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APPROXIMATION PROPERTIES OF POISSON INTEGRALS FOR ORTHOGONAL EXPANSIONS

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Abstract. In the present paper we introduce Poisson type integrals for orthogonal expansions. We first give some direct computations for the moments and compute the rates of convergence by means of the modulus of continuity and the Lipschitz functionals; and also we prove that our results are stronger and more general than the results obtained by Toczek and Wachnicki [J. Approx. Theory 116 (2002), 113-125]. We obtain a statistical approximation theorem by using the concept of $T-$statistical convergence which is a (non-matrix) summability transformation. Furthermore, we give a general Voronovskaya type theorem for these operators. Finally, introducing a higher order generalization of Poisson integrals we discuss their approximation properties.

1. Introduction

In this study we introduce a sequence of positive linear Poisson type integrals for a system of orthogonal polynomials defined on any interval (bounded or unbounded) of the real line with respect to a positive measure. Then, we first show that these operators include many Poisson integrals for orthogonal expansions, especially the operators introduced by Muckenhoupt [18] and also considered by Toczek and Wachnicki [22]. The second section of this paper gives some direct computations for the moments while in the third section we study the rates of convergence of our operators by means of modulus of continuity and the elements of Lipschitz class functionals; and also we prove that our results are stronger and more general than the results obtained in [18] and [22]. However, in the forth section considering the concept of $T-$statistical convergence (see, for details, [9, 13, 17]), where $T = (t_{jn})$

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is non-negative regular summability matrix, we obtain a statistical approximation result. We should remark that using this type of convergence method in approximation theory settings provides us more powerful results than the classical aspects (see, e.g., [6, 7]). The fifth section addresses a general Voronovskaya type theorem. In the last section we give a higher order generalization of our operators and discuss their approximation properties.

Before proceeding further we recall some notation used throughout paper.

Let I be an arbitrary interval (bounded or unbounded) of the real line, and let $\{P_k\}_{k\in\mathbb{N}_0}$ be a sequence of real-valued orthogonal polynomials P_k (with $P_0(y) = 1$) of degree k defined on I with respect to a finite measure μ, i.e.,

$$\int_I P_k(y)P_m(y)d\mu(y) = 0 \quad \text{for } k \neq m.$$

We should refer that the classical orthogonal polynomials and their generalizations may be found in Srivastava and Manocha [19].

Now, for each $k \in \mathbb{N}_0$, by $\|P_k\|$ we mean that

$$\|P_k\| := \left(\int_I P_k^2(y)d\mu(y)\right)^{\frac{1}{2}}.$$

By $L^p(I;\mu)$, $(p \geq 1)$, we denote the following space

$$L^p(I;\mu) := \left\{f : I \to \mathbb{R} : \int_I |f(y)|^p\, d\mu(y) < \infty\right\}.$$

With this terminology we now introduce the following operators:

$$L_n(f;x) := L_n(f;r_n;x) = \int_I J(x;y;r_n)f(y)d\mu(y), \tag{1.1}$$

where $0 < r_n < 1$, $f \in L^p(I,\mu)$ and $J(x;y;r_n)$ is non-negative for all $x, y \in I$, $n \in \mathbb{N}$ and defined by

$$J(x;y;r_n) := \sum_{k=0}^{\infty} \frac{P_k(x)P_k(y)}{\|P_k\|^2} r_n^k. \tag{1.2}$$

Then it is clear that the operators L_n given by (1.1) are positive and linear. Also taking into consideration that $P_0(x) = 1$, it follows from (1.1) and (1.2) that

$$L_n(1;x) = 1 \text{ for all } x \in I \text{ and } n \in \mathbb{N}.$$

Applications

(1) Let $I = [0,\infty)$. Choose $r_n = r$ for all $n \in \mathbb{N}$, $(0 < r < 1)$, and $P_k(x) =$

$L_k^{(\alpha)}(x)$ $(\alpha > -1)$ where $L_k^{(\alpha)}(x)$ is the Laguerre polynomials of degree k (see [19, p. 74]). Now consider the cumulative function F given by

$$F(y) = \int_0^y t^{\alpha} e^{-t} dt, \ 0 \le y < \infty.$$

Then the function F is monotone increasing and continuous on the right. So, it is well-known that there exists a unique Borel measure μ corresponding to F, which satisfies the following Lebesgue-Steiltjes integral equality

$$\int_I g(y) dF(y) = \int_I g(y) d\mu(y) \text{ for all } g \in L^p(I;\mu). \tag{1.3}$$

