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LIAOWISE REDUCED REORDERING AND SPECTRUM THEOREMS FOR DIFFERENTIAL SYSTEMS ON COMPACT MANIFOLDS AND APPLICATIONS

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Abstract. For any differential system $\vec{V}$ of class C^1 on an n-dimensional compact, smooth, and boundaryless riemannian manifold M, we consider the Liao frame skew-product flow on the reduced orthonormal frame bundle $\mathcal{C}^\sharp(M, \vec{V})$ naturally induced by $\vec{V}$ and, using some technical ideas due to S. Liao, we prove a 'reduced reordering' theorem and a 'reduced spectrum' theorem. As consequences, we also provide a reordering lemma for the natural skew-product flow $(\mathcal{F}(k), \{\mathcal{V}_t\})$ on the flag bundles $\mathcal{F}(k)$ of the tangent bundle *TM*, and give two characteristic spectra for parallelepiped. In addition, we obtain the uniformity of some non-uniformly expanding (resp. contracting) sets.

1. Introduction

Let M be an n-dimensional compact, smooth, riemannian manifold without boundary, $n \geq 2$. Assume that a smooth Riemann structure $\langle \cdot, \cdot \rangle$ on M and its induced norm $\|\cdot\|$ on *TM* have been fixed.

Let $\vec{V}$ be an arbitrarily given differential system of class C^1; namely, a C^1 vector field on M. Then, in a natural way, $\vec{V}$ induces a C^1-flow on M, written as

$$\{V_t\}_{t\in\mathbb{R}} \colon M \to M \tag{1.1}$$

where $\mathbb{R}$ denotes the real t-time axis. We identify $(M, \vec{V})$ with $(M, \{V_t\})$. It further induces another one-parameter group of transformations

$$\{\mathcal{V}_t\}_{t\in\mathbb{R}} \colon TM \to TM \tag{1.2}$$

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on the tangent bundle *TM*, where $\mathcal{V}_t = TV_t\colon TM \to TM$ is the spatial derivative of the diffeomorphism $V_t\colon M \to M$ for each $t \in \mathbb{R}$.

For any given integer $\ell, 1 \le \ell \le n$, as usual in Liao theory, let U_ℓ, F_ℓ and $F_\ell^\sharp$, respectively, stand for the ℓ-frame bundle, orthogonal ℓ-frame bundle, and orthonormal ℓ-frame bundle of tangent space *TM*, see [12, 5, 6]. The well-known Gram-Schmidt orthogonalizing process gives rise to an orthogonalizing map, written as

$$\mathrm{Ort}\colon U_\ell \to F_\ell. \tag{1.3}$$

Put

$$\mathrm{Pr}_i\colon U_\ell \to TM;\ (x; \vec{u}_1, \dots, \vec{u}_\ell) \mapsto \vec{u}_i \tag{1.4}$$

for each $i = 1, \dots, \ell$. Based on $(M, \vec{V})$, Liao in [12] introduced the following frame skew-product flows:

$$\begin{aligned} &\{V_t\}_{t\in\mathbb{R}}\colon U_\ell \to U_\ell \\ &\{\mathfrak{V}_t\}_{t\in\mathbb{R}}\colon F_\ell \to F_\ell \\ &\{\mathfrak{V}_t^\sharp\}_{t\in\mathbb{R}}\colon F_\ell^\sharp \to F_\ell^\sharp \end{aligned} \tag{1.5}$$

defined, respectively, by

$$\begin{aligned} &V_t(x; \vec{\gamma}) = (V_t(x); \mathcal{V}_t \circ \mathrm{Pr}_1(x; \vec{\gamma}), \dots, \mathcal{V}_t \circ \mathrm{Pr}_\ell(x; \vec{\gamma})) \\ &\mathfrak{V}_t(x; \vec{\gamma}') = \mathrm{Ort} \circ V_t(x; \vec{\gamma}') \\ &\mathfrak{V}_t^\sharp(x; \vec{\gamma}'') = \left(V_t(x); \frac{\mathrm{Pr}_1 \circ \mathfrak{V}_t(x; \vec{\gamma}'')}{\|\mathrm{Pr}_1 \circ \mathfrak{V}_t(x; \vec{\gamma}'')\|}, \dots, \frac{\mathrm{Pr}_\ell \circ \mathfrak{V}_t(x; \vec{\gamma}'')}{\|\mathrm{Pr}_\ell \circ \mathfrak{V}_t(x; \vec{\gamma}'')\|}\right) \end{aligned}$$

for all $(x; \vec{\gamma}) \in U_\ell, (x; \vec{\gamma}') \in F_\ell$ and $(x; \vec{\gamma}'') \in F_\ell^\sharp$. These skew-product flows are all well defined; see [12, 18, 5, 6] for the details. For the sake of simplicity, sometimes we will write a frame $(x; \vec{\gamma})$ as $\vec{\gamma}_x$.

Since the vector field $\vec{V}$ is of class C^1, the so-called *Liao qualitative functions*

$$\omega_k\colon F_\ell^\sharp \to \mathbb{R}, \quad k = 1, \dots, \ell \tag{1.6}$$

given by

$$\omega_k(\vec{\gamma}_x) = \frac{d}{dt}|_{t=0} \|\mathrm{Pr}_k \circ \mathfrak{V}_t(\vec{\gamma}_x)\| \quad \forall\, (x; \vec{\gamma}) \in F_\ell^\sharp, \tag{1.6a}$$

all are well defined and continuous with respect to $(x; \vec{\gamma}) \in F_\ell^\sharp$; see [12, 6]. The importance of the functions $\{\omega_1, \dots, \omega_\ell\}$ may be seen via Liao's Reordering Lemma

([14, Theorem 4.1]) and Liao's Spectrum Theorem ([14, Theorem A] or [15, Proposition 2.1] or see [6, Corollary 5.4]), restated in Section 2 below. We realize little by little that Liao's Spectrum Theorem is a powerful tool to the theory of skew-product flows; see [6-8, 15, 16] for some applications.

In the study of ordinary differential equations, it is a very useful technique reducing the order of systems. In this paper, we will give a Liaowise reduced reordering theorem and a reduced spectrum theorem for $\vec{V}$ (Theorem 1.1 below).

Set

$$\mathcal{C}^\sharp(M,\vec{V}) = \bigcup_{x\in M} \mathcal{C}^\sharp(x,\vec{V}), \tag{1.7}$$

whose fiber over x is defined as follows:

(1.7a)

$$\mathcal{C}^\sharp(x,\vec{V}) = \left\{(x;\vec{\gamma}) \in F_{n-1}^\sharp(x) \mid \langle \vec{V}(x), \mathrm{Pr}_k(\vec{\gamma}_x)\rangle_x = 0 \text{ for } 1 \le k \le n-1\right\}.$$

It is clear that $\mathcal{C}^\sharp(M,\vec{V})$ is a compact subset of $F_{n-1}^\sharp$. By an abuse of symbols, we denote all the bundle projections by $\pi\colon (x;\vec{\gamma}) \mapsto x$. Note that, unlike $F_{n-1}^\sharp$, $\mathcal{C}^\sharp(M,\vec{V})$ is not a vector bundle when $\vec{V}$ has singularities, but $(\mathcal{C}^\sharp(M,\vec{V}),\pi,M)$ is a topological subbundle of $(F_{n-1}^\sharp,\pi,M)$. It is called the *reduced orthonormal* $(n-1)$*-frame bundle of* $\vec{V}$. $\mathcal{C}^\sharp(M,\vec{V})$ will be a useful space when dealing with Liao theory; an example is to extend the concept of Liao reduced style number [11] to the closed invariant set meeting singularities. In the present paper, we use this bundle together with some technical ideas from [13, 14] to prove the reduced Liao reordering and spectrum theorem. This theorem guarantees that the reduced system will keep all the nontrivial Lyapunov exponents and one can rearrange them in a good order (here by trivial exponent we mean what given by the vector field $\vec{V}(x)$ directions for a.e. $x \in M$).

For any $(x;\vec{\gamma}) \in \mathcal{C}^\sharp(M,\vec{V})$, we arbitrarily take a frame $(x;\widetilde{\gamma}) \in F_n^\sharp$ such that

$$(x;\vec{\gamma}) = (x;\mathrm{Pr}_2(\widetilde{\gamma}_x),\dots,\mathrm{Pr}_n(\widetilde{\gamma}_x)). \tag{1.8}$$

If we put

$$\begin{aligned}\Theta_t(x;\vec{\gamma}) &= (V_t(x);\mathrm{Pr}_2\circ\mathfrak{V}_t(\widetilde{\gamma}_x),\dots,\mathrm{Pr}_n\circ\mathfrak{V}_t(\widetilde{\gamma}_x))\\ \Theta_t^\sharp(x;\vec{\gamma}) &= (V_t(x);\mathrm{Pr}_2\circ\mathfrak{V}_t^\sharp(\widetilde{\gamma}_x),\dots,\mathrm{Pr}_n\circ\mathfrak{V}_t^\sharp(\widetilde{\gamma}_x))\end{aligned} \tag{1.9}$$

where $\mathfrak{V}_t\colon F_n \to F_n$ and $\mathfrak{V}_t^\sharp\colon F_n^\sharp \to F_n^\sharp$ as in (1.5), respectively, induced naturally by $\vec{V}$ in the case $\ell = n$, we then obtain another well-defined skew-product flow based on $(M,\{V_t\})$:

$$\{\Theta_t^\sharp\}_{t\in\mathbb{R}}\colon \mathcal{C}^\sharp(M,\vec{V}) \to \mathcal{C}^\sharp(M,\vec{V}), \tag{1.10}$$

since $\Theta_t^\sharp(x;\vec{\gamma})$ does not depend upon the explicit choice of $\widetilde{\gamma}_x$ corresponding to $\vec{\gamma}_x$ as in (1.8). If we define

$$\omega_k^*\colon \mathcal{C}^\sharp(M,\vec{V}) \to \mathbb{R} \quad (k=1,\dots,n-1) \tag{1.11}$$

in the way

$$\omega_k^*(\vec{\gamma}_x) = \frac{d}{dt}|_{t=0}\|\mathrm{Pr}_k \circ \Theta_t(\vec{\gamma}_x)\| \quad \forall\,(x;\vec{\gamma}) \in \mathcal{C}^\sharp(M,\vec{V}), \tag{1.11a}$$

then each of $\omega_k^*, k=1,\dots,n-1$, makes sense and is continuous. Call $\omega_1^*,\dots,\omega_{n-1}^*$ the *reduced qualitative functions* of $\vec{V}$. It is easy to see that

$$\omega_k^*(\vec{\gamma}_x) = \frac{d}{dt}|_{t=0}\|\mathrm{Pr}_{k+1} \circ \mathfrak{V}_t(\widetilde{\gamma}_x)\| = \omega_{k+1}(\widetilde{\gamma}_x) \tag{1.11b}$$

and

$$\frac{1}{T}\log\|\mathrm{Pr}_k \circ \Theta_T(\vec{\gamma}_x)\| = \frac{1}{T}\int_0^T \omega_k^*(\Theta_t^\sharp(\vec{\gamma}_x))\,dt. \tag{1.11c}$$

For any given $\mu \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$, the set of all ergodic measures of $(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$, write

$$\vartheta_k^*(\mu) = \int_{\mathcal{C}^\sharp(M,\vec{V})} \omega_k^*(\vec{\gamma}_x)\,d\mu(\vec{\gamma}_x) \quad (k=1,\dots,n-1). \tag{1.12}$$

The following is our main theorem.

