TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 3, pp. 421-431, September 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ON THE JACOBIAN CONJECTURE

Lih-Chung Wang

Abstract. In 1983, T.T. Moh gave a computer-search algorithm to find the possible counter-example of Jacobian conjecture for polynomials of degree less than or equal to 100. His algorithm claimed that there are only six possible counter-examples. When we re-code his algorithm, there seems to be exceptions other than his six cases. We modified Moh's algorithm with his approximate root theory to fulfill his claim.

1. INTRODUCTION

Let $f, g \in \mathbf{C}[x, y]$ be two polynomials with Jacobian equal to 1. The well-known Jacobian conjecture states "$\mathbf{C}[x, y] = \mathbf{C}[f, g]$." After the contribution of Abhyankar, Moh, Nagata et al., the possible counter-example can be written as follows:

$$g = (x + y)^p y^{n-p} + \text{ lower degree terms}$$

$$f = (x + y)^q y^{m-q} + \text{ lower degree terms}$$

where $0 < p < n - p$ and $(n - p)/p = (m - q)/q$. Note that $n - p$ can not be equal to p ([2], [5]).

In 1983, Moh had established certain important properties belonging to a Jacobian pair of polynomials. These quantitative properties made a computer checking program possible and Moh succeeded in proving that the Jacobian problem is true for polynomials of degree less than or equal to 100. These subtle consequences all result from solving a sequence of differential equations discovered by Moh who effectively uses the Jacobian condition.

Our work originates from an attempt to understand Moh's algorithm. However, in re-coding his algorithm, there seems to be more exceptions than expected. In

Accepted December 26, 2003.
Communicated by Jing Yu.
2000 *Mathematics Subject Classification*: 14R15, 68W99.
Key words and phrases: Jacobian conjecture, Approximate root.
Paper partially supported by National Science Council (NSC-91-2115-M-259-003).

order to make the computer checking more efficient, we modified the computation of automorphism degree and had added two checking conditions which also comes from Moh's work. This is the main result of our work. We think that the unexpected result of Moh's algorithm maybe is caused by some typos or careless writing. We hope that this paper can make the whole procedure become clear.

Section 2 aims to give a survey of Moh's work. The original definition of A_i will be modified. In section 3, we list the exceptions and use the modified algorithm to reproduce Moh's result. Section 4 is devoted to the proof of the new criteria.

2. Survey of Moh's Theory

Following S.S. Abhyankar and T.T. Moh we consider the expansion of $\{f, g\}$ at ∞ as follows ([5], p.149-151). Analytically, let $\mathbf{C}[x]((\eta))$ denote the ring of Laurent series with variable η and coefficients in $\mathbf{C}[x]$. We have

$$g(x, y) = \eta^{-n},$$

$$\eta = g(x, y)^{-\frac{1}{n}} = y^{-1} + \alpha_2(x)y^{-2} + \cdots \in \mathbf{C}[x]((y^{-1})) = \mathbf{C}[x]((\eta))$$

then

$$f(x, y) = \eta^{-m} + \sum_{j>-m} f_j(x)\eta^j \in \mathbf{C}[x]((\eta)).$$

Recall [4, p.150]

$$\begin{aligned} d_1 &= n, \\ M_j &= min\{i \mid f_i(x) \neq 0, d_j \nmid i\}, \\ d_{j+1} &= g.c.d.(n, M_1, \cdots, M_j), \\ M_{h+1} &= \infty. \end{aligned}$$

The set of numbers $\{M_j, d_j\}$ are called the $characteristic\ data$ of the pair $\{f, g\}$ (with respect to x).

In the above, $n - 2$ must be a M_s in the characteristic data, where either $s = h$ or $s = h - 1$ ([2], [5]). We shall call s the *last effective index*. Moh had shown that a counter-example will have $s \geq 3$.

