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SINGULAR INTEGRALS ON LIPSCHITZ AND SOBOLEV SPACES

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Abstract. We consider the boundedness of Calderón–Zygmund operators on Lipschitz space and Sobolev space without assuming cancellation condition $T1 = 0$, and we apply our results to Calderón's commutator.

1. Introduction

Many authors have considered the boundedness of generalized singular integrals (non-convolution operators)

$$Tf(x) = \int_{R^n} K(x, y)f(y)dy,$$

on several function spaces (see, for example, [1-3,5,6,8]). But they assume the condition that $T1 = 0$. This condition is very strong and Calderón's commutator

$$C_a f(x) = \int_{R^1} \frac{a(x) - a(y)}{(x - y)^2} f(y)dy,$$

which is a typical example of generalized singular integral operator, does not satisfy this in general. Meyer [5,6], proved the boundedness of generalized singular integrals on Lipschitz and Sobolev spaces when $T1 = 0$.

In this paper we consider the boundedness of these operators by assuming that $T1$ belongs to some Lipschitz class. Our results are applicable to Calderón's commutator.

2. Definitions and Notations

The following notation is used: For a set $E \subset R^n$ we denote the characteristic

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function of E by χ_E and $|E|$ is the Lebesgue measure of E. We denote a ball of radius r centered at x by $B(x,r) = \{y; |x-y| < r\}$.

First we define some classical function spaces which we shall consider in this paper (see, for example, [6,7]).

Definition 1. Let $0 < \alpha < 1$. We define homogeneous Lipschitz space by

$$\dot{\Lambda}_\alpha(R^n) = \left\{ f; \|f\|_{\dot{\Lambda}_\alpha} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty \right\}.$$

Definition 2. Let $0 < \alpha < 1$. We define inhomogeneous Lipschitz space by

$$\Lambda_\alpha(R^n) = \left\{ f; \|f\|_{\Lambda_\alpha} = \|f\|_{L^\infty} + \|f\|_{\dot{\Lambda}_\alpha} < \infty \right\}.$$

Definition 3. Let $\lambda \geq 0$. We define Morrey space by

$$L^{1,\lambda}(R^n) = \left\{ f; \|f\|_{L^{1,\lambda}} = \sup_{\substack{x \in R^n \\ r>0}} \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)| dy < \infty \right\}.$$

Remark. $L^{1,0} = L^1, L^{1,n} = L^\infty$ and $L^{1,\lambda} = \{0\}$ where $\lambda > n$.

Definition 4. We define BMO by

$$BMO(R^n) = \left\{ f; \|f\|_{BMO} = \sup_{\substack{x \in R^n \\ r>0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_B| dy < \infty \right\},$$

where $f_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy$.

The following proposition is well-known (see [6, p. 213]).

Proposition. *Let* $0 < \alpha < 1$ *and* $1 \leq p < \infty$. *Then*

$$\sup_{\substack{x \in R^n \\ r>0}} \inf_c \frac{1}{r^\alpha} \Big(\frac{1}{r^n} \int_{B(x,r)} |f(y) - c|^p dy \Big)^{1/p} \approx \|f\|_{\dot{\Lambda}_\alpha}.$$

Remark. Because of this proposition, we can consider $BMO = \dot{\Lambda}_0$.

Definition 5. Let $0 < s < 1$. We define homogeneous Sobolev space by

$$\dot{B}^s(R^n) = \left\{ f; \|f\|_{\dot{B}^s} = \Big(\iint \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}} dx dy \Big)^{1/2} < \infty \right\}.$$

Definition 6. Let $0 < s < 1$. We define inhomogeneous Sobolev space by

$$H^s(R^n) = \left\{ f; \|f\|_{H^s} = \|f\|_{L^2} + \|f\|_{\dot{B}^s} < \infty \right\}.$$

Next we define new function spaces.

Definition 7. Let $\lambda, \mu \geq 0$. We define generalized Morrey space by

$$L^{1,(\lambda,\mu)}(R^n) = \Big\{ f; \|f\|_{L^{1,(\lambda,\mu)}} = \sup_{\substack{x\in R^n\\ 0<r\leq 1}} \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|dy + \sup_{\substack{x\in R^n\\ r\geq 1}} \frac{1}{r^\mu} \int_{B(x,r)} |f(y)|dy < \infty \Big\}.$$

Remark. $L^{1,(\lambda,\lambda)} = L^{1,\lambda}$. If $\lambda \leq n \leq \mu$ then $L^\infty \subset L^{1,(\lambda,\mu)}$.