Theorem 1.1. *For any given C^1 differential system $\vec{V}$ on M, the following properties are satisfied.*

(1) *For any $\mu \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$ and for any permutation $k \mapsto p(k)$ of $\{1,\dots,n-1\}$, there exists some $\mu' \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$, such that*

   $\pi_*(\mu) = \pi_*(\mu') \in \mathcal{M}_{erg}(M,\vec{V})$, where $\pi\colon \mathcal{C}^\sharp(M,\vec{V}) \to M$.

   $\vartheta_k^*(\mu') = \vartheta_{p(k)}^*(\mu)$ for $k=1,\dots,n-1$.

(2) *If $\nu \in \mathcal{M}_{erg}(M,\vec{V})$ and $\mu \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$ satisfy $\pi_*(\mu)=\nu$, then*

$$\{\vartheta_k^*(\mu) \mid 1 \le k \le n-1\} \subset \mathsf{Sp}(\vec{V};\nu)$$

   where $\mathsf{Sp}(\vec{V};\nu)$ denotes the Oseledec characteristic spectrum of ν counted with multiplicity.

(3) *For any given non-atomic measure* $\nu \in \mathcal{M}_{erg}(M, \vec{V})$*, if* μ_1 *and* μ_2 *both belong to* $\mathcal{M}_{erg}(\mathcal{C}^\sharp(M, \vec{V}), \{\Theta_t^\sharp\})$ *with* $\pi_*(\mu_1) = \pi_*(\mu_2) = \nu$*, then*

$$\mathsf{Sp}^*(\vec{V}; \nu) := \{\vartheta_k^*(\mu_1) \mid 1 \le k \le n-1\} = \{\vartheta_k^*(\mu_2) \mid 1 \le k \le n-1\}$$

counting with multiplicity and ignoring the order.

We call the statement (1) of Theorem 1.1 above the *reduced reordering lemma*, (2) and (3) the *reduced spectrum theorem* of $\vec{V}$.

Corollary 1.2. *If* $\nu \in \mathcal{M}_{erg}(M, \vec{V})$*, with* $\nu(\{\text{singularities of } \vec{V}\}) = 0$*, then* ν *is non-uniformly hyperbolic if and only if* $\mathsf{Sp}^*(\vec{V}; \nu)$ *contains no zero.*

Liao in [14] proved the non-reduced reordering lemma, which is an important tool to Liao theory as stated before. In his proof, one needs the natural linear one-parameter group $\{\mathcal{V}_t\}\colon TM \to TM$. For the reduced case, however, one can not construct such a corresponding global linear one-parameter group on the reduced tangent bundle

$$\vec{V}^\perp = \{\vec{v} \in T_xM \mid x \in M, \langle \vec{V}(x), \vec{v}\rangle_x = 0\}. \tag{1.13}$$

If one restricts oneself to the regular point set

$$R(\vec{V}) = \left\{x \in M \mid \vec{V}(x) \neq \vec{0}_x\right\}, \tag{1.14}$$

although there exists a natural linear skew-product flow

$$\{\mathcal{V}_t^*\}_{t\in\mathbb{R}}\colon \vec{V}^\perp(R(\vec{V})) \to \vec{V}^\perp(R(\vec{V})), \tag{1.15}$$

where

$$\vec{V}^\perp(R(\vec{V})) = \left\{(x; \vec{v}) \in \vec{V}^\perp \mid x \in R(\vec{V})\right\}, \tag{1.15a}$$

introduced by Liao in [12], $\vec{V}^\perp(R(\vec{V}))$ and further $\mathcal{C}^\sharp(R(\vec{V}), \vec{V})$ both are not compact when $\vec{V}$ has singularities. This point is crucial because it causes that one can not construct the "good" measure required by the given permutation in Theorem 1.1 by the standard results and techniques of the classic ergodic theory [19, Chap. 6], and the Liao Measure Lifting Lemma [14] is not valid as well. Despite there is no natural linear skew-product flow on the reduced tangent bundle corresponding to the compact system $(\mathcal{C}^\sharp(M, \vec{V}), \{\Theta_t^\sharp\})$, along every given orbit, say $\mathrm{Orb}_x = \{V_t(x) \mid t \in \mathbb{R}\}$, we can always construct a family of natural linear isomorphisms

$$\mathcal{V}_{x,t}^*\colon \mathbb{H}(\vec{\gamma}_x) \to \mathbb{H}(\Theta_t^\sharp(\vec{\gamma}_x)) \quad (\forall\, t \in \mathbb{R}) \tag{1.16}$$

corresponding to $\Theta_t^\sharp\colon \mathcal{C}^\sharp(x,\vec{V}) \to \mathcal{C}^\sharp(V_t(x),\vec{V})$ and a reduced $(n-1)$-frame $(x;\vec{\gamma})$ in $\mathcal{C}^\sharp(M,\vec{V})$, where $\mathbb{H}(\vec{\gamma}_x)$ means the linear subspace generated by the frame $\vec{\gamma}_x$ in T_xM. That is the most key idea for the proof of our reduced reordering lemma in Section 3.

In the case where ν satisfies that $\nu(R(\vec{V}) = 1$; i.e. ν is non-atomic, Liao, in his survey [13], announced that $\mathsf{Sp}^*(\vec{V};\nu)$ is independent of the choice of the lifting measure μ in the statement (3) of Theorem 1.1 above. But no proof is available, since he did not publish his proof. Under the condition of the statement (3), one can regard ν as in $\mathcal{M}_{erg}(R(\vec{V}),\{V_t\})$ if identifying ν with the conditional measure $\nu|_{R(\vec{V})}$, but we still have to consider the compact bundle $\mathcal{C}^\sharp(M,\vec{V})$ by the same reason of non-compactness. On the other hand, despite we finally gain $\mu_1(\mathcal{C}^\sharp(R(\vec{V}),\vec{V})) = \mu_2(\mathcal{C}^\sharp(R(\vec{V}),\vec{V})) = 1$ and then both μ_1 and μ_2 naturally induce measures μ_1^* and μ_2^* lying in $\mathcal{M}_{erg}(F_n^\sharp,\{\mathfrak{V}_t^\sharp\})$ respectively, covering ν, the fact $\int_{F_n^\sharp}\omega_1\,d\mu_1^* = \int_{F_n^\sharp}\omega_1\,d\mu_2^*$ is still not trivial.

We call $\mathsf{Sp}^*(\vec{V};\nu)$, defined by Theorem 1.1, the *reduced spectrum* of $\vec{V}$ w.r.t ν, or simply, the reduced spectrum of ν. The reduced and non-reduced spectrum theorems (Theorems 1.1 above and 2.1 below) provide convinced evidence for the computation of Lyapunov exponents by continuous Gram-Schmidt orthonormalization in mechanics, for example, Christiansen-Rugh [4].

In order to show the non-trivialness of Liaowise spectrum theorems, let's consider the following linear system on $\mathbb{R}^2$:

Example 1.1. Let

$$\mathcal{V}_t = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{-t} \end{bmatrix} \text{ and } A(t) = \frac{d\mathcal{V}_t}{dt}\mathcal{V}_t^{-1} \quad \forall\, t \in \mathbb{R}. \tag{1.17}$$

Then, the differential equations

$$\frac{dy}{dt} = A(t)y \quad ((t,y) \in \mathbb{R}\times\mathbb{R}^2) \tag{1.18}$$

has the fundamental matrix solution $\mathcal{V}_t$. As usual, let H_A be the hull of the real 2×2-matrix-valued function $A(t)$. Then, based on the translation flow $(H_A,\{T_t\})$ there is a natural linear skew-product flow $(H_A\times\mathbb{R}^2,\{\widehat{T}_t\})$ by putting

$$\widehat{T}_t(x;y) = (T_tx;\varPhi_x(t)y) \quad \forall\, t\in\mathbb{R}, \forall\,(x;y)\in H_A\times\mathbb{R}^2.$$

Where, $\varPhi_x(t)$ denotes the standard fundamental matrix solution of the equations

$$\frac{dy}{dt} = x(t)y \quad ((t,\vec{v})\in\mathbb{R}\times\mathbb{R}^2)$$

for any $x \in H_A$.

Now, in the situation of Example 1.1, we consider the case at the base point $x = A$. Then, $\Phi_x(t) = \mathcal{V}_t\mathcal{V}_0^{-1}$. Clearly, for the standard basis frame $\vec{\gamma}_x = (\vec{e}_1, \vec{e}_2)$, where $\vec{e}_1 = (1,0)^{\mathrm{T}}$ and $\vec{e}_2 = (0,1)^{\mathrm{T}}$ in $\mathbb{R}^2$, we have

$$\lim_{t\to\infty}\frac{1}{t}\log\|\Phi_x(t)(\vec{e}_1)\| = 1 \tag{1.19}$$

$$\lim_{t\to\infty}\frac{1}{t}\log\|\Phi_x(t)(\vec{e}_2)\| = 2 \tag{1.20}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\log\|\mathrm{Pr}_2\circ\mathrm{Ort}\circ\Phi_x(t)(\vec{\gamma}_x)\| = -1 \not\in \{1,2\}. \tag{1.21}$$

For any orthonormal frame $\vec{\gamma}$ of $\mathbb{R}^2$, by the action of $\Phi_x(t)$, we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathrm{vol}_{\vec{\gamma}}(t) = 0. \tag{1.22}$$

As applications of the reordering and spectrum theorems, we will, in Section 2, provide a reordering lemma (Theorem 2.3 below) for the natural skew-product flow $(\mathcal{F}, \{\mathcal{V}_t\})$ on the flag bundles $\mathcal{F}$ of the tangent bundle *TM*, which is not obvious from the classical smooth ergodic theory. In addition, we shall consider the characteristic spectra of parallelepiped in the tangent bundle, see Theorems 4.1 and 4.2 below in Section 4. This kind of spectrum theorems are very different from the existent, such as the one [1, §5.3.2] which says that, almost all k-dimensional parallelepiped $\Pi(\vec{\gamma}_x)$ spanned by frames $(x;\vec{\gamma}) \in F_k^\sharp, 1 \le k \le n$, have the Lyapunov exponent of volume; that is to say,

$$\lim_{t\to\pm\infty}\frac{1}{t}\log\ \mathrm{vol}(\Pi(V_t(\vec{\gamma}_x))) = \lambda^{(k)} \tag{1.23}$$

where $\lambda^{(k)}$ is some real number determined by the frames $\vec{\gamma}_x$. In fact, $\lambda^{(k)}$ is exactly the sum of some k numbers in $\mathsf{Sp}(\vec{V};\nu)$. Our Theorems 4.1 and 4.2 are, however, devoted to the converse problem whether, for any given sum $\lambda^{(k)}$ of k exponents in $\mathsf{Sp}(\vec{V};\nu)$, there are many $\Pi(\vec{\gamma}_x)$, $(x;\vec{\gamma}) \in F_k^\sharp$, exactly having the Lyapunov exponent of volume $\lambda^{(k)}$.

As the end of this Introduction, let us state the main reason why we are interested in the reduced reordering and spectrum theorem.