(see, for instance, [16]). Observe that

$$dF(y) = y^{\alpha} e^{-y} dy. \tag{1.4}$$

Now, for each $x \in [0,\infty)$, taking $g(y) =: J(x;y;r_n) f(y)$ and using (1.3) and (1.4), our operators $L_n(f;x)$ given by (1.1) turn out to be the linear positive operators

$$\begin{aligned} &A(f;r;x) \\ &= \int_0^{\infty} \left(\sum_{k=0}^{\infty} \frac{n! L_k^{(\alpha)}(x) L_k^{(\alpha)}(y) r^k}{\Gamma(\alpha+n+1)} \right) f(y) y^{\alpha} e^{-y} dy \\ &= \int_0^{\infty} \frac{(rxy)^{-\frac{\alpha}{2}}}{1-r} \exp\left(\frac{-r(x+y)}{1-r} \right) I_{\alpha} \left(\frac{2(rxy)^{\frac{1}{2}}}{1-r} \right) f(y) y^{\alpha} e^{-y} dy, \end{aligned} \tag{1.5}$$

where I_{α} is the modified Bessel function of the first kind (see, for instance, [19, p. 39]). Notice that the operators $A(f;r;x)$ were introduced by Muckenhoupt [18] and also considered by Toczek and Wachnicki [22].

(2) Let $I = (-\infty,\infty)$. If we choose $r_n = r$ for all $n \in \mathbb{N}$, $(0 < r < 1)$, and $P_k(x) = H_k(x)$ where $H_k(x)$ is the Hermite polynomials of degree k (see [19, p. 73]); and, as in the above technique, define an appropriate positive measure μ such that $d\mu(y) = e^{-y^2} dy$, then our operators immediately reduce to the operators

$$\begin{aligned} &B(f;r;x) \\ &= \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{H_k(x) H_k(y) r^k}{\sqrt{\pi} 2^n n!} \right) f(y) e^{-y^2} dy, \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi(1-r^2)}} \exp\left(\frac{-r^2x^2 + 2rxy - r^2y^2}{1-r^2} \right) f(y) e^{-y^2} dy, \end{aligned} \tag{1.6}$$

which were considered in [8] and also studied in [22].

2. Direct Computations for the Moments

In this section we compute the values $L_n(f_m;x)$ where f_m is the $m-$th moment function given by $f_m(y)=y^m$ for each $m\in\mathbb{N}_0$. We should remark that each function f_m can be written as a linear combination of the orthogonal polynomials P_j $(j=0,1,2,...,m)$, i.e.,

$$f_m(y)=y^m=\sum_{j=0}^{m}a_{m,j}P_j(y).$$

In this case observe that

$$a_{m,j}=\frac{\int_I f_m(y)P_j(y)d\mu(y)}{\|P_j\|^2},\ \ (j=0,1,2,...,m \text{ and } m\in\mathbb{N}_0) \tag{2.1}$$

and also since $P_0(y)=1$, we get

$$a_{0,0}=1.$$

Lemma 2.1. *For each* $m\in\mathbb{N}_0$, *we have*

$$L_n(f_m;x)=a_{m,0}+a_{m,1}r_nP_1(x)+...+a_{m,m}r_n^mP_m(x),$$

where the coefficients $a_{m,j}$ $(j=0,1,...,m)$ *are given as* (2.1).

Proof. Let $m\in\mathbb{N}_0$ be fixed. Then using linearity of L_n we get

$$\begin{aligned}L_n(f_m;x) &= \int_I\sum_{k=0}^{\infty}\frac{P_k(x)P_k(y)}{\|P_k\|^2}r_n^k\,y^m d\mu(y)\\ &= \sum_{k=0}^{\infty}\frac{P_k(x)r_n^k}{\|P_k\|^2}\int_I y^mP_k(y)d\mu(y),\end{aligned}$$

and also considering (2.1) and using orthogonality of P_k's we may write that

$$\begin{aligned}L_n(f_m;x) &= \sum_{k=0}^{\infty}\frac{P_k(x)r_n^k}{\|P_k\|^2}\sum_{j=0}^{m}a_{m,j}\int_I P_k(y)P_j(y)d\mu(y)\\ &= \sum_{j=0}^{m}\frac{a_{m,j}P_j(x)r_n^j}{\|P_j\|^2}\int_I P_j^2(y)d\mu(y)\\ &= \sum_{j=0}^{m}a_{m,j}\,r_n^jP_j(x)\\ &= a_{m,0}+a_{m,1}r_nP_1(x)+...+a_{m,m}r_n^mP_m(x)\end{aligned}$$

