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APOLLONIUS CIRCLES OF THE TRIANGLE IN AN ISOTROPIC PLANE

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Abstract. The concept of Apollonius circle and Apollonius axes of an allowable triangle in an isotropic plane will be introduced. Some statements about relationships between introduced concepts and some other previously studied geometric concepts about triangle will be investigated in an isotropic plane and some analogies with the Euclidean case will be also considered.

1. Introduction

Each allowable triangle in an isotropic plane can be set, by a suitable choice of coordinates, into the so called *standard position*, i.e. that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ where $a + b + c = 0$. With the labels $p = abc$, $q = bc + ca + ab$ it can be shown that the equalities $q = bc - a^2$, $(c - a)(a - b) = 2q - 3bc$, $(a - b)(b - c) = 2q - 3ca$, $(b - c)(c - a) = 2q - 3ab$, $(b - c)^2 = -(q + 3bc)$ are valid.

The standard triangle ABC has according to [5] the centroid $G = (0, -\frac{2q}{3})$, orthic axis with the equation $y = -\frac{q}{3}$, the side $\overline{BC}$ has the midpoint A_m with the abscissa $-\frac{a}{2}$ and the equation $y = -ax - bc$, and owing to [4] it has the symmedian AK with the equation

$$y = -\frac{2q}{3a}x + bc - \frac{q}{3}, \tag{1}$$

then the Brocard's diameter with the equation $x = \frac{3p}{2q}$ and the Lemoine line with the equation

$$y = \frac{3p}{q}x + \frac{q}{3}. \tag{2}$$

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Each circle in the isotropic plane has the equation of the form $2\rho y = x^2 + ux + w$, where ρ is its radius (see [6]). With $\rho \neq 0$ it is a "proper" circle, and with $\rho = 0$ it is "unproper" circle, which in case $u^2 > 4w$ falls into two isotropic lines. If these two lines have the equations $x = x_1$ and $x = x_2$ then $(x - x_1)(x - x_2) = 0$ is the equation of this circle. Any point with the abscissa $x_0 = \frac{1}{2}(x_1 + x_2)$ has the property that, by implementing the sign, it is equally distant from all the points of the circle. Therefore the isotropic line with the equation $x = x_0$ is called *the axis*, and the distance $\frac{1}{2}(x_2 - x_1)$ is *the radius* of this "unproper" circle.

In the Euclidean geometry Apollonius circle $\mathcal{A}_a$ of the triangle ABC is the set of points T such that $|TB| : |TC| = |AB| : |AC|$. It goes through the point A and through the intersections of the line BC with the bisectors of the angle A, and its center is at the intersection of the line BC with the Lemoine line of the triangle ABC.

We will define the Apollonius circle $\mathcal{A}_a$ of an allowable triangle ABC in the isotropic plane as the set of points T such that $TB : TC = AB : AC$ or $TB : TC = -AB : AC$. The first set of points is the isotropic line $\mathcal{A}$ through the point A, and the second set of points is the isotropic line $\mathcal{D}$ with the equation $x = d$, where in case of the standard triangle ABC we have the equation

$$(b - d) : (c - d) = -(b - a) : (c - a), \tag{3}$$

i.e. $(d - b)(a - c) + (d - c)(a - b) = 0$, so therefore we get

$$d = \frac{b(a - c) + c(a - b)}{2a - b - c} = \frac{q - 3bc}{3a} = -\frac{2q + 3a^2}{3a} = -a - \frac{2q}{3a}.$$

With $x = d$ from the equation $y = -ax - bc$ of the line BC we get

$$y = -a \cdot \frac{q - 3bc}{3a} - bc = -\frac{q}{3},$$

and the achieved point $BC \cap \mathcal{D}$ lies on the orthic axis of the triangle ABC. That triangle has two more analogous Apollonius circles $\mathcal{A}_b$ and $\mathcal{A}_c$. We have proved:

Theorem 1. *Apollonius circles $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{A}_c$ of an allowable triangle ABC consist of two by two isotropic lines $\mathcal{A}$, $\mathcal{D}$; $\mathcal{B}$, $\mathcal{E}$; $\mathcal{C}$, $\mathcal{F}$, while the lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ pass through the points A, B, C, and the lines $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ pass through the intersection of the lines BC, CA, AB with the orthic axis $\mathcal{H}$ of the triangle ABC. In case of the standard triangle ABC the lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ have the equations $x = a$, $x = b$, $x = c$, $x = d$, $x = e$, $x = f$, where*

$$d = -a - \frac{2q}{3a} = \frac{q - 3bc}{3a}, \quad e = -b - \frac{2q}{3b} = \frac{q - 3ca}{3b}, \quad f = -c - \frac{2q}{3c} = \frac{q - 3ab}{3c}. \tag{4}$$