According to Pesin theory, if the vector field $\vec{V}$ is of class $C^{1+\alpha}, 0 < \alpha \le 1$ and $\nu \in \mathcal{M}_{erg}(M,\vec{V})$ is non-uniformly hyperbolic; i.e., the Lyapunov exponents $\lambda_k(\nu) \neq 0$, then for ν-a.e. Oseledec regular points $x \in M$, there exist the C^1 stable and unstable manifolds $W^{s/u}(x)$. In the case of class C^1, Pugh's counterexample [19] shows that the stable and unstable manifolds are not necessarily existent at certain regular non-uniformly hyperbolic point. Liao [13] showed that Pugh's "bad" point does not lie in his regular point set constructed for the multiplicative ergodic theorem. He had suggested us to consider the existence of the stable and unstable manifolds at the points contained in his regular set for the C^1 vector field $\vec{V}$.

To attack this problem stated above, hopefully Liao's reduced standard systems of differential equations of $\vec{V}$ will be helpful. Based on Liao's theory, for any $(x;\vec{\gamma}) \in \mathcal{C}^\sharp(M,\vec{V})$ there is an associated nonlinear standard system

$$\frac{dy}{dt} = \widetilde{R}_{\vec{\gamma}_x}(t)y + \widetilde{V}_{\mathrm{Rem}(\vec{\gamma}_x)}(t,y), \tag{1.24}$$

see [18, Chap. 2] or [9]. Here $\widetilde{R}_{\vec{\gamma}_x}(t)$ is a upper triangular $n-1$ by $n-1$ matrix whose diagonal entries are just the reduced qualitative functions $\omega_k^*(\Theta_t^\sharp(\vec{\gamma}_x)), k = 1,\dots,n-1$. Therefore, if the reduced spectrum $\mathsf{Sp}^*(\vec{V};\nu)$ has no zero-value, then the reduced Liao linearization equations along the orbit Orb_x under the movable frames $\{\Theta_t^\sharp(\vec{\gamma}_x) \mid t \in \mathbb{R}\}$

$$\frac{dy}{dt} = \widetilde{R}_{\vec{\gamma}_x}(t)y \quad ((t,y) \in \mathbb{R}\times\mathbb{R}^{n-1}) \tag{1.25}$$

is non-uniformly hyperbolic. Using the Reduced Reordering Lemma proved in the present paper, we can choose frames $(x;\vec{\gamma})$ so that the diagonal entries of $\widetilde{R}_{\vec{\gamma}_x}(t)$ have a "good" order for all $t \in \mathbb{R}$.

On the other hand, in the study of the qualitative theory of $\vec{V}$, we often need to consider the Poincaré cross sections $S_{V_t(x)}$ along a regular orbit Orb_x and the naturally induced cross-section maps $P_t\colon S_x \to S_{V_t(x)}$. It turns out that (1.16) is its linearization and under the moving reduced frame $\Theta_t^\sharp(\vec{\gamma}_x)$, (1.16) is represented as (1.25). As mentioned above, any hyperbolicity of $\vec{V}$ is reflected to (1.25) by our reduced spectrum theorem. Moreover, its small perturbed equation (1.24) really embodies the structure of orbits near Orb_x from Liao's theory of standard systems of differential equations [18, 9]. Therefore, comparing with Liao's non-reduced version (Theorems 2.1 and 2.2 below), although the Liao's is powerful for general smooth linear skew-product flows over a compact system, the reduced one is more convenience in the studying of the qualitative theory of vector fields. The following is an explicit simple example.

Corollary 1.3. *Let M_0 be a $\vec{V}$-invariant closed subset in M, which contains no singularities.*

(1) *If every $\nu \in \mathcal{M}_{erg}(M,\vec{V})$ supported on M_0 is non-uniformly contracting in the sense $\mathsf{Sp}^*(\vec{V},\nu) \subset (-\infty,0)$, then M_0 is a uniformly contracting subset of $\vec{V}$.*

(2) *If every* $\nu \in \mathcal{M}_{erg}(M, \vec{V})$ *supported on* M_0 *is non-uniformly expanding in the sense* $\mathsf{Sp}^*(\vec{V}, \nu) \subset (0, +\infty)$*, then* M_0 *is a uniformly expanding subset of* $\vec{V}$.

This result is a continuous-time version of the main Theorem A of [2, 3]. However, with the aids of the reduced qualitative functions ω_k^* and the reduced spectrum Sp^*, we can give a very simple proof in Section 5.

2. On Liao's Reordering and Spectrum Theorem

To avoid using Oseledec's theorem in proving his spectrum theorem below, Liao first proved his reordering lemma (Theorem 2.2 below). We now will give a simple proof of Liao's spectrum theorem based on Oseledec's theorem.

Theorem 2.1. (Liao's spectrum theorem). *For any given* $\nu \in \mathcal{M}_{erg}(M, \vec{V})$*, if* $\mu \in \mathcal{M}_{erg}(F_n^\sharp, \{\mathfrak{V}_t^\sharp\})$ *covers* ν *via* π*; that is to say,* $\pi_*(\mu) = \nu$*, then*

$$\mathsf{Sp}(\vec{V}; \nu) = \left\{ \int_{F_n^\sharp} \omega_k \, d\mu \,|\, k = 1, \ldots, n \right\}$$

where $\pi \colon F_n^\sharp \ni (x; \vec{\gamma}) \mapsto x \in M$ *is the bundle projection,* ω_k *as in (1.6) in the case* $\ell = n$.

Proof. Let $\mu \in \mathcal{M}_{erg}(F_n^\sharp, \{\mathfrak{V}_t\})$ is a lifting of $\nu \in \mathcal{M}_{erg}(M, \vec{V})$ via the bundle projection π. From the Birkhoff ergodic theorem, we can take a $\{\mathfrak{V}_t^\sharp\}$-invariant set V_μ of μ-full measure lying in $F_n^\sharp$ such that for each $(x; \vec{\gamma}) \in V_\mu$

$$\vartheta_k(\mu) := \int_{F_n^\sharp} \omega_k \, d\mu = \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t \omega_k(\mathfrak{V}_\tau^\sharp(\vec{\gamma}_x)) \, d\tau \quad k = 1, \ldots, n. \tag{2.1}$$

Let $\varGamma(\nu)$ be the Oseledec regular subset of $(\vec{V}; \nu)$ in M. Put

$$L(\mu) = V_\mu \bigcap \pi^{-1}(\varGamma(\nu)). \tag{2.2}$$

Obviously, for every $(x; \vec{\gamma}) \in L(\mu)$, the Liao global linearization equations along the orbit Orb_x under the base $(x; \vec{\gamma})$

$$\frac{dy}{dt} = R_{\vec{\gamma}_x}(t) y \quad ((t, y) \in \mathbb{R} \times \mathbb{R}^n) \tag{2.3}$$

defined in [12] (or see [6]), is Lyapunov-Perron regular. This implies

$$\mathsf{Sp}(\vec{V}; \nu) = \{\vartheta_k(\mu) \,|\, k = 1, \ldots, n\} \tag{2.4}$$

and the proof is thus complete. ■

Theorem 2.2. (Liao's Reordering Lemma). *If* $\mu \in \mathcal{M}_{erg}(F_\ell^\sharp, \{\mathfrak{V}_t^\sharp\}), 2 \leq \ell \leq n$, *then, to any permutation* $p\colon \{1, \ldots, \ell\} \to \{1, \ldots, \ell\}$, *there corresponds some* $\bar{\mu} \in \mathcal{M}_{erg}(F_\ell^\sharp, \{\mathfrak{V}_t^\sharp\})$ *such that* $\pi_*(\mu) = \pi_*(\bar{\mu})$ *and* $\vartheta_{p(k)}(\mu) = \vartheta_k(\bar{\mu})$ *for* $k = 1, \ldots, \ell$. *Here* ϑ_k *is defined as in (2.1).*

We next present an equivalent formulation for Liao's Reordering Lemma using the terms of flag bundle. Let $\mathcal{F}$ be the flag bundle whose fiber $\mathcal{F}_x$ over $x \in M$ is the set of flags $\mathbb{F}_{1x} \subset \cdots \subset \mathbb{F}_{nx} = T_xM$ with $\dim \mathbb{F}_{ix} = i$. Then there is a natural flow on $\mathcal{F}$ induced by $(M, \vec{V})$, written simply as $(\mathcal{F}, \{\mathcal{V}_t\})$. We now define the projection

$$P\colon F_n^\sharp \to \mathcal{F} \tag{2.5}$$

by

$$P(x; \vec{\gamma}) = (x; \mathbb{H}(\mathrm{Pr}_1\vec{\gamma}_x) \subset \mathbb{H}(\mathrm{Pr}_1\vec{\gamma}_x, \mathrm{Pr}_2\vec{\gamma}_x) \subset \cdots \subset \mathbb{H}(\vec{\gamma}_x)) \tag{2.5a}$$

where $\mathbb{H}(x; \vec{v}_1, \ldots, \vec{v}_i)$ means the linear subspace of T_xM spanned by the frame $(x; \vec{v}_1, \ldots, \vec{v}_i)$. Clearly P is surjective and continuous. In addition, we easily see that

$$\begin{array}{ccc} F_n^\sharp & \xrightarrow{\mathfrak{V}_t^\sharp} & F_n^\sharp \\ \Big\downarrow P & & P \Big\downarrow \\ \mathcal{F} & \xrightarrow{\mathcal{V}_t} & \mathcal{F} \\ \Big\downarrow \pi & & \pi \Big\downarrow \\ M & \xrightarrow{V_t} & M \end{array} \qquad \forall\, t \in \mathbb{R}$$

is commutative and hence P is a semi-conjugacy from $(F_n^\sharp, \{\mathfrak{V}_t^\sharp\})$ to $(\mathcal{F}, \{\mathcal{V}_t\})$. Using Liao's measure lifting lemma [14] we have

$$P_*(\mathcal{M}_{erg}(F_n^\sharp, \{\mathfrak{V}_t^\sharp\})) = \mathcal{M}_{erg}(\mathcal{F}, \{\mathcal{V}_t\}). \tag{2.6}$$

For given $\nu \in \mathcal{M}_{erg}(M, \vec{V})$, let $\mathsf{Sp}(\vec{V}; \nu) = \{\lambda_1 \geq \cdots \geq \lambda_n\}$ and $\mu \in \mathcal{M}_{erg}(\mathcal{F}, \{\mathcal{V}_t\})$ covering ν. Then, by Theorems 2.1 and 2.2, there is a corresponding permutation $p\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that for μ-a.e. flag $(x; \mathbb{F}_1 \subset \cdots \subset \mathbb{F}_n)$, there exists a frame $(x; \vec{v}_1, \ldots, \vec{v}_n)$ in $F_n^\sharp$ which generates the flag (i.e., $\{\vec{v}_1, \ldots, \vec{v}_i\}$ generates $\mathbb{F}_{ix}$ for $i = 1, \ldots, n$) corresponding to the exponents $\lambda_{p(i)}$, i.e., $\lambda(\vec{v}_i) = \lambda_{p(i)}$.

If a permutation p has been chosen in advance, then can we find some $\mu \in \mathcal{M}_{erg}(\mathcal{F}, \{\mathcal{V}_t\})$ realizing it? The answer is positive.