whence the result. ∎

Lemma 2.2. *For each $m \in \mathbb{N}_0$, we have*

$$L_n(\varphi_m;x) = \sum_{j=0}^{m}\sum_{s=0}^{j}(-1)^{m-j}\binom{m}{j}x^{m-j}a_s\, r_n^s P_s(x),$$

where $\varphi_m(y) = (y-x)^m$.

Proof. By the definition of L_n and using the well-known binomial expansion

$$\begin{aligned}
L_n(\varphi_m;x) &= \int_I J(x;y;r_n)(y-x)^m d\mu(y)\\
&= \int_I J(x;y;r_n)\left\{\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}y^j x^{m-j}\right\}d\mu(y)\\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}x^{m-j}\int_I J(x;y;r_n)\,y^j d\mu(y)\\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}x^{m-j}L_n(f_j;x).
\end{aligned}$$

Now by Lemma 2.1 we conclude that

$$L_n(\varphi_m;x) = \sum_{j=0}^{m}\sum_{s=0}^{j}(-1)^{m-j}\binom{m}{j}x^{m-j}a_{j,s}\, r_n^s P_s(x),$$

which completes the proof. ∎

We notice that, in particular, if we take $m=1$ in Lemma 2.2, we have

$$\begin{aligned}
L_n(\varphi_1;x) &= L_n\left((y-x);x\right)\\
&= -x + a_{1,0} + a_{1,1}r_nP_1(x)\\
&= -\left(a_{1,0}+a_{1,1}P_1(x)\right) + a_{1,0} + a_{1,1}r_nP_1(x),
\end{aligned}$$

which yields that

$$L_n(\varphi_1;x) = -a_{1,1}(1-r_n)P_1(x). \tag{2.2}$$

Also, if $m=2$ in Lemma 2.2, then we get

$$\begin{aligned}
L_n(\varphi_2;x) &= L_n\left((y-x)^2;x)\right)\\
&= L_n(f_2;x) - 2xL_n(f_1;x) + x^2\\
&= L_n(f_2;x) - 2x\left(x + L_n(\varphi_1;x)\right) + x^2\\
&= L_n(f_2;x) - x^2 - 2xL_n(\varphi_1;x)
\end{aligned}$$

and hence, by (2.2) and Lemma 2.1

$$\begin{aligned} L_n(\varphi_2; x) =\ & a_{2,0} + a_{2,1} r_n P_1(x) + a_{2,2} r_n^2 P_2(x) \\ & - a_{2,0} - a_{2,1} P_1(x) - a_{2,2} P_2(x) \\ & + 2a_{1,1} x(1 - r_n) P_1(x). \\ =\ & -(1 - r_n) a_{2,1} P_1(x) - (1 - r_n^2) a_{2,2} P_2(x) \\ & + 2a_{1,1} x(1 - r_n) P_1(x). \end{aligned}$$

Therefore it follows that

$$\begin{aligned} L_n(\varphi_2; x) &= (1 - r_n)\{2a_{1,1} x P_1(x) \\ &\quad - a_{2,1} P_1(x) - (1 + r_n) a_{2,2} P_2(x)\}. \end{aligned} \tag{2.3}$$

Notice that if we choose $r_n = r$ and $P_k(x) = H_k(x)$, the Hermite polynomials, as in Application 2, then our operators turn out to be the operators $B(f; r; x)$ defined by (1.6). In this case observe that $H_0(x) = 1$, $H_1(x) = 2x$ and $H_2(x) = 4x^2 - 2$. Furthermore, since $x = \frac{1}{2} H_1(x)$ and $x^2 = \frac{1}{4} H_2(x) + \frac{1}{2} H_0(x)$, we can determine the coefficients a_{ij} $(i, j = 0, 1, 2)$ as

$$\begin{aligned} a_{1,0} &= 0,\ a_{1,1} = \frac{1}{2}, \\ a_{2,0} &= \frac{1}{2},\ a_{2,1} = 0,\ a_{2,2} = \frac{1}{4}. \end{aligned}$$