Definition 8. Let $0 < \alpha < 1$ and $\lambda, \mu \geq 0$. We define generalized inhomogeneous Lipschitz space by

$$\Lambda_\alpha^{(\lambda,\mu)}(R^n) = \left\{ f; \|f\|_{\Lambda_\alpha^{(\lambda,\mu)}} = \|f\|_{L^{1,(\lambda,\mu)}} + \|f\|_{\dot{\Lambda}_\alpha} < \infty \right\}.$$

We write $\Lambda_\alpha^{(\lambda,\lambda)} = \Lambda_\alpha^\lambda$.

Remark. $\Lambda_\alpha^{(n,n)} = \Lambda_\alpha$ and $\Lambda_\alpha \subset \Lambda_\alpha^{(\lambda,\mu)}$ where $\lambda \leq n \leq \mu$.

Definition 9. Let $0 < s < 1$ and $1 \leq p < \infty$. We define generalized inhomogeneous Sobolev space by

$$H^{s,p}(R^n) = \left\{ f; \|f\|_{H^{s,p}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s} < \infty \right\}.$$

Remark. $H^{s,2} = H^s$.

Next we define singular integrals.

Definition 10. Let T be a bounded linear operator from $\mathcal{S}$ to $\mathcal{S}'$. T is called an ε-Calderón–Zygmund operator ($CZO(\varepsilon)$), where $0 < \varepsilon \leq 1$, if T extends to a continuous operator on L^2 and there exists a function $K(x,y)$ defined on $\{(x,y) \in R^n \times R^n; x \neq y\}$, which satisfies the following:

$$|K(x,y)| \leq \frac{C}{|x-y|^n},$$

$$|K(x,y) - K(x',y)| \leq \frac{C|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}} \quad \text{if} \quad 2|x-x'| < |x-y|,$$

$$(Tf,g) = \int K(x,y)f(y)g(x)dydx \quad \text{for} \quad f,g \in \mathcal{D} \quad \text{with disjoint supports.}$$

Remark. Note that we only assume the regularity with respect to x variable for $K(x,y)$.

Throughout this paper C is a positive constant which is independent of essential parameters and not necessarily same at each occurrence.

We shall give an example of $CZO(\varepsilon)$ which is not convolution operator.

Definition 10. (Calderón's commutator)

$$C_a f(x) = \text{p.v.} \int_{R^1} \frac{a(x)-a(y)}{(x-y)^2} f(y)dy.$$

Remark. If $a' \in L^\infty$, then C_a is a $CZO(1)$ (see [6, p. 402]).

3. Results

Meyer [6] and Lemarié [4] proved the following:

Theorem A. *Let $0<\alpha<\varepsilon\le 1$. If T is a $CZO(\varepsilon)$ and $T1=0$, then T is bounded on $\dot{\Lambda}_\alpha$.*

Theorem B. *Let $0<s<\varepsilon\le 1$. If T is a $CZO(\varepsilon)$ and $T1=0$, then T is bounded on $\dot{B}^s$.*

Remark. For the meaning of $T1$, see [1, Chapter 8] or [6, p. 412]. Note that $C_a(1)\neq 0$ in general.

The following our results Corollary 1 and 3 are variants of Theorem A and B respectively. To prove these corollaries, we shall prove more general results.

Theorem 1. *Let $0<\alpha<\varepsilon\le 1$ and $0<\beta<1$. If T is a $CZO(\varepsilon)$ and $T1\in\dot{\Lambda}_\beta$, then T is bounded from $\Lambda_\alpha^{(\lambda,\mu)}$ to $\dot{\Lambda}_\alpha$ where $\lambda\ge n+\alpha-\beta$ and $\mu<n+\alpha$.*

Remark. If $\alpha\le\beta$, then we can take λ and μ such that $\Lambda_\alpha\subset\Lambda_\alpha^{(\lambda,\mu)}$.