The equality (3) means that the statement $H(\mathcal{AD},\mathcal{BC})$ of harmonic relation of the pairs of lines $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{B}$, $\mathcal{C}$ is valid. We will also prove the statement $H(\mathcal{AD},\mathcal{EF})$, i.e. the equality

$$(a-e)(d-f)+(a-f)(d-e)=0. \tag{5}$$

Because of (4) we get

$$a-e=a+b+\frac{2q}{3b}=\frac{2q}{3b}-c=\frac{2q-3bc}{3b},\quad a-f=\frac{2q-3bc}{3c},$$

$$d-e=b-a+\frac{2q}{3ab}(a-b)=\frac{a-b}{3ab}(2q-3ab)=\frac{a-b}{3ab}(b-c)(c-a),$$

$$d-f=\frac{a-c}{3ac}(2q-3ac)=-\frac{c-a}{3ac}(a-b)(b-c),$$

so the equality (5) is obviously valid. Therefore it is valid:

Theorem 2. *For the lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ from Theorem 1 the following statements*

$$H(\mathcal{AD},\mathcal{BC}),\quad H(\mathcal{BE},\mathcal{CA}),\quad H(\mathcal{CF},\mathcal{AB}),\quad H(\mathcal{AD},\mathcal{EF}),\quad H(\mathcal{BE},\mathcal{FD}),\quad H(\mathcal{CF},\mathcal{DE})$$

are valid.

The circle $\mathcal{A}_a$ has the equation $(x-a)(x-d)=0$, i.e. $x^2-(a+d)x+ad=0$. Owing to (4) we get

$$a+d=-\frac{2q}{3a},\quad ad=\frac{1}{3}(q-3bc),$$

so that equation is the first out of three analogous equations

$$\begin{aligned}\mathcal{A}_a\ \ldots\ x^2+\frac{2q}{3a}x+\frac{1}{3}(q-3bc)=0,\\ \mathcal{A}_b\ \ldots\ x^2+\frac{2q}{3b}x+\frac{1}{3}(q-3ca)=0,\\ \mathcal{A}_c\ \ldots\ x^2+\frac{2q}{3c}x+\frac{1}{3}(q-3ab)=0.\end{aligned} \tag{6}$$

We have:

Theorem 3. *Apollonius circles of the standard triangle have the equations (6).*

From the previous proof it follows

$$a_o=\frac{1}{2}(a+d)=-\frac{q}{3a},$$

and with $x = a_o$ from the equation $y = -ax - bc$ of the line BC we get

$$y = a \cdot \frac{q}{3a} - bc = \frac{q}{3} - bc,$$

so the line $\mathcal{A}_o$ with the equation $x = a_o$ meets the line BC at the point

$$A_o = \left(-\frac{q}{3a}, \frac{q}{3} - bc\right). \tag{9}$$

Because of

$$y - \frac{3p}{q}x - \frac{q}{3} = \frac{q}{3} - bc - \frac{3p}{q}\left(-\frac{q}{3a}\right) - \frac{q}{3} = 0$$

that point lies on the Lemoine line (2). We have proved:

Theorem 4. *The axes of the Apollonius circles $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{A}_c$ of the allowable triangle ABC are the isotropic lines $\mathcal{A}_o$, $\mathcal{B}_o$, $\mathcal{C}_o$ through the intersections of the lines BC, CA, AB with the Lemoine line of that triangle. In the standard triangle ABC the lines $\mathcal{A}_o$, $\mathcal{B}_o$, $\mathcal{C}_o$ have the equations $x = a_o$, $x = b_o$, $x = c_o$, where*

$$a_o = -\frac{q}{3a}, \quad b_o = -\frac{q}{3b}, \quad c_o = -\frac{q}{3c}. \tag{8}$$

The lines $\mathcal{A}_o$, $\mathcal{B}_o$, $\mathcal{C}_o$ from Theorem 4 will be called *Apollonius axes* of the allowable triangle ABC.

Haberland [2, 3] has proved a number of results about radii of the Apollonius circles in Euclidean geometry. Some of these results can be transferred in an isotropic plane in the form of Theorems 6, 7, 8.