Theorem 2.3. *Suppose ν lies in $\mathcal{M}_{erg}(M, \vec{V})$. If $\mathsf{Sp}(\vec{V}; \nu) = \{\lambda_1 \geq \cdots \geq \lambda_n\}$ and $p\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation, there is $\mu^* \in \mathcal{M}_{erg}(\mathcal{F}, \{\mathcal{V}_t\})$ such that*

(1) *$\pi_*(\mu^*) = \nu$;*

(2) *for μ^*-a.e. flag $(x; \mathbb{F}_1 \subset \cdots \subset \mathbb{F}_n) \in \mathcal{F}$, there exists a frame $(x; \vec{w}_1, \ldots, \vec{w}_n) \in F_n^\sharp$ which generates the flag and such that $(\vec{w}_1, \ldots, \vec{w}_n)$ is a normal base of T_xM corresponding to the exponents $\lambda_{p(i)}$.*

Proof. For the permutation p, we choose some $\mu \in \mathcal{M}_{erg}(F_n^\sharp, \{\mathfrak{V}_t^\sharp\})$ covering ν realizing it, by using Theorems 2.1 and 2.2; that is to say,

$$\vartheta_i(\mu) = \lambda_{p(i)} \quad i = 1, \ldots, n. \tag{2.7}$$

Take $\mu^* = P_*(\mu)$, where $P\colon F_n^\sharp \to \mathcal{F}$. Obviously $\pi_*(\mu^*) = \nu$. From Oseledec's theorem and the partial linearization equations along orbits under partial movable frames [6], the statement (2) follows easily. ∎

For the flag bundle $\mathcal{F}(\ell), 1 \leq \ell < n$, there is the reordering lemma similarly.

With $(\mathcal{F}, \{\mathcal{V}_t\})$ as a bridge, to Theorem 2.1 there is a more straightforward proof using [1, Theorems 6.3.1 and 5.3.1]. In addition, this idea also works for the reduced spectrum theorem. However, it is not enough to Theorem 2.3.

Indeed, we assume the situation described in Theorem 2.3. By [1, Theorem 6.3.1], for $\lambda^{(k)} = \lambda_{p(1)} + \cdots + \lambda_{p(k)}$ there exists a $\{\varphi_k^t\}_{t \in \mathbb{R}}$-invariant measure μ_k which realizes $\lambda^{(k)}$, where $\{\varphi_k^t\}\colon G_k(n) \to G_k(n)$ is the natural flow on the Grassmannian bundle $G_k(n)$ induced by $\{\mathcal{V}_t\}$ as in Arnold [1]. But it is the point, that it is not necessarily right that there exists some $\mu^* \in \mathcal{M}_{erg}(\mathcal{F}, \{\mathcal{V}_t\})$ which just covers every μ_k chosen by the natural projection $P_k\colon \mathcal{F} \to G_k(n)$; $(x; \mathbb{F}_1 \subset \cdots \subset \mathbb{F}_n) \mapsto (x; \mathbb{F}_k)$ for $k = 1, \ldots, n$. Maybe the important [1, Theorem 1.6.13] will have ones try to prove Theorem 2.3 as follows. Let ν be an ergodic measure for the flow $(M, \vec{V})$, and let $\lambda_1 \geq \cdots \geq \lambda_n$ be the corresponding Lyapunov exponents. Consider the flow on $\Omega = M$, with the invariant measure $\mathbb{P} = \nu$. Over this dynamical system, we take the RDS (random dynamical system) $(\mathcal{F}, \mathcal{V}_t)$ on the flag bundle. Suppose a permutation p is given. For ν-a.e. $x \in M$, let $K_x \subset \mathcal{F}_x$ be the set of flags $(x; \mathbb{F}_1 \subset \cdots \subset \mathbb{F}_n)$ which satisfies the assumptions of the statement (2) of Theorem 2.3. *This defines a random compact set K* which satisfies the assumptions of [1, Theorem 1.6.13]. Let $\mathcal{I}(\mathcal{V}_t|K)$ be the set of probability measures on $\mathcal{F}$, supported on K (see [1, Definition 1.6.4]), which project to ν, and which are invariant by the flow $\{\mathcal{V}_t\}$. That theorem then says that $\mathcal{I}(\mathcal{V}_t|K)$ is compact, convex and non-empty. By the Krein-Milman theorem, $\mathcal{I}(\mathcal{V}_t|K)$ contains an ergodic measure. This completes the proof. However, there is in fact an essential gap in the discussion above. If the ergodic system $(M, \{V_t\}; \nu)$ has the simple Lyapunov spectrum, K is obviously a random compact set. In general case, the assertion that K is a random set is nontrivial! The key difficulty is that we do not know how to show that the mapping $M \ni x \mapsto K_x \subset \mathcal{F}_x$ is measurable.

Conversely, Theorem 2.3 trivially implies Liao's Reordering Lemma. However, in the case of the reduced one stated in Theorem 1.1, it becomes very nontrivial. We shall explain this as follows.

We now define the projection $\bar{P}\colon \mathcal{C}^\sharp(M, \vec{V}) \to \mathcal{F}$ as follows: For any given $\vec{\gamma}_x = (x; \vec{v}_1, \ldots, \vec{v}_{n-1}) \in \mathcal{C}^\sharp(M, \vec{V})$, take some $\tilde{\gamma}_x = (x; \vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{n-1}) \in F_n^\sharp$; Let $\bar{P}(\vec{\gamma}_x) = P(\tilde{\gamma}_x)$ as in (2.5). Then $\bar{P} \circ \Theta_t^\sharp = \mathcal{V}_t \circ \bar{P}$ for all $t \in \mathbb{R}$. But we note that here $\bar{P}$ is not surjective and hence the similar formula to (2.6) does not exist. In view of this, for the permutation $p\colon \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$ in Theorem 1.1, although there exists some $\bar{\mu}^* \in \mathcal{M}_{erg}(\mathcal{F}, \{\mathcal{V}_t\})$ projecting on $\pi_*(\bar{\mu})$ realizing it, it is not necessary to exist some $\bar{\mu}' \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M, \vec{V}), \{\Theta_t^\sharp\})$ projecting on $\bar{\mu}^*$ via $\bar{P}$.

3. The Reduced Reordering and Spectrum Theorem

This section is devoted to the proofs of the main results. Before starting with the proof, we recall some notations for ergodicity. Suppose that X is a given compact metrizable space and $\{f_t\}\colon X \to X$ a given one-parameter group of transformations. For each $m \in \mathcal{M}_{erg}(X, \{f_t\})$, $Q_m(X, \{f_t\})$ denotes the set of all quasi-regular points corresponding to m; i.e., $x \in Q_m(X, \{f_t\})$ if and only if for all $\varphi \in C(X)$, the set of all real continuous functions on X, there exist

$$\lim_{T \to \pm\infty} \frac{1}{T} \int_0^T \varphi(f_t(x))\, dt = \int_X \varphi\, dm. \tag{3.1}$$

According to Birkhoff ergodic theorem [19], $Q_m(X, \{f_t\})$ is an $\{f_t\}$-invariant Borel subset of X with $m(Q_m(X, \{f_t\})) = 1$.

We first prove the reduced reordering theorem:

Theorem 3.1. (Reduced Reordering Lemma). *For any $\bar{\mu} \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M, \vec{V}), \{\Theta_t^\sharp\})$ and any permutation $k \mapsto p(k)$ of $\{1, \ldots, n-1\}$, there is $\bar{\mu}'$ in $\mathcal{M}_{erg}(\mathcal{C}^\sharp(M, \vec{V}), \{\Theta_t^\sharp\})$, such that*

(a) $\pi_*(\bar{\mu}) = \pi_*(\bar{\mu}') \in \mathcal{M}_{erg}(M, \vec{V})$.

(b) $\vartheta_k^*(\bar{\mu}') = \vartheta_{p(k)}^*(\mu)$ *for* $k = 1, \ldots, n-1$.

Where ϑ_k^* is defined as in (1.12).

Proof. For any $(x;\vec{\gamma})\in\mathcal{C}^\sharp(M,\vec{V})$, denote by $\mathbb{H}(\vec{\gamma}_x)$ the linear subspace of T_xM generated by the frame $\vec{\gamma}_x$. Write

$$G_{n-1}=\left\{(x;\mathbb{H}(\vec{\gamma}_x))\,|\,(x;\vec{\gamma})\in\mathcal{C}^\sharp(M,\vec{V})\right\} \tag{3.2}$$

the Grassmannian bundle whose fiber over $x\in M$ is

$$G_{n-1}(x)=\left\{(x;\mathbb{H}(\vec{\gamma}_x))\,|\,(x;\vec{\gamma})\in\mathcal{C}^\sharp(x,\vec{V})\right\},$$

where we regard $(x;\mathbb{H}(\vec{\gamma}_x))$ as a point of G_{n-1}. G_{n-1} is a compact metrizable space. Else, for any $(x;\mathbb{H})\in G_{n-1}$, if $\mathbb{H}=\mathbb{H}(\vec{\gamma}_x)$ for some $(x;\vec{\gamma})\in\mathcal{C}^\sharp(M,\vec{V})$, we then have $(V_t(x);\mathbb{H}(\Theta_t^\sharp(\vec{\gamma}_x))\in G_{n-1}(V_t(x))$ for all $t\in\mathbb{R}$. If $(x;\vec{\gamma}')\in\mathcal{C}^\sharp(M,\vec{V})$ is another frame such that $\mathbb{H}(\vec{\gamma}_x)=\mathbb{H}(\vec{\gamma}_x')$, we then easily see that $\mathbb{H}(\Theta_t^\sharp(\vec{\gamma}_x))=\mathbb{H}(\Theta_t^\sharp(\vec{\gamma}_x'))$. Setting

$$\varphi_t(x;\mathbb{H})=(V_t(x);\mathbb{H}(\Theta_t^\sharp(\vec{\gamma}_x))$$

we then obtain a well-defined skew-product flow based on $(M,\{V_t\})$

$$\{\varphi_t\}_{t\in\mathbb{R}}\colon G_{n-1}\to G_{n-1}. \tag{3.3}$$

In order to prove Theorem 3.1, we first show the following special case.

Lemma 3.1. *For any integer $1\leq\ell<n-1$ and for $\mu\in\mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$, there exists $\bar{\mu}\in\mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$, such that*

(*i*) *$\pi_*(\bar{\mu})=\pi_*(\mu)\in\mathcal{M}_{erg}(M,\vec{V})$, where $\pi\colon\mathcal{C}^\sharp(M,\vec{V})\to M$.*

(*ii*) *$\vartheta_k^*(\bar{\mu})=\vartheta_{r(k)}^*(\mu)$, where*

$$r(k)=\begin{cases} k, & k=1,\cdots,\ell-1,\ell+2,\cdots,n-1\\ \ell+1, & k=\ell\\ \ell, & k=\ell+1\end{cases}$$

Note that one may although think of $r\colon\mathcal{C}^\sharp(M,\vec{V})\to\mathcal{C}^\sharp(M,\vec{V})$ as a homeomorphism defined by $r(x;\vec{v}_1,\ldots,\vec{v}_{n-1})=(x;\vec{v}_{r(1)},\ldots,\vec{v}_{r(n-1)})$ for $(x;\vec{v}_1,\ldots,\vec{v}_{n-1})\in\mathcal{C}^\sharp(M,\vec{V})$, the difficulty is $\Theta_t^\sharp\circ r\neq r\circ\Theta_t^\sharp$ in general.

Proof of Lemma 3.1. Our proof here follows the frame of that of [14, Theorem 4.1]. Since in the case $n=2$ it is trivial, now let $q=n-3\geq 0$.