Theorem 2. *Let $0<s<\varepsilon\le 1$ and $s<\beta<1$. If T is a $CZO(\varepsilon)$ and $T1\in\dot{\Lambda}_\beta$, then T is bounded from $H^{s,p}$ to $\dot{B}^s$ where $\max\big(1,2n/(n+2(\beta-s))\big)<p\le 2$.*

As corollaries of our theorems we obtain the following:

Corollary 1. *Let $0<\alpha<\varepsilon\le 1$. If T is a $CZO(\varepsilon)$ and $T1\in\dot{\Lambda}_\alpha$, then T is bounded from Λ_α to $\dot{\Lambda}_\alpha$*

Proof. Let $\alpha = \beta$ and $\lambda = \mu = n$ in Theorem 1. ■

Corollary 2. (Calderón's commutator) *Let* $0 < \alpha < 1$. *If* $a' \in L^\infty(R^1)$ *and* $a' \in \dot{\Lambda}_\alpha(R^1)$, *then* C_a *is bounded from* $\Lambda_\alpha(R^1)$ *to* $\dot{\Lambda}_\alpha(R^1)$.

Proof. C_a is a $CZO(1)$ and $C_a(1) = -H(a')$ where H is the Hilbert transform. Because the Hilbert transform is bounded on $\dot{\Lambda}_\alpha$, we obtain $C_a(1) \in \dot{\Lambda}_\alpha$. ■

Remark. Corollary 2 is deduced from Meyer's result. In fact we can write

$$\begin{aligned} C_a f(x) &= \int \frac{a(x) - a(y) - (x-y)a'(y)}{(x-y)^2} f(y)dy + H(a'f)(x) \\ &= \widetilde{C}_a f(x) + H(a'f)(x), \end{aligned}$$

where $\widetilde{C}_a$ is a $CZO(1)$ and $\widetilde{C}_a(1) = 0$. So $\widetilde{C}_a$ is bounded on $\dot{\Lambda}_\alpha$. We also have $a'f \in \dot{\Lambda}_\alpha$ if $f \in \Lambda_\alpha$ and obtain $H(a'f) \in \dot{\Lambda}_\alpha$.

But if $a' \in \dot{\Lambda}_\beta$ where $\beta < \alpha$, we can not apply Meyer's theorem. Theorem 1 is applicable to these cases.

Corollary 3. *Let* $0 < s < \varepsilon \le 1$ *and* $s < \beta < 1$. *If* T *is a* $CZO(\varepsilon)$ *and* $T1 \in \dot{\Lambda}_\beta$, *then* T *is bounded from* H^s *to* H^s.

Proof. Note that T is bounded on L^2. ■

Corollary 4. (Calderón's commutator) *Let* $0 < s < \beta < 1$. *If* $a' \in L^\infty(R^1)$ *and* $a' \in \dot{\Lambda}_\beta(R^1)$, *then* C_a *is bounded from* $H^s(R^1)$ *to* $H^s(R^1)$

4. Proof of Theorem 1

First we note that T is bounded from L^∞ to BMO, so $T1 \in BMO$ (see [1, p. 20]). Therefore if $T1 \in \dot{\Lambda}_\beta$ then $T1 \in \dot{\Lambda}_\gamma$ for all $\gamma < \beta$.

Let $B(x_0, r)$ be fixed. We shall show

$$\frac{1}{r^{n+\alpha}} \int_{B(x_0,r)} |Tf(x) - c_B| dx \le C \|f\|_{\Lambda_\alpha^{(\lambda,\mu)}},$$

for some constant c_B.

We write

$$\begin{aligned} f(x) &= (f(x) - f_B)\chi_{B(x_0,2r)}(x) + (f(x) - f_B)\chi_{\ B(x_0,2r)}(x) + f_B \\ &= f_1(x) + f_2(x) + f_B, \end{aligned}$$

where $f_B = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy$.

The estimates of Tf_1 and Tf_2 are routine.