Theorem 5. *Apollonius circles $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{A}_c$ of the standard triangle ABC have the radii ρ_a, ρ_b, ρ_c given by formulae*

$$\rho_a = \frac{1}{3a}(c-a)(a-b), \quad \rho_b = \frac{1}{3b}(a-b)(b-c), \quad \rho_c = \frac{1}{3c}(b-c)(c-a). \tag{9}$$

Proof. Because of (4) we get for example

$$\rho_a = \frac{1}{2}(d-a) = -a - \frac{q}{3a} = -\frac{1}{3a}[q + 3(bc-q)] = \frac{2q-3bc}{3a} = \frac{1}{3a}(c-a)(a-b).$$

■

From (9) it follows

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} = \frac{3}{(b-c)(c-a)(a-b)}[a(b-c) + b(c-a) + c(a-b)] = 0$$

and because of

$$CA^2 - AB^2 = (a-c)^2 - (b-a)^2 = (2a-b-c)(b-c) = 3a(b-c)$$

it is for example

$$\rho_a = \frac{b-c}{CA^2 - AB^2} \cdot (c-a)(a-b) = -\frac{BC \cdot CA \cdot AB}{CA^2 - AB^2},$$

so we have:

Theorem 6. *For the radii ρ_a, ρ_b, ρ_c of the Apollonius circles of the allowable triangle the following equalities*

$$\rho_a = -\frac{BC \cdot CA \cdot AB}{CA^2 - AB^2}, \quad \rho_b = -\frac{BC \cdot CA \cdot AB}{AB^2 - BC^2}, \quad \rho_c = -\frac{BC \cdot CA \cdot AB}{BC^2 - CA^2},$$

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} = 0$$

are valid.

Theorem 7. *The distances of the Apollonius axes of the allowable triangle are inverse proportional to the radii of their corresponding Apollonius circles.*

Proof. Axes $\mathcal{B}_o$ and $\mathcal{C}_o$ from Theorem 4 have the distance

$$c_o - b_o = \frac{q}{3b} - \frac{q}{3c} = -\frac{q}{3bc}(b-c), \tag{10}$$

so because of (9) we get for example

$$(c_o - b_o)\rho_a = -\frac{q}{9p}(b-c)(c-a)(a-b).$$

Therefore $(c_o - b_o)\rho_a = (a_o - c_o)\rho_b = (b_o - a_o)\rho_c$. ∎

Theorem 8. *The squares of the distances from Theorem 7 are, respectively, equal to $\rho_b^2 + \rho_b\rho_c + \rho_c^2$, $\rho_c^2 + \rho_c\rho_a + \rho_a^2$, $\rho_a^2 + \rho_a\rho_b + \rho_b^2$, where ρ_a, ρ_b, ρ_c are the radii of Apollonius circles of the considered triangle.*

Proof. Because of (9) and (10) we get for example

$$\begin{aligned}
\rho_b^2 + \rho_b\rho_c + \rho_c^2 &= (\rho_b + \rho_c)^2 - \rho_b\rho_c \\
&= \frac{(b-c)^2}{9b^2c^2}[c(a-b) + b(c-a)]^2 - \frac{1}{9bc}(b-c)^2(c-a)(a-b) \\
&= \frac{(b-c)^2}{9b^2c^2}[a^2(b-c)^2 - bc(c-a)(a-b)]
\end{aligned}$$

$$= \frac{(b-c)^2}{9b^2c^2}[(q-bc)(q+3bc)-bc(2q-3bc)]$$

$$= \frac{(b-c)^2}{9b^2c^2}q^2 = (c_o - b_o)^2. \qquad \blacksquare$$

Let us mention that in Euclidean geometry the analogous equality $\rho_b^2 + \rho_b\rho_c + \rho_c^2 = S_bS_c^2$ means that Apollonius circles (S_b, ρ_b) and (S_c, ρ_c) of the triangle ABC intersect at the angle $\frac{\pi}{3}$.

The following two theorems transfer in an isotropic plane some results of COURT [1] from Euclidean geometry.

