $\ell(S_x) = R_0[x;\alpha,\delta] \circ (ex)$ for some idempotent $e \in R$, by Lemma 1.5. For each $sx \in S_x$, $0 = (ex) \circ (sx) = sex$. Therefore $e \in r_R(S)$. Now, let $a \in r_R(S)$. Then $(ax) \circ (sx) = sax = 0$ for each $sx \in S_x$. Thus $ax \in \ell(S_x) = R_0[x;\alpha,\delta] \circ (ex) = e \cdot R_0[x;\alpha,\delta]$. Hence $a = ea \in eR$. Thus $r_R(S) = eR$. Therefore $R \in \mathcal{B}_{r1}$.

(2) Assume $R \in \mathcal{B}_{r2}$. Let S be a nonempty subset of $R_0[x;\alpha,\delta]$. By a similar construction to that used in (1), we have $r_R(T) = r_R(e)$ for some idempotent $e \in R$. We claim $\ell(S) = \ell(ex)$. Let $(x)g = \sum_{j=1}^n b_j x^j \in \ell(ex)$. Then $0 = (x)g \circ ex = e \cdot (x)g$. Hence $eb_j = 0$ for all $1 \le j \le n$. Consequently, $b_j \in r_R(e) = r_R(T)$, for all $1 \le j \le n$. Let $(x)f = \sum_{i=1}^m a_i x^i \in S$. Then by using Lemma 1.2, $(x)g \circ (x)f = \sum_{i=1}^m a_i (\sum_{j=1}^n b_j x^j)^i = 0$. Therefore $\ell(ex) \subseteq \ell(S)$. Now, let $(x)g = \sum_{j=1}^n b_j x^j \in \ell(S)$. Then $b_j \in r_R(T) = r_R(e)$ for all $1 \le j \le n$, by Proposition 1.3. Thus $(x)g \circ (ex) = e \cdot (x)g = 0$. Therefore $\ell(S) = \ell(ex)$ and so $R_0[x;\alpha,\delta] \in \mathcal{B}_{\ell 2}$.

Assume $R_0[x;\alpha,\delta] \in \mathcal{B}_{\ell 2}$. Let S be a nonempty subset of R and let $S_x = \{sx | s \in S\}$. Then $\ell(S_x) = \ell((x)E)$ for some idempotent $(x)E = ex \in R_0[x;\alpha,\delta]$, by Lemma 1.5. We show that $r_R(S) = r_R(e)$. Let $a \in r_R(S)$. Then $ax \circ sx = sax = 0$ for all $sx \in S_x$. Hence $ax \in \ell(S_x) = \ell((x)E)$. Thus $ax \circ ex = eax = 0$ and that $a \in r_R(e)$. Therefore $r_R(S) \subseteq r_R(e)$. Now, let $b \in r_R(e)$. Then $bx \circ ex = ebx = 0$ and that $bx \in \ell(S_x)$. Thus $bx \circ sx = sbx = 0$ for all $s \in S$. Hence $b \in r_R(S)$. Therefore $R \in \mathcal{B}_{r2}$.

The following example shows that there exists a Baer ring R but $R_0[x;\alpha] \notin \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$. So "$\alpha$-rigid condition on R" in Proposition 1.9 is not superfluous.

Example 1.10. Let F be a filed and consider the polynomial ring $R =$

$(F[y], +, \cdot)$ over F. Then R is a commutative domain and so R is Baer. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha(f(y)) = f(0)$. Then

(I) R is not α-rigid;
Since $y\alpha(y) = 0$ but $y \neq 0$.