Suppose first $q>0$. Let

$$N_{n-1}=\left\{(x;\mathbb{H}(\vec{u}_1,\vec{u}_2);\vec{v}_1,\ldots,\vec{v}_q)\,|\,(x;\vec{u}_1,\vec{u}_2,\vec{v}_1,\ldots,\vec{v}_q)\in\mathcal{C}^\sharp(M,\vec{V})\right\} \tag{3.4}$$

which is endowed with the relative topology. Obviously N_{n-1} is a compact and metrizable space. We define the mapping

$$\tau\colon \mathcal{C}^\sharp(M,\vec{V}) \to N_{n-1} \tag{3.5}$$

by

$$\tau(\vec{\gamma}_x) = (x;\mathbb{H}(\mathrm{Pr}_\ell(\vec{\gamma}_x),\mathrm{Pr}_{\ell+1}(\vec{\gamma}_x));\vec{\alpha}) \tag{3.5a}$$

where

$$\vec{\alpha} = (\mathrm{Pr}_1(\vec{\gamma}_x),\dots,\mathrm{Pr}_{\ell-1}(\vec{\gamma}_x),\mathrm{Pr}_{\ell+2}(\vec{\gamma}_x),\dots,\mathrm{Pr}_{n-1}(\vec{\gamma}_x)) \tag{3.5b}$$

for any $(x;\vec{\gamma}) \in \mathcal{C}^\sharp(M,\vec{V})$. It is easy to see that the mapping τ is surjective.

For any $(x;\mathbb{H};\vec{\alpha}) \in N_{n-1}$, arbitrarily take some $(x;\vec{\gamma}) \in \mathcal{C}^\sharp(M,\vec{V})$ such that $\tau(\vec{\gamma}_x) = (x;\mathbb{H};\vec{\alpha})$ and put

$$\bar{\chi}_t(x;\mathbb{H};\vec{\alpha}) = \tau(\Theta_t^\sharp(\vec{\gamma}_x)) \quad \forall\, t \in \mathbb{R}. \tag{3.6}$$

We can easily check that $\bar{\chi}_t(x;\mathbb{H};\vec{\alpha})$ is defined independently of the choice of $\vec{\gamma}_x$, and further

$$\{\bar{\chi}_t\}_{t\in\mathbb{R}}\colon N_{n-1} \to N_{n-1} \tag{3.7}$$

is a skew-product flow based on $(M,\{V_t\})$ too. Since N_{n-1} is compact and metrizable, $\mathcal{M}_{erg}(N_{n-1},\{\bar{\chi}_t\})$ is non-void. Let $\rho\colon N_{n-1} \to M$ denote the natural bundle projection; that is, it projects each $(x;\mathbb{H};\vec{\alpha})$ onto its base point x in M.

If $q = 0$; i.e., $n = 3$, let $N_2 = G_{n-1} = G_2$, $\bar{\chi}_t = \varphi_t$, and let $\rho = \pi$ the bundle projection from G_2 onto M. Then, no matter when $q > 0$ or $q = 0$, we see that for all $t \in \mathbb{R}$ the commutativity holds in the following diagrams

$$\begin{array}{ccc} N_{n-1} & \xrightarrow{\bar{\chi}_t} & N_{n-1} \\ \rho\downarrow & & \downarrow\rho \\ M & \xrightarrow{V_t} & M \end{array} \qquad \begin{array}{ccc} \mathcal{C}^\sharp(M,\vec{V}) & \xrightarrow{\Theta_t^\sharp} & \mathcal{C}^\sharp(M,\vec{V}) \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow{V_t} & M \end{array} \qquad \begin{array}{ccc} \mathcal{C}^\sharp(M,\vec{V}) & \xrightarrow{\Theta_t^\sharp} & \mathcal{C}^\sharp(M,\vec{V}) \\ \tau\downarrow & & \downarrow\tau \\ N_{n-1} & \xrightarrow{\bar{\chi}_t} & N_{n-1} \end{array}$$

and hence the commutativity holds in the following diagram

$$\begin{array}{ccc} \mathcal{M}_{erg}(M,\vec{V}) & \xrightarrow{Id} & \mathcal{M}_{erg}(M,\vec{V}) \\ \pi_*\uparrow & & \uparrow\rho_* \\ \mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\}) & \xrightarrow{\tau_*} & \mathcal{M}_{erg}(N_{n-1},\{\bar{\chi}_t\}) \end{array} \tag{3.8}$$

When $q > 0$, we define functions $\bar{\omega}_k(x;\mathbb{H};\vec{\alpha})$ on N_{n-1} for $k = 1,\dots,q$ as follows: For $(x;\mathbb{H};\vec{\alpha}) \in N_{n-1}$ take a frame $\vec{\gamma}_x \in \mathcal{C}^\sharp(M,\vec{V})$ such that $\tau(\vec{\gamma}_x) = (x;\mathbb{H};\vec{\alpha})$, then put

$$\bar{\omega}_k(x;\mathbb{H};\vec{\alpha}) = \begin{cases} \omega_k^*(\vec{\gamma}_x), & \text{for } 1 \le k \le \ell - 1 \\ \omega_{k+2}^*(\vec{\gamma}_x), & \text{for } \ell \le k \le q \end{cases} \tag{3.9}$$

where ω_k^* is defined as in (1.11a). From the definitions of τ and ω_k^* we easily see that $\bar{\omega}_k$ is well defined and continuous on N_{n-1}.

For $\mu \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$, it is clear that

$$\int_{N_{n-1}} \bar{\omega}_k \, d(\tau_*\mu) = \begin{cases} \vartheta_k^*(\mu), & \text{for } 1 \le k \le \ell - 1 \\ \vartheta_{k+2}^*(\mu), & \text{for } \ell \le k \le q \end{cases} \tag{3.10}$$

Combining this with the commutativity in (3.8), we see that it suffices to show the existence of some $\bar{\mu} \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$, satisfying the following (3.11, 3.12 and 3.13):

$$\tau_*(\bar{\mu}) = \tau_*(\mu) \in \mathcal{M}_{erg}(N_{n-1},\{\bar{\chi}_t\}), \tag{3.11}$$

$$\vartheta_\ell^*(\bar{\mu}) = \vartheta_{\ell+1}^*(\mu), \tag{3.12}$$

$$\vartheta_{\ell+1}^*(\bar{\mu}) = \vartheta_\ell^*(\mu). \tag{3.13}$$

In fact, if $\vartheta_\ell^*(\mu) = \vartheta_{\ell+1}^*(\mu)$, we then take $\bar{\mu} = \mu$ which satisfies all the requirements. In the following, we consider the case $\vartheta_\ell^*(\mu) < \vartheta_{\ell+1}^*(\mu)$. Let us take $(x_0;\vec{\beta}_0) \in Q_\mu(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$ and $(x_0;\vec{\gamma}_0) \in \tau^{-1}(Q_{\tau_*\mu}(N_{n-1},\{\bar{\chi}_t\}))$ such that

$$\tau(\vec{\beta}_{0x_0}) = \tau(\vec{\gamma}_{0x_0}) \text{ and } \mathrm{Pr}_{\ell+1}\vec{\beta}_{0x_0} = \mathrm{Pr}_\ell\vec{\gamma}_{0x_0}$$

where τ as in (3.5) and Pr_ℓ as in (1.4). Then, applying Liao's Measure Lifting Lemma ([14, Lemma 2.2]) to the semi-conjugate $\tau\colon \mathcal{C}^\sharp(M,\vec{V}) \to N_{n-1}$ and $\omega_\ell^*\colon \mathcal{C}^\sharp(M,\vec{V}) \to \mathbb{R}$, and using (1.11c), we take some $\bar{\mu} \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\})$ such that (3.11) and

$$\begin{aligned} \vartheta_\ell^*(\bar{\mu}) &\ge \limsup_{T\to\infty} \frac{1}{T}\int_0^T \omega_\ell^*(\Theta_t^\sharp(\vec{\gamma}_{0x_0}))\,dt \\ &= \limsup_{T\to\infty} \frac{1}{T}\log\|\mathrm{Pr}_\ell\Theta_T(\vec{\gamma}_{0x_0})\| \\ &\ge \limsup_{T\to\infty} \frac{1}{T}\log\|\mathrm{Pr}_{\ell+1}\Theta_T(\vec{\beta}_{0x_0})\| \\ &= \vartheta_{\ell+1}^*(\mu) \end{aligned}$$

where $\Theta_t(\vec{\gamma}_x)$ is defined as in (1.9). Therefore

$$\vartheta_\ell^*(\bar{\mu}) \geq \vartheta_{\ell+1}^*(\mu) > \vartheta_\ell^*(\mu). \tag{3.14}$$

From (3.11) we gain

$$\tau_*\mu\left(\tau(Q_\mu(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\}))\bigcap\tau(Q_{\bar{\mu}}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\}))\right)=1$$

and hence we can take some

$$(x';\vec{\beta}')\in Q_\mu(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\}) \text{ and } (x';\vec{\gamma}')\in Q_{\bar{\mu}}(\mathcal{C}^\sharp(M,\vec{V}),\{\Theta_t^\sharp\}) \tag{3.15}$$

such that $\tau(\vec{\beta}'_{x'})=\tau(\vec{\gamma}'_{x'})$, which implies $\mathbb{H}(\vec{\beta}'_{x'})=\mathbb{H}(\vec{\gamma}'_{x'})$.

For such $\vec{\beta}'_{x'}$ and $\vec{\gamma}'_{x'}$, since $\mathbb{H}(\vec{\beta}'_{x'})=\mathbb{H}(\vec{\gamma}'_{x'})$, we may take a unit vector say $\vec{\mathbf{a}}\in T_{x'}M$, such that $\langle\vec{\mathbf{a}},\mathrm{Pr}_k(\vec{\beta}'_{x'})\rangle=\langle\vec{\mathbf{a}},\mathrm{Pr}_k(\vec{\gamma}'_{x'})\rangle=0$ for $k=1,\dots,n-1$. For the convenience, we write

$$\mathbb{H}_{x't}=\mathbb{H}(\Theta_t^\sharp(\vec{\beta}'_{x'})). \tag{3.16}$$

We put

$$\mathcal{V}^*_{\vec{\mathbf{a}},t}(\vec{u})=\mathcal{V}_t(\vec{u})-\frac{\langle\mathcal{V}_t(\vec{u}),\mathcal{V}_t(\vec{\mathbf{a}})\rangle}{\|\mathcal{V}_t(\vec{\mathbf{a}})\|^2}\mathcal{V}_t(\vec{\mathbf{a}})$$

for all $\vec{u}\in\mathbb{H}_{x'0}$ and $t\in\mathbb{R}$. Then $\mathcal{V}^*_{\vec{\mathbf{a}},t}(\vec{u})\in\mathbb{H}_{x't}$ and

$$\mathcal{V}^*_{\vec{\mathbf{a}},t}\colon\mathbb{H}_{x'0}\to\mathbb{H}_{x't}\quad\forall\, t\in\mathbb{R} \tag{3.17}$$

is a family of linear mappings.