So equalities (2.2) and (2.3) give the results obtain in [22] as follows:

$$B(\varphi_1; r; x) = -(1 - r)x$$

and

$$\begin{aligned} B(\varphi_2; r; x) &= (1 - r)\left\{2x^2 - \frac{1}{4}(1 + r)(4x^2 - 2)\right\} \\ &= (1 - r)\left\{x^2(1 - r) + \frac{1}{2}(1 + r)\right\}. \end{aligned} \tag{2.4}$$

On the other hand, setting $r_n = r$ and $P_k(x) = L_k^{(\alpha)}(x), (\alpha > -1)$, the Laguerre polynomials, and considering Application 1 we conclude from (2.3) and (2.3) that

$$A(\varphi_1; r; x) = (1 - r)(1 + \alpha - x)$$

and

$$\begin{aligned} A(\varphi_2; r; x) &= (1 - r)((x^2 + \alpha^2 + 3\alpha + 2)(1 - r) \\ &\quad + 2x((\alpha + 2)r - \alpha - 1))), \end{aligned} \tag{2.5}$$

which are also proved in [22].

3. Rates of Convergence

In this section we compute the rates of the approximation of $L_n(f;x)$ defined by (1.1) to $f(y)$ by means of the modulus of continuity and the elements of Lipschitz class functionals.

Let $f \in L^p(I;\mu) \cap C(I)$, where I is any interval of the real line and $p \geq 1$. The modulus of continuity of f, denoted by $w(f,\delta)$, is defined to be

$$w(f,\delta) = \sup_{x,y\in I\,,\ |x-y|<\,\delta} |f(x) - f(y)|\,, \quad (\delta > 0).$$

It is known that for any constants $c > 0$, $\delta > 0$,

$$w(f,c\delta) \leq (1+c)w(f,\delta).$$

(see [1], [5] and [14], for details). Hence, for the modulus of continuity, we may write that

$$|f(y) - f(x)| \leq w(f,\delta)\left\{1 + \frac{|y-x|}{\delta}\right\}, \tag{3.1}$$

where δ is any positive number and $f \in L^p(I;\mu) \cap C(I)$; and also $x, y \in I$.

Theorem 3.1. *For all $f \in L^p(I;\mu) \cap C(I)$, $(p \geq 1)$, we have*

$$|L_n(f;x) - f(x)| \leq 2w(f;\delta_n),$$

where

$$\delta_n := \delta_n(x) = \sqrt{L_n(\varphi_2;x)} \tag{3.2}$$

Proof. By (3.1) and monotonicity of L_n, we can write, for any $\delta > 0$, that

$$\begin{aligned}
|L_n(f;x) - f(x)| &\leq L_n\left(|f(y) - f(x)|\,;x\right) \\
&= \int_I J(x;y;r_n)\,|f(y) - f(x)|\,d\mu(y) \\
&\leq w(f,\delta)\int_I J(x;y;r_n)\left\{1 + \frac{|y-x|}{\delta}\right\}d\mu(y) \\
&= w(f,\delta)\left\{1 + \frac{1}{\delta}\int_I J(x;y;r_n)\,|y-x|\,d\mu(y)\right\}.
\end{aligned}$$

Hence, applying the Cauchy-Schwarz-Bunyakowsky inequality we conclude that

$$\begin{aligned}
|L_n(f;x) - f(x)| &\leq w(f,\delta)\left\{1 + \frac{1}{\delta}\left(\int_I J(x;y;r_n)(y-x)^2 d\mu(y)\right)^{\frac{1}{2}}\right\} \\
&= w(f,\delta)\left\{1 + \frac{1}{\delta}\sqrt{L_n(\varphi_2;x)}\right\}.
\end{aligned}$$

Now choosing $\delta = \delta_n$ given as in (3.2) the proof follows. ■

Remarks. According to Theorem 3.1, by (2.3) we have, for all $f \in L^p(I;\mu) \cap C(I)$, $(p \geq 1)$,

$$|L_n(f;x) - f(x)| \leq 2w(f, \delta_n), \tag{3.3}$$

where

$$\delta_n = \sqrt{(1-r_n)\left\{2a_{1,1}xP_1(x) - a_{2,1}P_1(x) - (1+r_n)a_{2,2}P_2(x)\right\}}. \tag{3.4}$$