In order to understand the fine structure of the points at infinity of the curve $g(x, y) = 0$, Moh had factored $g(t^{-1}, y)$ into $(y - \tau_1)(y - \tau_2) \cdots (y - \tau_n)$ in the algebraically closed field $\bigcup_{k=1}^{\infty} \mathbf{C}((t^{\frac{1}{k}}))$. Recall that the leading form of $g(x, y)$ is $(x+y)^p y^{n-p}$, we can assume without loss of generality that the order of $\tau_1, \cdots, \tau_{n-p}$ are larger that -1 and $\tau_{n-p+1}, \cdots, \tau_n$ are of the form $-t^{-1}$+higher terms in t. Moh had defined inductively a sequence $\{1, 2, \cdots, n\} \supset I_s \supset I_{s-1} \supset \cdots \supset I_2$ by the following rules.

- $I_s = \{1, 2, \cdots, n-p\}$.
- Suppose that I_{i+1} has been defined. Define V_{i+1} and δ_i as follows.

$$V_{i+1} \equiv \frac{d_{i+1}\,|I_{i+1}|}{n}$$

and

$$\delta_i \equiv 1 - \frac{(n-M_i)\prod_{j=i+1}^{s}[V_j(n-M_j)-d_j]}{\prod_{j=i+1}^{s}[V_j(n-M_{j-1})-d_j]}.$$

Then $\mathrm{ord}_t(\tau_j-\tau_k) \geq \delta_i$ for all $j,k \in I_{i+1}$ and I_{i+1} will be decomposed into disjoint union of subsets $I_{i+1,1}, I_{i+1,2}, \cdots$ such that j and k belong to the same subset if and only if $\mathrm{ord}_t(\tau_j-\tau_k) > \delta_i$.
- $I_{i+1,l}$ is called $major$ if $|I_{i+1,l}| > \frac{n}{n-M_i}$, otherwise it is called $minor$. Choose I_i to be any major one.

Moh's main result in [5] is that under the Jacobian condition, the sequence exists. In solving $g(t^{-1}, y)$, Moh introduced the concept of a 'π-root'

$$\sigma = \sum_{j<\delta} a_j t^j + \pi t^\delta$$

where $\sum_{j<\delta} a_j t^j$ is a partial sum of the expansion of some τ_k. If $\sum_{j<\delta_i} a_j t^j$ is the common part of the expansion of τ_k for $k \in I_{i+1}$ and we substitute the π-root $\sigma_i = \sum_{j<\delta_i} a_j t^j + \pi t^{\delta_i}$ into $g(t^{-1}, y)$, we obtain

$$g(t^{-1}, \sigma_i) = g_{\sigma_i}(\pi) t^{n\lambda_i} + \text{ higher terms in } t.$$

Moh proved that the order $n\lambda_i$ is $n\frac{-1+\delta_i}{n-M_i}$ and $g_{\sigma_i}(\pi)$ is of the form $p_i(\pi)^{\frac{n}{d_i}}$ with $\deg p_i(\pi) = V_{i+1}\frac{d_i}{d_{i+1}}$ ([5]). Denote by c_i the coefficient of t^{δ_i} in the expansion of τ_k for $k \in I_i$. c_i will be a root of $p_i(\pi)$. However, since the δ_i's are mostly fractional, the π-roots have conjugations due to the Galois extension $\mathbf{C}(t, c_{s-1}t^{\delta_{s-1}}, \cdots, c_{i+1}t^{\delta_{i+1}}, t^{\delta_i})$ over $\mathbf{C}(t, c_{s-1}t^{\delta_{s-1}}, \cdots, c_{i+1}t^{\delta_{i+1}})$. Consider the degree of the extension

$$A_i = [\mathbf{C}(t, c_{s-1}t^{\delta_{s-1}}, \cdots, c_{i+1}t^{\delta_{i+1}}, t^{\delta_i}) : \mathbf{C}(t, c_{s-1}t^{\delta_{s-1}}, \cdots, c_{i+1}t^{\delta_{i+1}})].$$

When $c_i \neq 0$, $p_i(\pi)$ will have the factor $(\pi^{A_i} - c_i^{A_i})^{V_i}$. If $c_i = 0$, $p_i(\pi)$ will have the factor π^{V_i} whose multiplicity V_i is congruent to $\deg p_i(\pi)$ modulo A_i.