Put

$$\lambda_k=\min_{\delta=\pm1}\|\mathrm{Pr}_\ell\Theta_k^\sharp(\vec{\beta}'_{x'})-\delta\mathrm{Pr}_\ell\Theta_k^\sharp(\vec{\gamma}'_{x'})\| \tag{3.18}$$

for any $k=0,1,\dots$. We then consider the sequence $\{\lambda_k\}$ and assert that

$$\limsup_{k\to\infty}\lambda_k>0. \tag{3.19}$$

Otherwise, by continuity of the function ω_ℓ^* we have

$$\lim_{k\to\infty}\frac{1}{k}\int_0^k[\omega_\ell^*(\Theta_t^\sharp(\vec{\beta}'_{x'}))-\omega_\ell^*(\Theta_t^\sharp(\vec{\gamma}'_{x'}))]\,dt=0$$

which by the choice of $\vec{\beta}'_{x'}$ and $\vec{\gamma}'_{x'}$ would lead to a contradiction to (3.14). It then follows from the definition of the map τ that $\mathrm{Pr}_\ell\vec{\beta}'_{x'}\neq\pm\mathrm{Pr}_\ell\vec{\gamma}'_{x'}$ and $\mathrm{Pr}_{\ell+1}\vec{\beta}'_{x'}\neq\pm\mathrm{Pr}_{\ell+1}\vec{\gamma}'_{x'}$.

Denote by v_k the projection of $\mathrm{Pr}_\ell\Theta^\sharp_k(\vec{\gamma}'_{x'})$ on the 1-dimensional linear space generated by the vector $\mathrm{Pr}_{\ell+1}\Theta^\sharp_k(\vec{\beta}'_{x'})$ for each $k = 0, 1, \dots$. Then, by the forgoing assertion we can take some real number λ with $0 < \lambda \leq 1$ and a subsequence $\{v_{k_j}\}$ of $\{v_k\}$, $k_j > 0$, such that

$$v_0 = s_0 \mathrm{Pr}_{\ell+1}\vec{\beta}'_{x'} \text{ with } \lambda \leq |s_0| \leq 1, \tag{3.20}$$

$$v_{k_j} = s_j \mathrm{Pr}_{\ell+1}\Theta^\sharp_{k_j}(\vec{\beta}'_{x'}) \text{ with } \lambda \leq |s_j| \leq 1,\ j = 1, 2, \cdots. \tag{3.21}$$

The equality (3.20) implies that $\mathcal{V}^*_{\mathbf{\bar{a}},k_j}(\mathrm{Pr}_\ell\vec{\gamma}'_{x'}) = u(k_j) + s_0\mathcal{V}^*_{\mathbf{\bar{a}},k_j}(\mathrm{Pr}_{\ell+1}(\vec{\beta}'_{x'}))$ and hence

$$\mathrm{Pr}_\ell\Theta_{k_j}(\vec{\gamma}'_{x'}) = v(k_j) + s_0\|\mathrm{Pr}_{\ell+1}\Theta_{k_j}(\vec{\beta}'_{x'})\|\mathrm{Pr}_{\ell+1}\Theta^\sharp_{k_j}(\vec{\beta}'_{x'}), \tag{3.22}$$

where $u(k_j)$ and $v(k_j)$ both in the linear space

$$H^* := \mathbb{H}(\mathrm{Pr}_1\Theta^\sharp_{k_j}(\vec{\beta}'_{x'}), \dots, \mathrm{Pr}_\ell\Theta^\sharp_{k_j}(\vec{\beta}'_{x'})).$$

From (3.21) we obtain $\mathrm{Pr}_\ell\Theta^\sharp_{k_j}(\vec{\gamma}'_{x'}) = w(k_j) + v_{k_j}$, where $w(k_j) \in H^*$, and hence

$$\mathrm{Pr}_\ell\Theta_{k_j}(\vec{\gamma}'_{x'}) = \|\mathrm{Pr}_\ell\Theta_{k_j}(\vec{\gamma}'_{x'})\|w(k_j) + s_j\|\mathrm{Pr}_\ell\Theta_{k_j}(\vec{\gamma}'_{x'})\|\mathrm{Pr}_{\ell+1}\Theta^\sharp_{k_j}(\vec{\beta}'_{x'}). \tag{3.23}$$

From (3.22) and (3.23), it follows that

$$\|\mathrm{Pr}_{\ell+1}\Theta_{k_j}(\vec{\beta}'_{x'})\| = \frac{|s_j|}{|s_0|}\mathrm{Pr}_\ell\Theta_{k_j}(\vec{\gamma}'_{x'}), \tag{3.24}$$

for each $j = 1, 2, \dots$. This implies that

$$\begin{aligned}\vartheta^*_{\ell+1}(\mu) &= \lim_{j\to\infty}\frac{1}{k_j}\log\|\mathrm{Pr}_\ell\Theta_{k_j}(\vec{\gamma}'_{x'})\| \\ &= \vartheta^*_\ell(\bar{\mu})\end{aligned}$$

which proved (3.12). If we let v_k be the projection of $\mathrm{Pr}_\ell\Theta^\sharp_k(\vec{\beta}'_{x'})$ on the direction determined by $\mathrm{Pr}_{\ell+1}\Theta^\sharp_k(\vec{\gamma}'_{x'})$, for $k = 0, 1, 2, \dots$, then in a similar way, we can prove (3.13). Therefore Lemma 3.1 is shown in the case $\vartheta^*_\ell(\mu) < \vartheta^*_{\ell+1}(\mu)$.

In the other case $\vartheta^*_{\ell+1} < \vartheta^*_\ell(\mu)$, we consider $\{V'_t\} = \{V_{-t}\}\colon M \to M$. In the same way, we can verify the statements (3.11, 3.12 and 3.13).

Thus, Lemma 3.1 is true. ■

By applying Lemma 3.1 repeatedly one completes the proof of Theorem 3.1. ■

We next prove the reduced spectrum theorem by a method different from that mentioned in Section 2.

Theorem 3.2. (Reduced Spectrum Theorem). *For any C^1 differential system $\vec{V}$ on M, the following properties hold.*

(A) *If* $\nu \in \mathcal{M}_{erg}(M, \vec{V})$ *and* $\bar{\mu} \in \mathcal{M}_{erg}(\mathcal{C}^\sharp(M, \vec{V}), \{\Theta_t^\sharp\})$ *satisfy* $\pi_*(\bar{\mu}) = \nu$*, then*

$$\{\vartheta_k^*(\bar{\mu}) \,|\, 1 \leq k \leq n-1\} \subset \mathsf{Sp}(\vec{V}; \nu),$$

where $\mathsf{Sp}(\vec{V}; \nu)$ *denotes the Oseledec characteristic spectrum of* ν *counted with the multiplicity.*

(B) *For any given non-atomic measure* $\nu \in \mathcal{M}_{erg}(M, \vec{V})$*, if* $\bar{\mu}_1$ *and* $\bar{\mu}_2$ *both belong to* $\mathcal{M}_{erg}(\mathcal{C}^\sharp(M, \vec{V}), \{\Theta_t^\sharp\})$ *with* $\pi_*(\bar{\mu}_1) = \pi_*(\bar{\mu}_2) = \nu$*, then*

$$\mathsf{Sp}^*(\vec{V}; \nu) = \{\vartheta_k^*(\bar{\mu}_1) \,|\, 1 \leq k \leq n-1\} = \{\vartheta_k^*(\bar{\mu}_2) \,|\, 1 \leq k \leq n-1\}$$

counting with multiplicity.

Proof. In order to prove the statement (A), we divide it into two cases.

At first, we let $\bar{\mu}(\mathcal{C}^\sharp(S(\vec{V}), \vec{V})) = 1$, where and in the sequel $S(\vec{V})$ stands for the set of singularities of $\vec{V}$. In this case, we define

$$\mathrm{Pr}_{2,\ldots,n} \colon F_n^\sharp|_{S(\vec{V})} \to \mathcal{C}^\sharp(S(\vec{V}), \vec{V})$$

by

$$\mathrm{Pr}_{2,\ldots,n}(x; \vec{\gamma}) = (x; \mathrm{Pr}_2(\vec{\gamma}_x), \ldots, \mathrm{Pr}_n(\vec{\gamma}_x))$$

for each $(x; \vec{\gamma}) \in F_n^\sharp|_{S(\vec{V})}$. Clearly, $\mathrm{Pr}_{2,\ldots,n}$ is continuous and surjective, and such that $\mathrm{Pr}_{2,\ldots,n} \circ \mathfrak{V}_t^\sharp = \Theta_t^\sharp \circ \mathrm{Pr}_{2,\ldots,n}$ on $F_n^\sharp|_{S(\vec{V})}$. Then, by the definition of ω_k^* and the Measure Lifting Lemma which guarantees

$$(\mathrm{Pr}_{2,\ldots,n})_* \mathcal{M}_{erg}(F_n^\sharp|_{S(\vec{V})}, \{\mathfrak{V}_t^\sharp\}) = \mathcal{M}_{erg}(\mathcal{C}^\sharp(S(\vec{V}), \vec{V}), \{\Theta_t^\sharp\}),$$

the desired result is true under this hypothesis.

Secondly, we assume that $\bar{\mu}(\mathcal{C}^\sharp(S(\vec{V}), \vec{V}) = 0$. Since $\mathcal{C}^\sharp(R(\vec{V}), \vec{V})$ is an open $\{\Theta_t^\sharp\}$-invariant subset of $\mathcal{C}^\sharp(M, \vec{V})$, hence $\bar{\mu}(\mathcal{C}^\sharp(R(\vec{V}), \vec{V}) = 1$. Moreover, we can naturally imbed it into $F_n^\sharp$ in the way: $(x; \vec{\gamma}) \mapsto (x; \vec{V}(x)/\|\vec{V}(x)\|, \vec{\gamma})$ for $(x; \vec{\gamma}) \in \mathcal{C}^\sharp(R(\vec{V}), \vec{V})$. From Liao's Spectrum Theorem, it easily follows that $\{\vartheta_k^*(\bar{\mu}) \,|\, k = 1, \ldots, n-1\}$ is contained in $\mathsf{Sp}(\vec{V}; \nu)$.

We thus proved the statement (A).

In order to prove the statement (B), we need an ergodicity lemma.

Lemma 3.2. *Let* $\{f_t\}_{t \in \mathbb{R}}$ *be a one-parameter group of transformations on a compact metric space* X*. For given* $\bar{\mu} \in \mathcal{M}_{erg}(X, \{f_t\})$*, there exists* $x_0 \in Q_{\bar{\mu}}(X, \{f_t\})$ *such that* $\bar{\mu}\left(\overline{\{f_t(x_0) \,|\, t \in \mathbb{R}\}}\right) = 1$*. In fact,* $\bar{\mu}$*-a.e.* $x_0 \in Q_{\bar{\mu}}(X, \{f_t\})$ *is such that* $\overline{\{f_t(x_0) \,|\, t \in \mathbb{R}\}} = \mathrm{supp}(\bar{\mu})$*.*

Proof. The argument is standard [19] and so we omit the details. ■

Corollary 3.3. *For any given $\bar{\mu} \in \mathcal{M}_{erg}(X, \{f_t\})$, if $\bar{\mu}(U) > 0$ for every open set U of X, then for $\bar{\mu}$-a.e. $x_0 \in X$, we have $X = \overline{\{f_t(x_0) | t \in \mathbb{R}\}}$.*

We now proceed the proof of the statement (B). Under the hypotheses of statement (B), it is easy to choose two reduced $(n-1)$-frames $(p_0; \vec{\gamma}_1)$ and $(p_0; \vec{\gamma}_2)$ in $\mathcal{C}^\sharp(M, \vec{V})$ such that $p_0 \in Q_\nu(M, \{V_t\})$ is a regular point (i.e. $\vec{V}(p_0) \neq \vec{0}_{p_0}$) and the requirements of Lemma 3.2 corresponding to $\bar{\mu}_1$ and $\bar{\mu}_2$ respectively are satisfied. For the given $(p_0; \vec{\gamma}_i), i = 1, 2$, we, respectively, take and then fix $(p_0; \widetilde{\gamma}_i) \in F_n^\sharp$ such that

$$\vec{\gamma}_i = \mathrm{Pr}_{2,\ldots,n}(\widetilde{\gamma}_{ip_0}) \text{ and } \mathrm{Pr}_1(\widetilde{\gamma}_{ip_0}) = \frac{\vec{V}(p_0)}{\|\vec{V}(p_0)\|}.$$

Define

$$\mathcal{I}\colon \{\Theta_t^\sharp(p_0; \vec{\gamma}_i) \,|\, t \in \mathbb{R}\} \to F_n^\sharp$$

by

$$\mathcal{I}(\Theta_t^\sharp(p_0; \vec{\gamma}_i)) = \mathfrak{V}_t^\sharp(p_0; \widetilde{\gamma}_i) \text{ for } s \in \mathbb{R}.$$

Clearly, $\mathcal{I}$ is continuous under the natural relative topologies. In a natural way one can define the mapping

$$\mathcal{I}\colon \overline{\mathrm{Orb}}_{\widetilde{\gamma}_{ip_0}} := \overline{\{\Theta_t^\sharp(p_0; \vec{\gamma}_i) \,|\, t \in \mathbb{R}\}} \to F_n^\sharp \tag{3.25}$$

such that $\mathfrak{V}_t^\sharp \circ \mathcal{I} = \mathcal{I} \circ \Theta_t^\sharp$ for all $t \in \mathbb{R}$.