(II) $R_0[x; \alpha] \notin \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$;
First we show that the only idempotents of nearing $R_0[x; \alpha]$ are 0 and x. Let $(x)e = f_1(y)x + \cdots + f_n(y)x^n$ be a nonzero idempotent of $R_0[x; \alpha]$. Then $(x)e \circ (x)e = (x)e$ and that $f_1(y)(f_1(y)x + \cdots + f_n(y)x^n) + \cdots + f_n(y)(f_1(y)x + \cdots + f_n(y)x^n)^n = f_1(y)x + \cdots + f_n(y)x^n$. Then $f_1(y)^2 = f_1(y)$ and that $f_1(y) = 0$ or $f_1(y) = 1$, since R is domain. If $f_1(y) = 0$, then by a simple calculation we can show that $(x)e = 0$, which is a contradiction. Hence $f_1(y) = 1$. Since $f_1(y)f_2(y) + f_2(y)f_1(y)\alpha(f_1(y)) = f_2(y)$ and $\alpha(f_1(y)) = 1$, so $f_2(y) = 0$. Continuing this process, we have $(x)e = 1$. Now we show that $R_0[x; \alpha] \notin \mathcal{B}_{\ell 1}$. Let $S = \{x^2\}$. Since $yx \circ x^2 = 0$ so $\ell_{R_0[x;\alpha]}(S) \neq 0 = R_0[x; \alpha] \circ 0$. Since $x \circ x^2 = x^2$ so $\ell_{R_0[x;\alpha]}(S) \neq R_0[x; \alpha] = R_0[x; \alpha] \circ x$. Therefore $R_0[x; \alpha] \notin \mathcal{B}_{\ell 1}$. By a similar argument one can show that $R_0[x; \alpha] \notin \mathcal{B}_{\ell 2}$.

Theorem 1.11. *Let R be an α-rigid ring. Then:*

(1) *If R is Baer, then $R_0[x; \alpha, \delta] \in \mathcal{B}_{r1} \cap \mathcal{B}_{r2} \cap \mathcal{B}_{\ell 1} \cap \mathcal{B}_{\ell 2}$.*

(2) *If $R_0[x; \alpha, \delta] \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$, then R is Baer.*

Proof. Assume that R is Baer. From Proposition 1.3, Lemmas 2.2 and 2.3 in [4], we need only show that $R_0[x; \alpha, \delta] \in \mathcal{B}_{\ell 1}$. Let S be a nonempty subset of $R_0[x; \alpha, \delta]$ and let $T = \cup_{f \in S} S_f^*$. Then $r_R(T) = r_R(e)$ for some idempotent $e \in R$, since R is Baer. We show that $\ell(S) = (1 - e)R_0[x; \alpha, \delta] = R_0[x; \alpha, \delta] \circ (1 - e)x$. Let $(x)f = \sum_{i=1}^m a_i x^i \in S$ and $(x)g = \sum_{j=1}^n b_j x^j \in \ell(S)$. Then $a_i b_j = 0$ for all $1 \leq i \leq m$, $1 \leq j \leq n$, by Proposition 1.3. Thus $eb_j = 0$ and that $b_j = (1 - e)b_j$ for all $1 \leq j \leq n$. Hence $(x)g = (1 - e)\sum_{j=1}^n b_j x^j \in (1 - e)R_0[x; \alpha, \delta]$ and that $\ell(S) \subseteq (1 - e)R_0[x; \alpha, \delta]$. Now, let $(x)g = (1 - e)\sum_{j=1}^n b_j x^j \in (1 - e)R_0[x; \alpha, \delta]$. Since $(1 - e)$ is an idempotent of R and R is α-rigid, so $\alpha(1 - e) = (1 - e)$, $\delta(1 - e) = 0$ and $(1 - e)$ is a central element of R. Hence for all $(x)f = \sum_{i=1}^m a_i x^i \in S$, we have $(x)g \circ (x)f = (\sum_{j=1}^n b_j x^j) \circ (\sum_{i=1}^m a_i(1 - e)x^i) = 0$. Thus $\ell(S) = (1 - e)R_0[x; \alpha, \delta]$. Therefore $R_0[x; \alpha, \delta] \in \mathcal{B}_{\ell 1}$.

Assume $R_0[x; \alpha, \delta] \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$. By Lemmas 2.2 and 2.3 in [4], $R_0[x; \alpha, \delta] \in \mathcal{B}_{\ell 2}$. From Proposition 1.9(2), $R \in \mathcal{B}_{r2}$. Proposition 1.4(1) in [5] and Proposition 1.3 yield that R has a unity. Therefore R is a Baer ring.