By using L^2 boundedness of T, we have

$$\begin{aligned}\frac{1}{r^{n+\alpha}}\int_{B(x_0,r)}|Tf_1(x)|dx &\le \frac{C}{r^\alpha}\left(\frac{1}{r^n}\int_{B(x_0,r)}|Tf_1(x)|^2dx\right)^{1/2}\\ &\le \frac{C}{r^\alpha}\left(\frac{1}{r^n}\int_{B(x_0,2r)}|f(x)-f_B|^2dx\right)^{1/2} \le C\|f\|_{\dot{\Lambda}_\alpha}.\end{aligned}$$

To estimate Tf_2, let $c_2 = \int K(x_0,y)f_2(y)dy$. For any $x \in B(x_0,r)$, we have

$$\begin{aligned}|Tf_2(x)-c_2| &= \left|\int (K(x,y)-K(x_0,y))f_2(y)dy\right|\\ &\le Cr^\varepsilon\int_{|x_0-y|\ge 2r}\frac{|f(y)-f_B|}{|x_0-y|^{n+\varepsilon}}dy \le Cr^\alpha\|f\|_{\dot{\Lambda}_\alpha} \quad \text{if} \quad \alpha<\varepsilon.\end{aligned}$$

So we have

$$\frac{1}{r^{n+\alpha}}\int_{B(x_0,r)}|Tf_2(x)-c_2|dx \le C\|f\|_{\dot{\Lambda}_\alpha}.$$

To estimate Tf_B, we use the condition for generalized Morrey space. We take γ such that $\mu \le n+\alpha-\gamma$ and $0<\gamma<\beta$. Let $x \in B(x_0,r)$. We have

$$\begin{aligned}|Tf_B(x)-Tf_B(x_0)| &= |f_B||T1(x)-T1(x_0)|\\ &\le \begin{cases}|f_B|\ \|T1\|_{\dot{\Lambda}_\beta}r^\beta, & \text{if} \quad r\le 1\\ |f_B|\ \|T1\|_{\dot{\Lambda}_\gamma}r^\gamma, & \text{if} \quad r\ge 1\end{cases}\\ &\le \begin{cases}\|T1\|_{\dot{\Lambda}_\beta}r^\alpha\left(r^{-\lambda}\int_{B(x_0,r)}|f(y)|dy\right)r^{\lambda-\alpha-n+\beta}, & \text{if} \quad r\le 1\\ \|T1\|_{\dot{\Lambda}_\gamma}r^\alpha\left(r^{-\mu}\int_{B(x_0,r)}|f(y)|dy\right)r^{\mu-\alpha-n+\gamma}, & \text{if} \quad r\ge 1\end{cases}\\ &\le r^\alpha\left(\|T1\|_{\dot{\Lambda}_\beta}+\|T1\|_{\dot{\Lambda}_\gamma}\right)\|f\|_{L^{1,(\lambda,\mu)}}.\end{aligned}$$

Therefore we obtain the desired result.

5. Proof of Theorem 2

We shall show

$$\iint \frac{|Tf(y)-Tf(x)|^2}{|x-y|^{n+2s}}dxdy \le C\|f\|_{H^{s,p}}^2.$$

Let $\xi \in \mathcal{D}$ be a radial function such that $\xi(u) = 1$ where $|u| \le 2$, and put $\eta(u) = 1 - \xi(u)$. As in [1, p. 119] (see also [5]), we write

$$\begin{aligned} Tf(y) - Tf(x) =& g_1(x,y) + g_2(x,y) + g_3(x,y) + g_4(x,y) \\ &+ f(x)\left(T1(y) - T1(x)\right), \end{aligned}$$

where

$$\begin{aligned} g_1(x,y) &= \int \big(K(y,u) - K(x,u)\big)\big(f(u) - f(x)\big)\eta\left(\frac{u-x}{|y-x|}\right) du, \\ g_2(x,y) &= -\int K(x,u)\big(f(u) - f(x)\big)\xi\left(\frac{u-x}{|y-x|}\right) du, \\ g_3(x,y) &= \int K(y,u)\big(f(u) - f(y)\big)\xi\left(\frac{u-x}{|y-x|}\right) du, \\ g_4(x,y) &= \big(f(y) - f(x)\big)\int K(y,u)\xi\left(\frac{u-x}{|y-x|}\right) du. \end{aligned}$$

We can also write

$$\begin{aligned} Tf(y) - Tf(x) =& - g_1(y,x) - g_2(y,x) - g_3(y,x) - g_4(y,x) \\ &+ f(y)\left(T1(y) - T1(x)\right). \end{aligned}$$

So we have

$$\begin{aligned} |Tf(y) - Tf(x)| \le& \sum_{i=1}^{4}\big(|g_i(x,y)| + |g_i(y,x)|\big) \\ &+ \min(|f(x)|, |f(y)|)\cdot|T1(y) - T1(x)|. \end{aligned}$$