Since

$$X_i := \mathcal{I}\left(\overline{\{\Theta_t^\sharp(p_0; \vec{\gamma}_i) \,|\, t \in \mathbb{R}\}}\right)$$

is a closed $\{\mathfrak{V}_t^\sharp\}$-invariant subset of $F_n^\sharp$, there exists $\bar{\mu}_i^* \in \mathcal{M}_{erg}(F_n^\sharp, \{\mathfrak{V}_t^\sharp\})$ such that

- $\pi_*(\bar{\mu}_i) = \pi_*(\bar{\mu}_i^*) = \nu$, where the first π is defined on $\mathcal{C}^\sharp(M, \vec{V})$ and the second on $F_n^\sharp$.
- $\bar{\mu}_i^* = \mathcal{I}_*(\bar{\mu}_i)$.
- $\bar{\mu}_i^*(X_i) = 1$.

From the definition of the reduced qualitative function ω_k^*, it follows that

$$\vartheta_k^*(\bar{\mu}_i) = \int_{F_n^\sharp} \omega_{k+1} d\bar{\mu}_i^*, \ k = 1, 2, \ldots, n-1; i = 1, 2.$$

Hence by the fact $\mathcal{I}(Q_{\bar{\mu}_i}(\overline{\mathrm{Orb}}_{\vec{\gamma}_{ip_0}}, \{\Theta_t^\sharp\})) \subseteq Q_{\bar{\mu}_i^*}(X_i, \{\mathfrak{V}_t^\sharp\})$, we have

$$\begin{aligned}
\textstyle\int_{F_n^\sharp} \omega_1 d\bar{\mu}_i^* &= \int_{X_i} \omega_1 d\bar{\mu}_i^* \\
&= \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T \omega_1(\mathfrak{V}_t^\sharp(p_0;\widetilde{\gamma}_i))dt \\
&= \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T \omega_1(\frac{\vec{V}(V_t(p_0))}{\|\vec{V}(V_t(p_0))\|})dt
\end{aligned}$$

for $i = 1, 2$. This implies

$$\mathsf{Sp}^*(\vec{V};\nu) = \{\vartheta_k^*(\bar{\mu}_1) \,|\, 1 \leq k \leq n-1\} = \{\vartheta_k^*(\bar{\mu}_2) \,|\, 1 \leq k \leq n-1\}.$$

Therefore, the statement (B) and hence Theorem 3.2 are shown. ■

Now, Theorem 1.1 comes immediately from Theorems 3.1 and 3.2.

4. The Spectra of Parallelepiped

In this section, we will use the reordering and spectrum theorems to study the characteristic spectra of parallelepiped in the tangent bundles. Using the projections $P_k\colon F_k^\sharp \to G_k(n), 1 \leq k \leq n$, defined by $P_k(x;\vec{\gamma}) = (x;\mathbb{H}(\vec{\gamma}_x))$ for $(x;\vec{\gamma}) \in F_k^\sharp$, the theorem 4.1 below is a direct corollary of Arnold [1, Theorems 6.3.1 and 5.3.1]. For the reduced case this approach, however, does not work. We will give a Liaowise proof for Theorem 4.1 below, since it still works in the reduced one.

Let $\vec{V}$ be a given vector field on M of class C^1.

4.1 Non-reduced cases.

For given positive integer ℓ with $1 \leq \ell \leq n$, we define

$$\mathrm{vol}\colon F_\ell^\sharp \times \mathbb{R} \to \mathbb{R}^+ \tag{4.1}$$

by

$$\mathrm{vol}(\vec{\gamma}_x, t) = \mathrm{volume}(\Pi(V_t(\vec{\gamma}_x))) \tag{4.1a}$$

for all $((x;\vec{\gamma}), t) \in F_\ell^\sharp \times \mathbb{R}$, where $\mathrm{volume}(\Pi(\vec{\gamma}_x))$ means the volume of the parallelepiped $\Pi(\vec{\gamma}_x)$ determined by the ℓ-frame $(x;\vec{\gamma}) \in U_\ell$ and, $V_t\colon U_\ell \to U_\ell$ is defined by $\vec{V}$ as in (1.5). For given $(x;\vec{\gamma}) \in F_\ell^\sharp$, we write simply

$$\mathrm{vol}_{\vec{\gamma}_x}(t) = \mathrm{vol}(\vec{\gamma}_x, t). \tag{4.1b}$$

For any given $\nu \in \mathcal{M}_{erg}(M, \vec{V})$ and $\mu \in \mathcal{M}_{erg}(F_n^\sharp, \{\mathfrak{V}_t^\sharp\})$ with $\nu = \pi_*(\mu)$, the Oseledec spectrum $\mathit{Sp}(\vec{V}; \nu)$ of ν coincides with

$$\left\{ \int_{F_n^\sharp} \omega_k(\vec{\gamma}_x) d\mu(\vec{\gamma}_x) \mid k = 1, 2, \ldots, n \right\} \tag{4.2}$$

from Liao's Spectrum Theorem (Theorem 2.1).

Theorem 4.1. *Let $\nu \in \mathcal{M}_{erg}(M, \vec{V})$ and $\mathit{Sp}(\vec{V}; \nu) = \{\lambda_1 \geq \cdots \geq \lambda_n\}$. Then, for any given $1 \leq i_1 < \cdots < i_k \leq n$, there exists a $\mu_k \in \mathcal{M}_{erg}(F_k^\sharp, \{\mathfrak{V}_t^\sharp\})$, such that*

(1) $\pi_*(\mu_k) = \nu$;

(2) *there exists a $\{\mathfrak{V}_t^\sharp\}$-invariant Borel subset $P(\nu; \mu_k)$ of $F_k^\sharp$ with μ_k-measure one such that*
$$\lim_{t \to \pm\infty} \frac{1}{t} \log \mathrm{vol}_{\vec{\gamma}_x}(t) = \sum_{j=1}^{k} \lambda_{i_j}$$
for all $(x; \vec{\gamma}) \in P(\nu; \mu_k)$.

Proof. By Liao's Reordering Lemma, we can choose an ergodic measure $\bar{\mu}_k$ in $\mathcal{M}_{erg}(F_n^\sharp, \{\mathfrak{V}_t^\sharp\})$ such that $\pi_*(\bar{\mu}_k) = \nu$ and

$$\lambda_{i_j} = \int_{F_n^\sharp} \omega_j d\bar{\mu}_k \quad j = 1, \ldots, k.$$

Define

$$\mathrm{Pr}_{1,\ldots,k} \colon F_n^\sharp \to F_k^\sharp \tag{4.3}$$

by

$$\mathrm{Pr}_{1,\ldots,k}(x; \vec{u}_1, \ldots, \vec{u}_n) = (x; \vec{u}_1, \ldots, \vec{u}_k). \tag{4.3a}$$

Then $\mathfrak{V}_t^\sharp \circ \mathrm{Pr}_{1,\ldots,k} = \mathrm{Pr}_{1,\ldots,k} \circ \mathfrak{V}_t^\sharp$ for all $t \in \mathbb{R}$, where the first $\mathfrak{V}_t^\sharp$ is on $F_k^\sharp$ and the second on $F_n^\sharp$. Furthermore, we may take $\mu_k \in \mathcal{M}_{erg}(F_k^\sharp, \{\mathfrak{V}_t^\sharp\})$ such that $\nu = \pi_*(\mu_k)$ and $\mu_k = (\mathrm{Pr}_{1,\cdots,k})_*(\bar{\mu}_k)$. Moreover,

$$\lambda_{i_j} = \int_{F_k^\sharp} \omega_j \, d\mu_k \quad j = 1, \ldots, k. \tag{4.4}$$

Define

$$h_j \colon F_k^\sharp \times \mathbb{R} \to \mathbb{R}^+$$

by

$$h_j(\vec{\gamma}_x, t) = \|\mathrm{Pr}_j \circ \mathfrak{V}_t(\vec{\gamma}_x)\|$$

and write $h_{\vec{\gamma},j}(t) = h_j(\vec{\gamma}, t)$. It is clear that

$$\mathrm{vol}_{\vec{\gamma}_x}(t) = h_{\vec{\gamma}_x,1}(t) \cdot \cdots \cdot h_{\vec{\gamma}_x,k}(t) \tag{4.6}$$

for $(\vec{\gamma}_x, t) \in F_k^\sharp \times \mathbb{R}$. Then the function vol satisfies the following properties:

- vol is a continuous function on $F_k^\sharp \times \mathbb{R}$.
- for any $(x; \vec{\gamma}) \in F_k^\sharp$, for any $s, t \in \mathbb{R}$, $\mathrm{vol}_{\mathfrak{V}_s^\sharp(\vec{\gamma}_x)}(t) = \frac{\mathrm{vol}_{\vec{\gamma}_x}(s+t)}{\mathrm{vol}_{\vec{\gamma}_x}(s)}$.
- $\vec{\gamma}_x \mapsto \frac{d}{dt}|_{t=0}\mathrm{vol}_{\vec{\gamma}_x}(t)$ exists and is continuous with respect to $(x; \vec{\gamma}) \in F_k^\sharp$.

From the Birkhoff Ergodic Theorem, it follows that there exists a $\{\mathfrak{V}_t^\sharp\}$-invariant subset $P(\nu; \mu_k)$ of $F_k^\sharp$ such that

- $\mu_k(P(\nu; \mu_k)) = 1$.
- for all $(x; \vec{\gamma}) \in P(\nu; \mu_k)$, the limit

$$\begin{aligned}\lim_{t\to\pm\infty} \tfrac{1}{t} \log \mathrm{vol}_{\vec{\gamma}_x}(t) &= \int_{F_k^\sharp} \frac{d}{dt}|_{t=0}\mathrm{vol}_{\vec{\gamma}_x}(t)\, d\mu_k \\ &= \int_{F_k^\sharp} \sum_{j=1}^k \omega_j \, d\mu_k\end{aligned}$$

exists.