Actually, in (3.4) if we take $r_n = r \in (0,1)$ for all $n \in \mathbb{N}$ and replace $L_n(f;x)$ by $B(f;r;x)$ (i.e., choose $P_k(x) = H_k(x)$; see Application 2), then using (3.3) we conclude that, for all $f \in L^p(I;\mu) \cap C(I)$ (here $I = (-\infty, \infty)$ and μ is the Borel measure such that $d\mu(y) = e^{-y^2}dy$ as considered in Application 2),

$$|B(f;r;x) - f(x)| \leq 2w(f, \delta(1)), \tag{3.5}$$

where

$$\delta(1) = \sqrt{(1-r)\left\{x^2(1-r) + \frac{1}{2}(r+1)\right\}}. \tag{3.6}$$

Notice that Toczek and Wachnicki obtained estimation (3.5) but they found the constant 3 instead of 2 (see Theorem 3.8 of [22]). In a similar manner it follows from (2.5), Theorem 3.1 and Application 1 that, for the $A(f;r;x)$ operators, we have

$$|A(f;r;x) - f(x)| \leq 2w(f, \delta(2)), \tag{3.7}$$

where

$$\delta(2) = \sqrt{(1-r)((x^2+\alpha^2+3\alpha+2)(1-r) + 2x((\alpha+2)r - \alpha - 1))}. \tag{3.8}$$

Again we may see that estimation (3.7) is better than Theorem 3.7 of [22].

We will now study the rate of convergence of the positive linear operators $L_n(f;x)$ with the help of the elements of the Lipschitz class $Lip_M(\beta)$, where $M > 0$ and $0 < \beta \leq 1$.

We recall that a function $f \in L^p(I;\mu) \cap C(I)$ belongs to $Lip_M(\beta)$ if the inequality

$$|f(y) - f(x)| \leq M\,|y-x|^{\beta}\,, \ (y, x \in I,\ 0 < \beta \leq 1) \tag{3.9}$$

holds. Then we have the following result.

Theorem 3.2. *Let $x \in I$ be fixed. For all $f \in Lip_M(\beta)$, $0 < \beta \leq 1$, we have*

$$|L_n(f;x) - f(x)| \leq M\delta_n^\beta,$$

where δ_n is the same as in (3.4).

Proof. Let $f \in Lip_M(\beta)$ and $x \in I$ be fixed, and let $0 < \beta \leq 1$. By linearity and monotonicity of L_n and using (3.9) we have

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq L_n(|(f(y) - f(x)|\,;x) \\ &= \int_I J(x;y;r_n)\,|f(y) - f(x)|\,d\mu(y) \\ &\leq M \int_I J(x;y;r_n)\,|y-x|^\beta\,d\mu(y). \end{aligned}$$

Applying the Hölder inequality with $u = \frac{2}{\beta}$, $v = \frac{2}{2-\beta}$ we get

$$|L_n(f;x) - f(x)| \leq M \left\{ \int_I J(x;y;r_n)(y-x)^2 d\mu_{n,k}(y) \right\}^{\frac{\beta}{2}}$$

and therefore

$$|L_n(f;x) - f(x)| \leq M \left\{L_n(\varphi_2;x)\right\}^{\frac{\beta}{2}}. \tag{3.10}$$

Setting $\delta_n := \sqrt{L_n(\varphi_2;x)}$ in (3.10) the proof is completed. ■

Now taking into consideration the above remarks and Applications 1 and 2, from Theorem 3.2 we easily get the next two results.

Corollary 3.3. *Let $x \in I = (-\infty,\infty)$ be fixed. For all $f \in Lip_M(\beta)$, $0 < \beta \leq 1$, we have*

$$|B(f;r;x) - f(x)| \leq M\left(\delta(1)\right)^\beta, \quad (0 < r < 1),$$

where $\delta(1)$ is given by (3.6).

Corollary 3.4. *Let $x \in I = [0,\infty)$ be fixed. For all $f \in Lip_M(\beta)$, $0 < \beta \leq 1$, we have*

$$|A(f;r;x) - f(x)| \leq M\left(\delta(2)\right)^\beta, \quad (0 < r < 1),$$

where $\delta(2)$ is given by (3.8).

4. Statistical Approximation

Most of the classical approximation operators tend to converge to the value of the function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the function. There are, however, some sharp exceptions such as the interpolation operator of Hermite-Fejer (see [2]). These operators do not converge at points of simple discontinuity. For such a misbehavior, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence (see [3]). The Cesáro summability method also corrects Gibbs phenomenon of some non-positive approximation operators such as the partial sums of Fourier series (see [15, 20, 21]). In recent years another form of regular (non-matrix) summability transformation has shown to be quite effective in "summing" non-convergent sequences which may have unbounded subsequences [9, 10]. The aim of this section is to investigate their use in approximation of our operators $L_n(f;x)$ to $f(x)$.