Note that Moh had defined A_i to be

$$[\mathbf{C}(t, t^{\delta_{s-1}}, \cdots, t^{\delta_i}) : \mathbf{C}(t, t^{\delta_{s-1}}, \cdots, t^{\delta_{i+1}})]$$

which is different from ours. If the coefficients $c_{s-1}, \cdots, c_{i+1}$ are not zero, the two definitions are the same. Otherwise, Moh's A_i will be a factor of ours.

Actually, Moh had considered not only $g(t^{-1}, y)$ but also $f(t^{-1}, y)$ together with a system of related polynomials (so-called approximate roots). By substituting the same π-root σ_i into $f(t^{-1}, y)$, Moh can prove that $f(t^{-1}, \sigma_i) = f_{\sigma_i}(\pi) t^{m\lambda_i}$+higher terms in $t = (p_i(\pi))^{\frac{m}{d_i}} t^{m\lambda_i}$+higher terms in t for all $i \geq 2$. On the other hand, for $i = 1$, $m g'_{\sigma_1}(\pi) f_{\sigma_1}(\pi) - n g_{\sigma_1}(\pi) f'_{\sigma_1}(\pi)$ is a nonzero constant ([5], p.187 and p.200). The coprimeness of g_{σ_1} and g'_{σ_1} (resp. f_{σ_1} and f'_{σ_1}) implies that g_{σ_1} (resp. f_{σ_1}) has only simple roots. Since g_{σ_1} and f_{σ_1} are coprime, they cannot both have π as a factor. However, they will be polynomials in π^{A_1} if both are coprime to π. This will contradict the coprimeness of g'_{σ_1} and f'_{σ_1}. We conclude that one of g_{σ_1} and f_{σ_1} is coprime to π and the other has π but not π^2 as a factor. That is, either

$$A_1 \mid \deg g_{\sigma_1}(\pi) = V_2 \frac{n}{d_2},$$

$$A_1 \mid (\deg f_{\sigma_1}(\pi) - 1) = V_2 \frac{m}{d_2} - 1$$

or

$$A_1 \mid V_2 \frac{n}{d_2} - 1,$$

$$A_1 \mid V_2 \frac{m}{d_2}.$$

Therefore, any possible counter-example has to satisfy the above conditions.

3. Moh's Algorithm

One application of Moh's theory is to give a search algorithm of minimal counter-example for polynomials of degrees less than or equal to 100. The searching goes as follows:

Consider a possible counter-example of minimal degree. We shall search for the sequence of integers $\{n, M_1, \cdots, M_s\}$ such that

- $\deg f(x, y) = m = -M_1 < \deg g(x, y) = n \leq 100$,
- $m \nmid n$,
- $-m = M_1 < M_2 < \cdots < M_s = n - 2$,
- $d_r = \gcd(n, M_1, \cdots, M_{r-1})$ and $d_s < \cdots < d_2$.

The constraints for s and d_s are

$$5 \geq s \geq 3$$

and

$$d_s \geq 4$$

([5], corollary 6.1).