By (4.4), we easily conclude the statement (2). ■

4.2 Reduced cases.

For the reduced case, by Theorem 1.1, there will be a similar consequence. Recall that $R(\vec{V}) = \{x \in M \,|\, \vec{V}(x) \neq \vec{0}_x\}$ is the regular point set of the given C^1 vector field $\vec{V}$ on the n-dimensional manifold M. For a positive integer k with $1 \leq k \leq n-1$, write

$$\mathcal{C}_k^\sharp(R(\vec{V})) = \Big\{(x; \vec{\gamma}) \in F_k^\sharp \,|\, x \in R(\vec{V}), \langle \vec{V}(x), \mathrm{Pr}_i(\vec{\gamma}_x)\rangle = 0, i = 1, \ldots, k\Big\}.$$

In a way similar to $\mathcal{C}^\sharp(M, \vec{V})$, we have a natural skew-product flow based on $(R(\vec{V}), \{V_t\})$

$$\{\Theta_t^\sharp\}_{t\in\mathbb{R}} \colon \mathcal{C}_k^\sharp(R(\vec{V})) \to \mathcal{C}_k^\sharp(R(\vec{V})). \tag{4.7}$$

For any $t \in \mathbb{R}$, define the mapping

$$\psi_t \colon \mathcal{C}_k^\sharp(R(\vec{V})) \to F_k$$

by

$$\psi_t(\vec{\gamma}_x) = \mathrm{Pr}_{2,\dots,k+1}(\mathfrak{V}_t(x; \vec{V}(x), \vec{\gamma}))$$

for $(x; \vec{\gamma}) \in \mathcal{C}_k^\sharp(R(\vec{V}))$, where

$$\mathrm{Pr}_{2,\dots,k+1} \colon (x; \vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k+1}) \mapsto (x; \vec{v}_2, \dots, \vec{v}_{k+1})$$

and $\{\mathfrak{V}_t\} \colon F_{k+1} \to F_{k+1}$ as in (1.5). We then define

$$\mathrm{vol}^* \colon \mathcal{C}_k^\sharp(R(\vec{V})) \times \mathbb{R} \to \mathbb{R}^+ \tag{4.8}$$

by

$$\mathrm{vol}^*(\vec{\gamma}_x, t) = \mathrm{volume}(\Pi(\psi_t(\vec{\gamma}_x)))$$

for $(\vec{\gamma}_x, t) \in \mathcal{C}_k^\sharp(R(\vec{V})) \times \mathbb{R}$.

Then we have a similar theorem.

Theorem 4.2. *Suppose $\nu \in \mathcal{M}_{erg}(M, \vec{V})$ satisfies $\nu(R(\vec{V})) = 1$. Then, to any given k exponents $\{\lambda_1^*, \dots, \lambda_k^*\} \subset \mathsf{Sp}^*(\vec{V}; \nu)$, there is $\mu_k^* \in \mathcal{M}_{erg}(\mathcal{C}_k^\sharp(R(\vec{V})), \{\Theta_t^\sharp\})$, such that*

(1) $\pi_*(\mu_k^*) = \nu$;

(2) *there exists a $\{\Theta_t^\sharp\}$-invariant Borel subset $P^*(\nu; \mu_k^*)$ of $\mathcal{C}_k^\sharp(R(\vec{V}))$ with μ_k^*-full measure such that*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \mathrm{vol}^*_{\vec{\gamma}_x}(t) = \sum_{i=1}^k \lambda_i^*$$

for all $(x; \vec{\gamma}) \in P^*(\nu; \mu_k^*)$.

Proof. Under the condition $\nu(R(\vec{V})) = 1$, we have the reordering lemma for the skew-product flow $(\mathcal{C}^\sharp(R(\vec{V})), \{\Theta_t^\sharp\})$. Then, the remaining procedure of the proof is parallel to that of Theorem 4.1. ∎

4.3 Comparison results.

For given l with $2 \le l \le n$. Define

$$\mathrm{vol}_i \colon F_l^\sharp \times \mathbb{R} \to \mathbb{R}^+ \quad i = 1, \dots, l \tag{4.9}$$

by

$$\mathrm{vol}_i(\vec{\gamma}_x, t) = \mathrm{vol}(\mathrm{Pr}_{1,\dots,i}(\vec{\gamma}_x), t)$$

for $(\vec{\gamma}_x, t) \in F_l^\sharp \times \mathbb{R}$, where $\mathrm{Pr}_{1,\dots,i}$ as in (4.3). For any given $(x; \vec{\gamma}) \in F_l^\sharp$, we simply write

$$\mathrm{vol}_{\vec{\gamma}_x,i}(t) = \mathrm{vol}_i(\vec{\gamma}_x, t).$$

Then, as a direct application of Liao's Comparison Theorem ([12, Theorem 6.1] or [18, Theorem 1.6.1]), we can obtain the following

Theorem 4.3. *Let $\mu \in \mathcal{M}_{erg}(F_l^\sharp, \{\mathfrak{V}_t^\sharp\}), n \geq l \geq 2$. Then there exists some permutation $i \mapsto p(i)$ of $\{1, \dots, l\}$ such that the set of points $(x; \vec{\gamma}) \in F_l^\sharp$ with the following properties:*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \frac{\min\{\mathrm{vol}_{\vec{\gamma}_x,p(1)}(t),\dots,\mathrm{vol}_{\vec{\gamma}_x,p(k-1)}(t)\}}{\max\{\mathrm{vol}_{\vec{\gamma}_x,p(k)}(t),\dots,\mathrm{vol}_{\vec{\gamma}_x,p(l)}(t)\}}dt > 0$$

$$\lim_{T\to-\infty} \frac{1}{T}\int_0^T \frac{\min\{\mathrm{vol}_{\vec{\gamma}_x,p(k)}(t),\dots,\mathrm{vol}_{\vec{\gamma}_x,p(l)}(t)\}}{\max\{\mathrm{vol}_{\vec{\gamma}_x,p(1)}(t),\dots,\mathrm{vol}_{\vec{\gamma}_x,p(k-1)}(t)\}}dt > 0$$

for $k = 2, \dots, l$, is a $\{\mathfrak{V}_t^\sharp\}$-invariant Borel subset of $F_l^\sharp$, which has μ-measure one.

For the reduced case, there is a similar result, but we omit the discussion.

5. Proof of Corollary 1.3

In order to prove the statement, we need a semi-uniform ergodic theorem which is a special case of [7, Lemma 3.1].

Theorem 5.1. *Suppose that $\{\Upsilon_t\}_{t\in\mathbb{R}}\colon Y \to Y$ is a C^0-flow on a compact metrizable space Y and $\varphi\colon Y \to \mathbb{R}$ a continuous function.*

(1) *If φ is such that there exists a constant $a \in \mathbb{R}$ with*

$$\int_Y \varphi\, d\mu < a \quad \forall\, \mu \in \mathcal{M}_{erg}(Y, \{\Upsilon_t\}),$$

then there exists some $\delta > 0$ such that

$$\int_Y \varphi\, d\mu \leq a - \delta \quad \forall\, \mu \in \mathcal{M}_{erg}(Y, \{\Upsilon_t\})$$

and given $\varepsilon > 0$, there is a $T_0 > 0$ such that for all $T \geq T_0$ we have

$$\frac{1}{T}\int_0^T \varphi(\Upsilon_t(y))\, dt < a - \delta + \varepsilon$$

for all $y \in Y$.

(2) *If φ is such that there exists a constant $b \in \mathbb{R}$ with*

$$\int_Y \varphi\, d\mu > b \quad \forall\, \mu \in \mathcal{M}_{erg}(Y, \{\Upsilon_t\}),$$

then, there exists some $\delta > 0$ such that

$$\int_Y \varphi\, d\mu \geq b + \delta \quad \forall\, \mu \in \mathcal{M}_{erg}(Y, \{\Upsilon_t\})$$

and given $\varepsilon > 0$, there is a $T_0 > 0$ such that for all $T \geq T_0$ we have

$$\frac{1}{T}\int_0^T \varphi(\Upsilon_t(y))\, dt > b + \delta - \varepsilon$$

for all $y \in Y$.

Let M_0 be given as in Corollary 1.3. Now, let

$$Y = \mathcal{C}^\sharp(M_0, \vec{V}) := \Big\{(x, \vec{\gamma}) \in \mathcal{C}^\sharp(M, \vec{V}) \mid x \in M_0\Big\} \tag{5.1}$$

and

$$\Upsilon_t = \Theta_t^\sharp | Y \quad \forall\, t \in \mathbb{R}. \tag{5.2}$$

The corollary will be proved under the guise of Theorems 5.2 and 5.3 below.

Theorem 5.2. *Let M_0 be a $\vec{V}$-invariant closed subset in M, which contains no singularities. If every $\nu \in \mathcal{M}_{erg}(M, \vec{V})$ supported on M_0 is non-uniformly contracting in the sense $\mathsf{Sp}^*(\vec{V}, \nu) \subset (-\infty, 0)$, then M_0 is a uniformly contracting subset of $\vec{V}$.*

Proof. Given any $\mu \in \mathcal{M}_{erg}(Y, \{\Upsilon_t\})$. By the assumptions of Theorem 5.2 and Theorem 1.1(3) we get that

$$\int_Y \omega_k^*\, d\mu = \vartheta_k^*(\mu) < 0 \quad k = 1, \ldots, n-1. \tag{5.3}$$

It follows from Theorem 5.1(1) and (1.11c) that there exist $\lambda < 0$ and $T_0 > 0$ such that for $k = 1, \ldots, n-1$

$$\|\mathrm{Pr}_k(\Theta_t(\vec{\gamma}_x))\| \leq e^{\lambda t} \quad \forall\, t \geq T_0 \text{ and } \forall\, (x, \vec{\gamma}) \in Y. \tag{5.4}$$

Since Y is compact and Θ_t is continuous, (5.4) implies that there is some constant $C > 0$ so that for $k = 1, \ldots, n-1$

$$\|\mathrm{Pr}_k(\Theta_t(\vec{\gamma}_x))\| \leq Ce^{\lambda t} \quad \forall\, t \geq 0 \text{ and } \forall\, (x, \vec{\gamma}) \in Y. \tag{5.5}$$

Particulary, consider the case $k = 1$. By the linearity of $\mathcal{V}_t^*\colon \vec{V}^{\perp}(M_0) \to \vec{V}^{\perp}(M_0)$ as in (1.15), we obtain

$$\|\mathcal{V}_t^*(\vec{v})\| \leq Ce^{\lambda t}\|v\| \quad \forall\, \vec{v} \in \vec{V}^{\perp}(M_0) \text{ and } \forall\, t \geq 0, \tag{5.6}$$

which implies that Theorem 5.2 is true. ■

Theorem 5.3. *Let M_0 be a $\vec{V}$-invariant closed subset in M, which contains no singularities. If every $\nu \in \mathcal{M}_{erg}(M, \vec{V})$ supported on M_0 is non-uniformly expanding in the sense $\mathsf{Sp}^*(\vec{V}, \nu) \subset (0, +\infty)$, then M_0 is a uniformly expanding subset of $\vec{V}$.*

Proof. There are two lines to prove this statement. One is considering $-\vec{V}$ instead of $\vec{V}$. The other is using Theorem 5.1(2). We omit the details. ■

Note that in the situations of Theorems 5.2 and 5.3, there are finite number of contracting and expanding periodic orbits nearby M_0 respectively. Furthermore, we guess that in such situations, M_0 consists of finite contracting and expanding periodic orbits, respectively.

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