Now we recall the concept of $T-$statistical convergence.

Let $T := (t_{jn})$, $j, n \in \mathbb{N}$, be a non-negative regular summability, i.e. $\lim Tx = L$ whenever $\lim x = L$, where $Tx := ((Tx)_j)$ is called $T-$transform of $x := (x_n)$ and is given by $(Tx)_j := \sum_{n=1}^{\infty} t_{jn}x_n$ provided that the series convergence for each $j \in \mathbb{N}$ (see [4]). Then a sequence $x := (x_n)$, is called $T-$statistical convergent to a number L if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n-L|\geq\varepsilon} t_{jn} = 0.$$

We denote this limit by $st_T - \lim x = L$ [9] (see also [13, 17]). If we take $T = C_1$, the Cesáro matrix of order one, then C_1-statistical convergence is equivalent to statistical convergence [8, 10]. Also replacing the matrix T by the identity matrix, $T-$statistical convergence coincides with the ordinary convergence. Kolk [13] proved that $T-$statistical convergence is stronger than ordinary convergence in the case of which $\lim_j \max_n |t_{jn}| = 0$. We should note that the concept of $T-$statistical convergence may also be given in normed spaces [12].

Then we first need the following lemma.

Lemma 4.1. *Let $T = (t_{jn})$ be a non-negative regular summability matrix. If $st_T - \lim_n r_n = 1$, then we have for every $x \in I$*

$$st_T - \lim_n L_n(\varphi_i; x) = 0, \quad (i = 1, 2).$$

Proof. Let $\varepsilon > 0$ be given. By (2.2), one can write, for every $x \in I$, that

$$U := \{n \in \mathbb{N} : |L_n(\varphi_1; x)| \geq \varepsilon\} = \left\{n \in \mathbb{N} : (1 - r_n) \geq \frac{\varepsilon}{|P_1(x)a_{1,1}|}\right\} =: V.$$

Then we get, for every $j \in \mathbb{N}$,

$$\sum_{n \in U} t_{jn} = \sum_{n \in V} t_{jn}. \tag{4.1}$$

Now letting $j \to \infty$ in (4.1) and using the hypothesis we obtain that $\lim_j \sum_{n \in U} t_{jn} = 0$. This means $st_T - \lim_n L_n(\varphi_1; x) = 0$. In a similar way, it follows from (2.3) that $st_T - \lim_n L_n(\varphi_2; x) = 0$. ■

Theorem 4.1. *Let $T = (t_{jn})$ be a non-negative regular summability matrix. If $st_T - \lim_n r_n = 1$, then for all $f \in L^p(I; \mu) \cap C(I)$, $(p \geq 1)$, and for every $x \in I$ we have*

$$st_T - \lim_n |L_n(f; x) - f(x)| = 0.$$

Proof. Since $st_T - \lim_n r_n = 1$, by Lemma 4.1 we have

$$st_T - \lim_n L_n(\varphi_2; x) = 0,$$

which implies that, for all $f \in L^p(I; \mu) \cap C(I)$, $(p \geq 1)$,

$$st_T - \lim_n w(f; \delta_n) = 0 \tag{4.2}$$

with $\delta_n = \sqrt{L_n(\varphi_2; x)}$. Now given $\varepsilon > 0$ define the following sets:

$$\begin{aligned} D_1 &: = \{n \in \mathbb{N} : |L_n(f; x) - f(x)| \geq \varepsilon\}, \\ D_2 &: = \left\{n \in \mathbb{N} : w(f; \delta_n) \geq \frac{\varepsilon}{2}\right\}. \end{aligned}$$

Then, by Theorem 3.1, it is obvious that $D_1 \subseteq D_2$. Thus we have

$$\sum_{n \in D_1} t_{jn} \leq \sum_{n \in D_2} t_{jn}. \tag{4.3}$$

Taking limit as $j \to \infty$ in (4.3) and using (4.2), we get

$$\lim_j \sum_{n \in D_1} t_{jn} = 0,$$

which gives the result. ■

If we replace the matrix $T = (t_{jn})$ in Theorem 4.2 by the identity matrix we immediately obtain the following result.