Assign a sequence of integers $\{V_s, \cdots, V_2\}$ for a hypothetically existing sequence of subfamilies with

$$V_{r+1}\frac{d_r}{d_{r+1}} \geq V_r > \frac{d_r}{n - M_r}.$$

Compute the orders $\delta_{s-1}, \cdots, \delta_1$ through

$$\delta_i = 1 - \frac{(n - M_i)\prod_{j=i+1}^{s}[V_j(n - M_j) - d_j]}{\prod_{j=i+1}^{s}[V_j(n - M_{j-1}) - d_j]}.$$

Let L be the

$$\mathrm{l.\,c.\,m.}\{\text{reduced denominators of } \delta_i, i = s-1, \cdots, r\}.$$

Denote the reduced denominator of $L\delta_{r-1}$ by A_{r-1}. Consider the division algorithm equation

$$V_r\frac{d_{r-1}}{d_r} = \Delta_{r-1}A_{r-1} + \Box_{r-1}.$$

The value of V_{r-1} is further restricted by

$$V_{r-1} \leq \Delta_{r-1}$$

if the corresponding factor of $p_{r-1}(\pi)$ is of the form $(\pi - \alpha)^{V_{r-1}}$ with $\alpha \neq 0$ or

$$A_{r-1} \mid V_{r-1} - \Box_{r-1}$$

if the corresponding factor of $p_{r-1}(\pi)$ is of the form $\pi^{V_{r-1}}$.

Finally, when $r = 2$ we need to check the following conditions.

Condition 1. Either $A_1 \mid V_2\frac{n}{d_2}$, $A_1 \mid V_2\frac{m}{d_2} - 1$ or $A_1 \mid V_2\frac{n}{d_2} - 1$, $A_1 \mid V_2\frac{m}{d_2}$ ([5], proposition 5.5).

Condition 2. The corresponding factor of $p_r(\pi)$ can not be of the form π^{V_r} for all $r \geq 2$ ([5], proposition 5.6).

Moh's result consists of two parts. First of all, there were six cases (four pairs of (n, m)) left in the computer counter-example-searching for degree ≤ 100. They are counter-example candidates.

n	$m = -M_1$	M_2	M_3	M_4	V_3	V_2	δ_2	δ_1
64	48	52	62	63	3	3	$\frac{1}{4}$	$\frac{9}{16}$
84	56	64[72]	82	83	3	2[5]	$\frac{2}{7}\left[\frac{1}{4}\right]$	$\frac{16}{21}\left[\frac{7}{12}\right]$
75	50	55	73		4	3 [2]	$\frac{1}{5}$	$\frac{1}{2}\left[\frac{2}{3}\right]$
99	66	77	97		8	8	$\frac{1}{3}$	$\frac{4}{9}$

Secondly, these six candidates can further be excluded by direct computation. ([5], p.207-211)

Nevertheless, in re-coding Moh's algorithm we found that there are other cases satisfying Moh's checking criteria. For example, for $s = 4$, we have

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	64	-16	84	94	95	3	3	2	$\frac{1}{4}$	$\frac{9}{16}$	$\frac{7}{12}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
84	56	70	77	82	83	5	10	5	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{7}{12}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	72	84	88	94	95	3	9	3	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{59}{80}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
72	48	12	56	70	71	3	9	2	$\frac{3}{11}$	$\frac{7}{22}$	$\frac{13}{33}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
90	60	10	45	88	89	4	8	3	$\frac{8}{35}$	$\frac{5}{21}$	$\frac{2}{7}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
90	60	10	45	88	89	4	8	1	$\frac{8}{35}$	$\frac{5}{21}$	$\frac{17}{42}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
90	60	45	70	88	89	4	2	3	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{2}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
90	60	45	80	88	89	4	5	3	$\frac{1}{7}$	$\frac{5}{14}$	$\frac{13}{28}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
90	60	45	80	88	89	4	5	2	$\frac{1}{7}$	$\frac{5}{14}$	$\frac{11}{21}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	64	48	68	94	95	3	2	3	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{2}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	64	48	68	94	95	3	2	1	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{3}{4}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	64	48	76	94	95	3	5	3	$\frac{2}{7}$	$\frac{5}{14}$	$\frac{13}{28}$

n	$m = -M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	64	48	76	94	95	3	5	2	$\frac{2}{7}$	$\frac{5}{14}$	$\frac{11}{21}$

n	$m=-M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	64	48	88	94	95	6	7	3	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{2}$