Theorem 9. *The potential axes of the circumscribed circle of the allowable triangle with its particular Apollonius circles are the symmedians of that triangle, which are simultaneously the polar lines of the intersections of its sides with its Lemoine line with respect to its circumscribed circle.*

Proof. The circle $\mathcal{A}_a$ from (6) and circumscribed circle with equation $y = x^2$ have the potential axis with the equation (1). The point (x_o, y_o) has with respect to circumscribed circle the polar line with equation $y + y_o = 2x_o x$. In case of the point A_o from (7) this equation gets the form of

$$y + \frac{q}{3} - bc = -\frac{2q}{3a}x,$$

and this is again the equation (1). $\blacksquare$

Theorem 9 implies that the other intersections of symmedians AK, BK, CK of the triangle ABC with its circumscribed circle lie on the lines $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, and then these intersections are the points $D = (d, d^2)$, $E = (e, e^2)$, $F = (f, f^2)$, where the abscissas d, e, f are given with (4).

The line BF, where $B = (b, b^2)$ and $F = (f, f^2)$, has obviously the equation $y = (b+f)x - bf$, and the line CE has the equation $y = (c+e)x - ce$. For this reason $L = BF \cap CE$ has the abscissa

$$l = \frac{bf - ce}{b + f - c - e}.$$

Since because of (4) we get

$$bf - ce = \frac{2q}{3bc}(c^2 - b^2) = \frac{2q}{3bc}a(b-c),$$

$$b + f - c - e = 2(b-c) - \frac{2q}{3bc}(b-c) = \frac{2(b-c)}{3bc}(3bc - q),$$

it follows

$$l = -\frac{aq}{q - 3bc}. \tag{11}$$

For the point L and analogously for the points M and N the following theorem is valid.

Theorem 10. *If the points D, E, F are the second intersections of symmedians AK, BK, CK of an allowable triangle ABC with its circumscribed circle, then the points $L = BF \cap CE$, $M = CD \cap AF$, $N = AE \cap BD$ lie on the fourth harmonic lines of Brocard's diameter of the triangle with respect to pairs of isotropic lines $\mathcal{A}$, $\mathcal{D}$; $\mathcal{B}$, $\mathcal{E}$; $\mathcal{C}$, $\mathcal{F}$, which present the Apollonius circles of that triangle.*

Proof. With $k = \frac{3p}{2q}$ it is necessary to prove the following equality

$$\frac{l-a}{l-d} : \frac{k-a}{k-d} = -1.$$

Owing to (11) and (4) we get

$$\begin{aligned}
l - a &= -\frac{aq}{q-3bc} - a = -a\frac{2q-3bc}{q-3bc}, \\
l - d &= -\frac{aq}{q-3bc} - \frac{q-3bc}{3a} = -\frac{3q(bc-q) + (q-3bc)^2}{3a(q-3bc)} \\
&= \frac{2q^2 + 3bcq - 9b^2c^2}{3a(q-3bc)} = \frac{(q+3bc)(2q-3bc)}{3a(q-3bc)}, \\
k - a &= \frac{3p}{2q} - a = -\frac{a}{2q}(2q-3bc), \\
k - d &= \frac{3p}{2q} - \frac{q-3bc}{3a} = \frac{9bc(bc-q) - 2q^2 + 6bcq}{6aq} \\
&= \frac{9b^2c^2 - 3bcq - 2q^2}{6aq} = -\frac{(q+3bc)(2q-3bc)}{6aq},
\end{aligned}$$

therefore

$$\frac{l-a}{l-d} = -\frac{3a^2}{q+3bc}, \quad \frac{k-a}{k-d} = \frac{3a^2}{q+3bc},$$

which proves the required equality. ∎

Corollary 1. *If $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ are isotropic lines through the points L, M, N from Theorem 10 and $\mathcal{T}$ Brocard's diameter of the triangle ABC, then the following statements $H(\mathcal{AD}, \mathcal{LT})$, $H(\mathcal{BE}, \mathcal{MT})$, $H(\mathcal{CF}, \mathcal{NT})$ are valid.*

Theorem 11. *The other intersections U and D' of the median AG of an allowable triangle ABC with its circumscribed circle and its Apollonius circle $\mathcal{A}_a$ are symmetrical with respect to midpoint A_m of its side $\overline{BC}$.*

Proof. The points $A = (a, a^2)$ and $U = (u, u^2)$ on circumscribed circle of the standard triangle ABC have the joint line with the equation $y = (a + u)x - au$. If $u = \frac{2q}{3a}$, then $-au = -\frac{2q}{3}$, and in this case the line AU passes through the centroid $G = (0, -\frac{2q}{3})$ of the triangle ABC. The point D' has abscissa d. Since, owing to (4) we have equation $d + u = -a$ then the midpoint of points U and D' has abscissa $-\frac{a}{2}$, and therefore it is the midpoint A_m of the side $\overline{BC}$, which lies on the median AU of the triangle ABC. ∎

Corollary 2. *The median AG of the standard triangle ABC has equation*

$$y = \left(a + \frac{2q}{3a}\right)x - \frac{2q}{3},$$

and its second intersection with circumscribed circle of that triangle is the point $U = (u, u^2)$, where $u = \frac{2q}{3a}$. Analogous statements are valid for medians BG and CG.