Meyer showed that for $1 \le i \le 4$,

$$\iint \frac{|g_i(x,y)|^2 + |g_i(y,x)|^2}{|x-y|^{n+2s}} dxdy \le C\|f\|^2_{\dot{B}^s} \quad \text{if} \quad s < \varepsilon.$$

Therefore we need to estimate

$$I = \iint \frac{\min(|f(x)|^2, |f(y)|^2)\cdot|T1(y) - T1(x)|^2}{|x-y|^{n+2s}} dxdy.$$

We take γ such that $0 < \gamma < \beta$ and $\gamma < s$. Because $T1 \in \dot{\Lambda}_\beta \cap \dot{\Lambda}_\gamma$, we have

$$|T1(y) - T1(x)| \le \begin{cases} \|T1\|_{\dot{\Lambda}_\beta}|x-y|^\beta, & \text{if} \quad |x-y| \le 1 \\ \|T1\|_{\dot{\Lambda}_\gamma}|x-y|^\gamma, & \text{if} \quad |x-y| \ge 1. \end{cases}$$

Let $q = p/(2-p)$ and $1/q + 1/q' = 1$. (When $p = 2$, we set $1/q = 0$ and $q' = 1$). Then we obtain

$$
\begin{aligned}
I \leq \int |f(x)|^p \Big\{ & \|T1\|^2_{\dot{\Lambda}_\beta} \int_{|x-y|\leq 1} |f(y)|^{2-p}|x-y|^{-n-2s+2\beta} dy \\
& + \|T1\|^2_{\dot{\Lambda}_\gamma} \int_{|x-y|\geq 1} |f(y)|^{2-p}|x-y|^{-n-2s+2\gamma} dy \Big\} dx \\
\leq \|f\|^p_{L^p} \|f\|^{p/q}_{L^p} \Big\{ & \|T1\|^2_{\dot{\Lambda}_\beta} \Big(\int_{|x|\leq 1} |x|^{(-n-2s+2\beta)q'} dx \Big)^{1/q'} \\
& + \|T1\|^2_{\dot{\Lambda}_\gamma} \Big(\int_{|x|\geq 1} |x|^{(-n-2s+2\gamma)q'} dx \Big)^{1/q'} \Big\} \\
\leq C \|f\|^2_{L^p} \big(& \|T1\|^2_{\dot{\Lambda}_\beta} + \|T1\|^2_{\dot{\Lambda}_\gamma} \big),
\end{aligned}
$$

because $(-n-2s+2\gamma)q' < -n < (-n-2s+2\beta)q'$.

References

1. M. Frazier, Y.-S. Han, B. Jawerth, and G. Weiss, The $T1$ Theorem for Triebel-Lizorkin spaces, Harmonic Analysis and Partial Differential Equations, Lecture Notes in Math. No. **1384**, *J. García-Cuerva,* ed., 168-181.
2. M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Reg. Conf. Ser. in Math. No. **79**, Amer. Math. Soc., Providence, RI, 1991.
3. J. E. Gilbert, Y.-S. Han, J. A. Hogan, J. D Lakey, D. Wiland, and G. Weiss, *Smooth Molecular Decompositions of Functions and Singular Integral Operators*, Memoirs of the AMS, No. **742**, 2002.
4. P. G. Lemarié, Continuité sur les espaces de Besov des opérateurs définis par des intégrales singulières, *Ann. Inst. Fourier* (*Grenoble*), **35** (1985), 175-187.
5. Y. Meyer, Continuité sur les espaces de Holder et de Sobolev des opérateurs définis par des intégrales singulières, *Recent Progress in Fourier Analysis*, Peral and Rubio de Francia, eds., (1985), 145-172.
6. Y. Meyer and R. Coifman, *Wavelets: Calderón-Zygmund and multilinear operators*, Cambridge Studies in Advanced Mathematics **48**, Cambridge Univ. Press, 1997.
7. A. Torchinsky, *Real Variable Methods in Harmonic Analysis*, Academic Press, 1986.
8. R.H. Torres, *Boundedness results for operators with singular kernels on distribution spaces*, Memoirs of the AMS, No. **442**, 1991.

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