n	$m=-M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	72	-60	56	94	95	3	9	1	$\frac{9}{29}$	$\frac{19}{58}$	$\frac{39}{116}$

n	$m=-M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	72	36	80	94	95	3	9	4	$\frac{3}{11}$	$\frac{7}{22}$	$\frac{4}{11}$

n	$m=-M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	72	36	80	94	95	3	9	1	$\frac{3}{11}$	$\frac{7}{22}$	$\frac{23}{44}$

n	$m=-M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	72	36	78	94	95	5	3	4	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{3}$

n	$m=-M_1$	M_2	M_3	M_4	M_5	V_4	V_3	V_2	δ_3	δ_2	δ_1
96	72	36	78	94	95	5	3	1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{2}$

and, for $s=5$, we have

n	m=$-M_1$	M_2	M_3	M_4	M_5	M_6	V_5	V_4	V_3	V_2	δ_4	δ_3	δ_2	δ_1
96	64	80	88	92	94	95	3	5	10	5	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{7}{12}$

n	m=$-M_1$	M_2	M_3	M_4	M_5	M_6	V_5	V_4	V_3	V_2	δ_4	δ_3	δ_2	δ_1
96	64	-48	-8	20	94	95	3	6	1	1	$\frac{9}{28}$	$\frac{25}{77}$	$\frac{5}{14}$	$\frac{3}{8}$

n	m=$-M_1$	M_2	M_3	M_4	M_5	M_6	V_5	V_4	V_3	V_2	δ_4	δ_3	δ_2	δ_1
96	64	48	88	92	94	95	3	6	7	3	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{2}$

We modify Moh's algorithm as follows. The idea originates from Moh's work and the proof will appear in the next section (proposition 4.2,4.4).

Consider a possible counter-example of minimal degree. We shall search for the sequence of integers $\{n, M_1, \cdots, M_s\}$ such that

- $\deg f(x,y) = m = -M_1 < \deg g(x,y) = n \leq 100$,
- $m \nmid n$,
- $-m = M_1 < M_2 < \cdots < M_s = n-2$,
- $d_r = \gcd(n, M_1, \cdots, M_{r-1})$ and $d_s < \cdots < d_2$.

The constraints for s and d_s are

$$5 \geq s \geq 3$$

and

$$d_s \geq 4.$$

Assign a sequence of integers $\{V_s, \cdots, V_2\}$ for a hypothetically existing sequence of subfamilies with

$$V_{r+1}\frac{d_r}{d_{r+1}} \geq V_r > \frac{d_r}{n-M_r}.$$

Compute the orders $\delta_{s-1}, \cdots, \delta_1$ through

$$\delta_i = 1 - \frac{(n-M_i)\prod_{j=i+1}^{s}[V_j(n-M_j)-d_j]}{\prod_{j=i+1}^{s}[V_j(n-M_{j-1})-d_j]}.$$

Let L be the

$$\mathrm{l.c.m.}\{\text{reduced denominators of } \delta_i \text{ with } V_i \leq \triangle_i, i = s-1, \cdots, r\}.$$

Denote the reduced denominator of $L\delta_{r-1}$ by A_{r-1}. Consider the division algorithm equation

$$V_r\frac{d_{r-1}}{d_r} = \Delta_{r-1}A_{r-1} + \square_{r-1}.$$

The value of V_{r-1} is restricted by

$$V_{r-1} \leq \Delta_{r-1}$$

if the corresponding factor of $p_{r-1}(\pi)$ is of the form $(\pi-\alpha)^{V_{r-1}}$ with $\alpha \neq 0$ or

$$A_{r-1} \mid V_{r-1} - \square_{r-1}$$

if the corresponding factor of $p_{r-1}(\pi)$ is of the form $\pi^{V_{r-1}}$.

Finally, when $r = 2$ we need to check the following conditions.