Corollary 4.2. *If* $\lim_n r_n = 1$*, then for all* $f \in L^p(I; \mu) \cap C(I)$, $(p \geq 1)$*, and for every* $x \in I$ *we have*

$$\lim_n |L_n(f; x) - f(x)| = 0.$$

We remark that since $T-$statistical convergence method is stronger than the ordinary convergence method in case of $\lim_j \sup_n\{t_{jn}\} = 0$, our $T-$statistical approximation in Theorem 4.1 is more general result than Corollary 4.2.

5. A Voronovskaya Type Theorem

In this section we obtain a Voronovskaya type theorem for the operators $L_n(f; x)$ defined by (1.1).

Theorem 5.1. *Let* $T = (t_{jn})$ *be a non-negative regular summability matrix and let* $st_T - \lim_n r_n = 1$*. Suppose that* $x \in I$ *and* $f \in L^p(I; \mu) \cap C(I)$, $(p \geq 1)$*. Assume further that* f *is of the class* C^1 *in a certain neighborhood of* x *and that* f'' *exists. If for each* $x \in I$

$$st_T - \lim_n \frac{1}{(1 - r_n)^2} L_n(\varphi_4; x) = \alpha(x), \tag{5.1}$$

where α *is a function defined on* I*, then we have*

$$\begin{aligned} st_T - \lim_n \frac{1}{1 - r_n} \{L_n(f; x) - f(x)\} &= -a_{1,1} P_1(x) f'(x) \\ &\quad + \{a_{1,1} x P_1(x) - \frac{a_{2,1}}{2} P_1(x) \\ &\quad - a_{2,2} P_2(x)\} f''(x). \end{aligned}$$

Proof. By Taylor's theorem

$$f(y) = f(x) + (y - x) f'(x) + \frac{(y - x)^2}{2} f''(x) + (y - x)^2 R(y, x),$$

where

$$R(x, x) = 0. \tag{5.2}$$

Then we have

$$L_n(f;x) - f(x) = L_n(\varphi_1;x)f'(x) + \frac{1}{2}L_n(\varphi_2;x)f''(x) + L_n((y-x)^2R(y,x);x). \tag{5.3}$$

Using the Cauchy-Schwarz-Bunyakowsky inequality we obtain

$$\left|L_n((y-x)^2R(y,x);x)\right| \le \{L_n(\varphi_4;x)\}^{\frac{1}{2}}\left\{L_n(R^2(y,x);x)\right\}^{\frac{1}{2}},$$

which yields that

$$\frac{1}{1-r_n}\left|L_n((y-x)^2R(y,x);x)\right| \le \left\{\frac{1}{(1-r_n)^2}L_n(\varphi_4;x)\right\}^{\frac{1}{2}} \times \left\{L_n(R^2(y,x);x)\right\}^{\frac{1}{2}}.$$

Hence it follows from (5.1) and (5.2) that

$$st_T - \lim_n \frac{1}{1-r_n}L_n((y-x)^2R(y,x);x) = 0.$$

Now considering (5.3), the desired result is obtained from (2.2) and (2.3). ■

Observe that specializing the operators $L_n(f;x)$ as in Applications 1 and 2, and choosing the identity maxtrix, our Theorem 4.1 reduces to Theorems 4.9 and 4.10 in [22].

6. Higher Order Generalization

Let $u \in \mathbb{N}_0$ and $p \ge 1$. Then we consider the space

$$L^{p,u}(I;\mu) = \left\{f : f^{(u)} \in L^p(I;\mu)\right\}.$$

If $u = 0$, then $L^{p,0}(I;\mu) = L^p(I;\mu)$. We now introduce $u-$th order generalization of the operators $L_n(f;x)$ as follows:

$$L_{n,u}(f;x) = \int_I \sum_{i=0}^{u} J(x;y;r_n)f^{(i)}(y)\frac{(x-y)^i}{i!}d\mu(y) \tag{6.1}$$

where $f \in L^{p,u}(I;\mu)$, $(u = 0,1,2,...)$, $n \in N$. This kind of generalization was also be considered in [11]. Note that taking $u = 0$ since $f^{(0)}(y) = f(y)$, we have $L_{n,0}(f;x) = L_n(f;x)$.