Example 1.10 also shows that "α-rigid condition on R" in Theorem 1.11 is not superfluous.

Corollary 1.12. *Let R be an α-rigid ring. Then the following are equivalent:*

(1) *R is Baer;*

(2) *$(R[x;\alpha,\delta],+,\cdot)$ is Baer;*

(3) *$(R_0[x;\alpha,\delta],+,\circ) \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$.*

Proof. This follows from [9, Theorem 11] and Theorem 1.11.

Proposition 1.13. *Let R be an α-rigid ring. Let S be the subnearring of $R[x;\alpha,\delta]$ generated by the set $\{ex|e^2 = e \in R\}$, and T a subnearring of $R[x;\alpha,\delta]$. If $S \subseteq T$ and $R[x;\alpha,\delta] \in \mathcal{B}_{ij}$, where $i \in \{r,\ell\}$ and $j \in \{1,2\}$, then $T \in \mathcal{B}_{ij}$.*

Proof. This follows from Proposition 1.5 in [4] and Lemma 1.5.

Example 1.14. Using Proposition 1.9, the following nearrings satisfy all the Baer-type annihilator conditions discussed in this paper when R is α-rigid Baer ring: (i) $\{ax \,|\, a \in R\}$; (ii) $\{(x)f = \sum_{i=1}^{n} a_{2i-1}x^{2i-1} \in R_0[x;\alpha] \,|\, a_{2i-1} \in R, n \in \mathbb{N}\}$; (iii) $E_0[x;\alpha,\delta]$, where E is a subring containig all idempotents of R.

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References

1. D. D. Anderson and V. Camillo, Armendariz rings and Gaussian rings. *Comm. Algebra*, **26(7)** (1998), 2265-2275.
2. E. P. Armendariz, A note on extensions of Baer and p.p.-rings, *J. Austral. Math. Soc.*, **18** (1974), 470-473.
3. S. K. Berberian, *Baer $*$-rings*, Springer-Verlag, Berlin, 1968.
4. G. F. Birkenmeier and F. K. Huang, Annihilator conditions on polynomials, *Comm. Algebra*, **29(5)** (2001), 2097-2112.
5. G. F. Birkenmeier and F. K. Huang, Annihilator conditions on formal power series, *Algebra Colloq.*, **9(1)** (2002), 29-37.
6. E. Hashemi and A. Moussavi, Polynomial extensions of quasi-Baer rings, *Acta Math. Hungar.*, **107(3)** (2005), 207-224.
7. E. Hashemi and A. Moussavi, Skew power series extensions of α-rigid p.p.-rings, *Bull. Korean Math. Soc.*, **41(4)** (2004), 657-665.
8. E. Hashemi and A. Moussavi, On (α,δ)-skew Armendariz Rings, *J. Korean Math. Soc.*, **42(2)** (2005), 353-363.

9. C. Y. Hong, Nam Kyun Kim, Tai Keun Kwak, Ore extensions of Baer and p.p.-rings, *J. Pure Appl. Algebra*, **151** (2000), 215-226.

10. C. Y. Hong, Nam Kyun Kim, Tai Keun Kwak, On skew Armendariz rings, *Comm. Algebra*, **31(1)** (2003), 103-122.

11. Chan Huh et al. Armendariz rings and semicommutative rings, *Comm. Algebra*, **30(2)** (2002), 751-761.

12. I. Kaplansky, *Rings of Operators*, Benjamin, New York, 1965.

13. N.K. Kim and Y. Lee, Armendariz rings and reduced rings, *J. Algebra*, **223** (2000), 477-488.

14. J. Krempa, Some examples of reduced rings, *Algebra Colloq.*, **3(4)** (1996), 289-300.

15. Tsiu-Kwen Lee and Tsai-Lien Wong, On Armendariz rings, *Houston J. Math.*, **29(3)** (2003), 583-593.

16. J. Matczuk, A characterization of α-rigid rings, *Comm. Algebra 32(1) (2004) 4333-4337.*

17. M. B. Rege and S. Chhawchharia, Armendariz rings, *Proc. Japan Acad. Ser. A Math. Sci.*, **73** (1997), 14-17.

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