Corollary 3. *If $\mathcal{U}$ is isotropic line through the point U from Theorem 11 and $\mathcal{D}$ the line from Theorem 1, then the lines $\mathcal{D}$ i $\mathcal{U}$ are symmetrical with respect to bisector of the side $\overline{BC}$*

Let us prove one lemma (see [6]).

Lemma 1. *If any line through the given point $T = (x, y)$ meets the circle with the equation*

$$2\rho y = x^2 + ux + v \tag{12}$$

at the points T_1 and T_2, then the product $T_1T \cdot T_2T$ is constant and the equality

$$T_1T \cdot T_2T = x^2 + ux + v - 2\rho y \tag{13}$$

is valid.

Proof. Let $T_i = (x_i, y_i)$ $(i = 1, 2)$. Then the equalities

$$2\rho y_i = x_i^2 + ux_i + v \quad (i = 1, 2) \tag{14}$$

are valid. The line with the equation

$$2\rho y = (x_1 + x_2 + u)x + v - x_1x_2 \tag{15}$$

passes through the points T_1 and T_2 because, for example, for the the point T_1 according to (14) we get

$$(x_1 + x_2 + u)x_1 + v - x_1x_2 = x_1^2 + ux_1 + v = 2\rho y_1.$$

The point $T = (x, y)$ lies on the line T_1T_2, and so the equation (15) is valid for it. Now we get

$$\begin{aligned} T_1T \cdot T_2T &= (x - x_1)(x - x_2) \\ &= x^2 - (x_1 + x_2)x + x_1x_2 = x^2 + ux + v - 2\rho y \end{aligned}$$

because of (15) the following equality $x_1x_2 - (x_1 + x_2)x = ux + v - 2\rho y$ holds. ■

The constant product from lemma is called *power* of the point T with respect to a considered circle.

Corollary 4. *The power* Π *of the point* $T = (x, y)$ *with respect to the circle with the equation (12), is given by the equality*

$$\Pi = x^2 + ux + v - 2\rho y.$$

We will say that circle (12) has radius ρ. Specially, the circumscribed circle of the standard triangle ABC has radius $R = \frac{1}{2}$, and the power of the point $T = (x, y)$ with respect to that circle with the equation $y = x^2$ is $x^2 - y$.

Corollary 5. *Let* T *be the intersection of the lines* T_1T_2 *and* T_3T_4*. The points* T_1*,* T_2*,* T_3*,* T_4 *lie on one circle if and only if the following equality* $TT_1 \cdot TT_2 = TT_3 \cdot TT_4$ *is valid.*

Corollary 6. *Let* T_1*,* T_2*,* T *be points on one line and let point* T_3 *does not lie on that line. The circle through the points* T_1*,* T_2*,* T_3 *touches the line* TT_3 *at the point* T_3 *if and only if the equality* $TT_1 \cdot TT_2 = TT_3^2$ *holds.*

With the labels from Theorem 11 according to Corollary 5 the following equalities hold

$$AA_m \cdot UA_m = BA_m \cdot CA_m = -\frac{1}{4}BC^2,$$

and since $UA_m = -D'A_m$ it follows

$$AA_m \cdot D'A_m = \frac{1}{4}BC^2 = BA_m^2.$$

According to Corollary 6 it means that circle through the points A, B, D' touches the line BA_m, i.e. the line BC, at the point B. Analogously, the circle through the points A, C, D' touches the line BC at the point C. Therefore, we have:

Theorem 12. *Circles through the point A, which touch the line BC at the points B and C, have for the second intersection the point D' from Theorem 11.*

In Euclidean case the statement of Theorem 12 can be found in Smeenk [7].

Corollary 7. *The Apollonius circle $\mathcal{A}_a$ of the allowable triangle ABC and two circles from Theorem 12 belong to one pencil of circles and potential axis of that pencil is the median AG of the triangle ABC. The point D' from Theorem 11 has with respect to these three circles the same power which equals to $\frac{1}{4}BC^2$.*

Haberland [3] has the statements of Theorem 12 and Corollary 7 for Euclidean case.

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