We now obtain the following approximation theorem for the operators $L_{n,u}(f;x)$ given by (6.1)

Theorem 6.1. *Let I be an arbitrary interval of the real line. Then for all $f \in L^{p,u}(I;\mu)$ such that $f^{(u)} \in Lip_M(\beta)$, $0 < \beta \leq 1$, and for each $x \in I$, we have*

$$|L_{n,u}(f;x) - f(x)| \leq CL_n(|x-y|^{\beta+u};x)$$

where

$$C = \frac{M\beta}{(\beta+u)} \frac{B(\beta,u)}{(u-1)!}, \tag{6.2}$$

and $B(\alpha,u)$ is the beta function; $r \in \mathbb{N}$.

Proof. By (6.1) we get

$$f(x) - L_{n,u}(f;x) = \int_I J(x;y;r_n) \left\{ f(x) - \sum_{i=0}^{u} f^{(i)}(y) \frac{(x-y)^i}{i!} \right\} d\mu(y). \tag{6.3}$$

From the Taylor's formula (see [11])

$$\begin{aligned} f(x) - \sum_{i=0}^{u} f^{(i)}(y) \frac{(x-y)^i}{i!} &= \frac{(x-y)^u}{(u-1)!} \int_0^1 (1-t)^{u-1} \\ &\quad \times \left\{ f^{(u)}(y+t(x-y)) - f^{(u)}(y) \right\} dt. \end{aligned} \tag{6.4}$$

Since $f^{(u)} \in Lip_M(\beta)$, we have

$$\left| f^{(u)}(y+t(x-y)) - f^{(u)}(y) \right| \leq Mt^\beta |x-y|^\beta. \tag{6.5}$$

Considering (6.5) in (6.4), and using the beta integral, we conclude

$$f(x) - \sum_{i=0}^{u} f^{(i)}(y) \frac{(x-y)^i}{i!} \leq |x-y|^{\beta+u} \frac{M\beta}{\beta+u} \frac{B(\beta,u)}{(u-1)!}. \tag{6.6}$$

By using (6.6) in (6.3), we get

$$\begin{aligned} |f(x) - L_{n,r}(f;x)| &\leq \frac{M\beta}{\beta+u} \frac{B(\beta,u)}{(u-1)!} \int_I J(x;y;r_n) |x-y|^{\beta+u} d\mu(y) \\ &= \frac{M\beta}{\beta+u} \frac{B(\beta,u)}{(u-1)!} L_n\left(|x-y|^{\beta+u};x\right), \end{aligned}$$

which gives the desired result. ■

Remarks. Let $T = (t_{jn})$ be a non-negative regular summability matrix. Then, setting $g(y) = |x - y|^{\beta+u}$, under the light of Theorem 4.2 if $st_T - \lim_n r_n = 1$, then one can get, for each $x \in I$, that

$$st_T - \lim_n L_n(g; x) = 0. \tag{6.7}$$

Hence, by (6.7) and Theorem 6.1 we obtain that if $st_T - \lim_n r_n = 1$, then for all $f \in L^{p,u}(I; \mu) \cap C(I)$,

$$st_T - \lim_n |L_{n,u}(f; x) - f(x)| = 0 \text{ for every } u \in \mathbb{N}_0. \tag{6.8}$$

Of course, if we replace $T = (t_{jn})$ in (6.8) by the identity matrix, then the condition $\lim_n r_n = 1$ implies

$$\lim_n |L_{n,u}(f; x) - f(x)| = 0 \text{ for every } r \in \mathbb{N}_0.$$

On the other hand, it follows from the definition of $L_{n,u}(f; x)$ given by (6.1), for sufficiently large n, $L_{n,u}(f; x)$ tends to $f(x)$ as $u \to \infty$.

Finally, the following results are natural consequences of Theorems 3.1, 3.2 and 6.1. So we omit their proofs.

Corollary 6.2. *Let I be an arbitrary interval of the real line, and let δ_n be the same as in* (6.8). *Then, for all $f \in L^{p,u}(I; \mu)$ such that $f^{(u)} \in Lip_M(\beta)$, $0 < \beta \le 1$, and for each $x \in I$, we have*

$$|L_{n,u}(f; x) - f(x)| \le 2Cw(|x - y|^{\beta+u}, \delta_n),$$

where C and δ_n are given by (6.2) *and* (3.4), *respectively.*

Corollary 6.3. *Under the conditions in Corollary* 6.2, *we have*

$$|L_{n,u}(f; x) - f(x)| \le CM\, \delta_n^{\beta},$$

where C and δ_n are the same as above.

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