Condition 1. Either $A_1 \mid V_2\frac{n}{d_2}$, $A_1 \mid V_2\frac{m}{d_2} - 1$ or $A_1 \mid V_2\frac{n}{d_2} - 1$, $A_1 \mid V_2\frac{m}{d_2}$ ([5], proposition 5.5).

Condition 2. If the corresponding factor of $p_r(\pi)$ is of the form π^{V_r} for $r = s, \cdots, i$, then $d_2(1-\delta_i) \geq n - M_i$ (See proposition 4.2).

Condition 3. If the corresponding factor of $p_r(\pi)$ is of the form π^{V_r} for $r = s, \cdots, 3$, then $\square_2$ can not be $\frac{d_2}{n-M_2}$ (See proposition 4.4).

Furthermore, if one qualified sequence $\{V_i\}$ satisfies $A_r \nmid V_r - \square_r$ and $\square_r > \frac{d_r}{n-M_r}$ for some r, another qualified sequence $\{V_s, \cdots, V_{r+1}, V_r', \cdots\}$ with $A_r \mid V_r' - \square_r$

under the same characteristic data $\{M_i, d_i\}$ should exist. Similarly, for any qualified sequence $\{V_i\}$ satisfying $V_{r+1}\frac{d_r}{d_{r+1}} > V_r > \triangle_r$ and $\frac{d_r}{n-M_r} < 1$ for some r, another qualified sequence $\{V_s, \cdots, V_{r+1}, V_r', \cdots\}$ with $V_r' \leq \triangle_r$ should exist.

With the modified algorithm, the six cases for $s = 3$ are still left and all of the cases for $s = 4$ or $s = 5$ are ruled out.

We'd like to point out that a counter-example should reveal a whole tree of major subfamilies. Supposed that a sequence of data satisfies the checking criteria, and at the i-th stage, after the V_i (resp. I_i) has been chosen, if the factorization of $p_i(\pi)$ ensures the existence of another V_i, then a second sequence of data must exist and satisfies the checking criteria also. This consideration has been included in our modified algorithm.

4. The New Checking Criteria

It is well known that there are a term of the form x^{l_n} in $g(x, y)$ and a term of the form x^{l_m} in $f(x, y)$ [3]. It is also well known that the Newton polygons of f and g are similar [3]. Hence, we have $\frac{n}{m} = \frac{l_n}{l_m}$ and

$$l_n = \frac{n}{m} l_m = \frac{\frac{n}{d_2}}{\frac{m}{d_2}} l_m \geq \frac{n}{d_2}$$

since d_2 is the g.c.d. of m and n. Let's summarize it as the following lemma.

Lemma 4.1. *There is a term of the form x^{l_n} in $g(x, y)$ and $l_n \geq \frac{n}{d_2}$.*

Proposition 2.2. *Suppose that $s \geq 2$ and the π-root σ_i is of the following form*

$$\sigma_i = \pi t^{\delta_i}$$

such that

$$d_2(1 - \delta_i) < n - M_i,$$

then (f, g) with such a π-root is not a minimal counter-example.

Proof. Suppose that (f, g) is a minimal counter-example. From the above lemma, we have $g(x, y) = x^{l_n} + h(x, y)$ for some $h(x, y)$. Consider $g(t^{-1}, \sigma_i) = g(t^{-1}, \pi t^{\delta_i}) = t^{-l_n} + h(t^{-1}, \pi t^{\delta_i})$. Each term in $h(x, y)$ is either x-monomial with degree less than l_n or a term with the y-variable. Therefore,

$$n\frac{-1 + \delta_i}{n - M_i} = \mathrm{ord}_t g(t^{-1}, \sigma_i) \leq -l_n \leq -\frac{n}{d_2}.$$

i.e.,

$$d_2(1-\delta_i) \geq n - M_i.$$ ■

Corollary 4.3. ([5], proposition5.6) *Suppose that $s \geq 2$ and the π-root σ_1 is of the following form*

$$\sigma_1 = \pi t^{\delta_1},$$

then (f,g) with such a π-root is not a minimal counter-example.

Proof. We have

$$\delta_1 > -1.$$

Hence

$$d_2(1-\delta_1) < 2d_2 < n+m = n - M_1.$$ ■

Proposition 4.4. *Suppose that $s \geq 3$ and that a sequence of subfamilies*

$$I_s \supset I_{s-1} \supset \cdots \supset I_3$$

has been constructed. If the corresponding factor of $p_r(\pi)$ is of the form π^{V_r} for $r \geq 3$ and $\square_2 = \frac{d_2}{n-M_2}$, then (f,g) with such a π-root is not a minimal counter-example.

Proof. Suppose that (f,g) with such a π-root is a minimal counter-example. The multiplicity of the root zero in $p_2(\pi)$ is at least $\square_2$. If it is strictly bigger, (f,g) is not a minimal counter-example by corollary 4.3. If it is $\square_2$, the corresponding π-root σ_1^* is of the form

$$\sigma_1^* = \pi t^{\delta_1^*}$$

(corresponding to a minor case). Moh has shown that $\delta_1^* \geq 1$ ([5], proposition 6.1). It follows that

$$\begin{aligned}
\mathrm{ord}_t g(t^{-1},\sigma_1^*) &= \mathrm{ord}_t g(t^{-1},\pi t^{\delta_2}) + V_2 \frac{n}{d_2}(\delta_1^* - \delta_2) \\
&= n\frac{-1+\delta_2}{n-M_2} + \frac{d_2}{n-M2}\frac{n}{d_2}(\delta_1^* - \delta_2) \\
&= n\frac{-1+\delta_2}{n-M_2} + n\frac{\delta_1^* - \delta_2}{n-M_2} \\
&= n\frac{-1+\delta_1^*}{n-M_2} \\
&\geq 0.
\end{aligned}$$

On the other hand, we have

$$\mathrm{ord}_t g(t^{-1},\sigma_1^*) \leq -l_n < 0$$

as in the proof of Proposition 4.2. This gives a contradiction. ■

5. Appendix

In order to help the readers in seeing the differences between Moh's algorithm and ours, we make a comparison table.

	Moh's algorithm	Our algorithm
Definition of L	$\mathrm{lcm}\{$denominators of $\delta_i\}$	$\mathrm{lcm}\{$denominators of δ_i with $V_i \leq \triangle_i\}$ Our L is a factor of Moh's L.
Definition of A_{r-1}	the denominator of $L\delta_{r-1}$	the denominator of $L\delta_{r-1}$ Our A_{r-1} is a multiple of Moh's A_{r-1}.
Condition 2	Corollary 4.3	Proposition 4.2 Our condition 2 is more general.
Condition 3	NA	Proposition 4.4 This is a new checking condition.

Acknowledgement

The author wishes to express his deep gratitude to Professor Moh for kindly explaining his work to him. The author also wishes to thank the referee for his suggestion.

References

1. S. S. Abhyankar,*On Expansion Techniques in Algebraic Geometry*, TATA Institute, Bombay 1977.
2. H.-C. Chang and L.-C. Wang, On Group structure associated to Jacobian Pairs with maixed leading forms, *J. Pure and App. Alg.* **119** (1997), 1-11.
3. C. Cheng and S. Wang, Radial Similarity of Newton Polygons, Automorphisms of Affine Spaces, *Kluwer Academic Publishers*, (1995), 157-167.
4. R. C. Heitmann, On the Jacobian conjecture, *J. Pure and App. Alg.* **64** (1990), 35-72.
5. T. T. Moh, On the Jacobian conjecture and the configuration of roots, *J. Reine Angew. Math.* **340** (1983), 140-212.

Lih-Chung Wang
Department of Applied Mathematics,
National Donghwa University,
Shoufeng, Hualien 974,
Taiwan, R.O.C.
E-mail: lcwang@mail.ndhu.edu.tw