TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 4, pp. 713-737, December 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

A RESOLVABLE $r \times c$ GRID-BLOCK PACKING AND ITS APPLICATION TO DNA LIBRARY SCREENING

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Abstract. For a v-set V, let $\mathcal{A}$ be a collection of $r \times c$ arrays with elements in V. A pair $(V, \mathcal{A})$ is called an $r \times c$ *grid-block packing* if every two distinct points i and j in V occur together at most once in the same row or in the same column of arrays in $\mathcal{A}$. And an $r \times c$ grid-block packing $(V, \mathcal{A})$ is said to be *resolvable* if the collection of arrays $\mathcal{A}$ can be partitioned into sub-classes $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_t$ such that every point of V is contained in precisely one array of each class. These packings have originated from the use of DNA library screening. In this paper, we give some constructions of resolvable $r \times c$ grid-block packings and give a brief survey of their application to DNA library screening.

1. Introduction

A *graph* G is a pair of sets (V, E), where V is a finite set and E is a set of unordered pairs of elements of V. The elements of V are called *vertices* or *points* of G and the elements of E are called *edges* of G. For vertices i and j of a graph G, we say that i is *adjacent* to j if there is an edge between i and j. The *complete graph*, denoted by K_v, is the graph with v vertices such that every vertex is adjacent to every other vertex. The *Cartesian product* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \times G_2$, is defined to be the graph with the vertex set $V = V_1 \times V_2$, and two vertices $i = (i_1, i_2)$ and $j = (j_1, j_2)$ are adjacent in the Cartesian product whenever $i_1 = j_1$ and i_2 is adjacent to j_2 in G_2 or symmetrically if $i_2 = j_2$ and i_1 is adjacent to j_1 in G_1. Note that the vertices of a Cartesian product $G_1 \times G_2$ can be arranged on the $|V_1| \times |V_2|$ array and each subgraph with vertices in a row (or in a column) of the array is isomorphic to G_2 (or G_1).

Received June 12, 2002; accepted September 2, 2003.
Communicated by Gerard J. Chang.
2000 *Mathematics Subject Classification*: 05B05.
Key words and phrases: Grid-block design, Grid-block packing, Lattice square design, DNA library screening.
*Research supported in part by JSPS Research Fellow 09978.

For a v-set V, let $\mathcal{A}$ be a collection of $r \times c$ arrays with elements in V, and we identify each $r \times c$ array in $\mathcal{A}$ with a Cartesian product of two complete graphs $K_r \times K_c$ whose vertices are labelled by the points in V. Each array is called a *grid-block*. A pair $(V, \mathcal{A})$ is called a $K_r \times K_c$ *packing into* K_v or an $r \times c$ *grid-block packing*, denoted by $P_{r\times c}(K_v)$, if every two distinct points i and j in V occur at most once in the same row or in the same column of a grid-block in $\mathcal{A}$. Moreover, if every two distinct points occur exactly once in the same row or in the same column, then $(V, \mathcal{A})$ is called a $K_r \times K_c$ *decomposition* or an $r \times c$ *grid-block design*, denoted by $D_{r\times c}(K_v)$. Here we used the terminology 'grid-block design' to avoid the confusion with the 'grid design' defined by Lamken and Wilson [12].

Especially, a $K_r \times K_1$ packing or a $K_1 \times K_r$ packing is called a *packing with block size* r and a $K_r \times K_1$ decomposition or a $K_1 \times K_r$ decomposition is called a *balanced incomplete block design* (*BIB*) *design*, denoted by $B(v, r; 1)$.

Two grid-blocks A and A' are said to be *equivalent* if there are permutation matrices P and Q such that $PAQ = A'$. For an $r \times c$ grid-block packing $(V, \mathcal{A})$, let σ be a permutation on V. For any $r\times c$ grid-block $A = (a_{ij}) \in \mathcal{A}$, let $A^\sigma = (a_{ij}^\sigma)$. If there is a permutation σ on V such that A^σ is contained in $\mathcal{A}$ for any grid-block $A \in \mathcal{A}$, then σ is called an *automorphism* of the $r \times c$ grid-block packing $(V, \mathcal{A})$. If there is an automorphism σ of order $v = |V|$, then the $r \times c$ grid-block packing is said to be *cyclic* .

For a cyclic grid-block packing $(V, \mathcal{A})$, the point set V can be identified with $\boldsymbol{Z}_v$. For a grid-block $A = (a_{ij})$, the *grid-block orbit* containing A is defined to be the set of the inequivalent grid-blocks

$$A^{\sigma^l} = A + l = (a_{ij} + l) \pmod{v}$$

for $l \in \boldsymbol{Z}_v$. If a grid-block orbit has v distinct grid-blocks, then the orbit is said to be *full*, otherwise *short*. Choose an arbitrarily fixed grid-block from each grid-block orbit and call it a *base grid-block*. Similarly, we define cyclic packings and cyclic BIBDs.

A grid-block packing $(V, \mathcal{A})$ is said to be *resolvable* if the collection of grid-blocks $\mathcal{A}$ can be partitioned into subclasses $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, ..., $\boldsymbol{R}_t$ such that every point of V is contained in precisely one grid-block of each class. The classes $\boldsymbol{R}_i$ are called *resolution classes* and $\mathcal{R} = \{\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_t\}$ is called a *resolution*. Such a grid-block packing is said to be an $r \times c$ *resolvable grid-block packing*. Similarly, we define resolvable packings and resolvable BIBDs.

Here we show examples of a cyclic $D_{2\times3}(K_{10})$, a resolvable $D_{3\times3}(K_9)$, a resolvable $P_{2\times2}(K_8)$ and a resolvable $P_{3\times3}(K_{18})$.

Example 1. The following five grid-blocks form a cyclic $D_{2\times3}(K_{10})$.

1	2	4
7	6	9

2	3	5
8	7	0

3	4	6
9	8	1

4	5	7
0	9	2

5	6	8
1	0	3

Example 2. The following two grid-blocks form a resolvable $D_{3\times3}(K_9)$.

1	2	3
4	5	6
7	8	9

1	6	8
9	2	4
5	7	3

Example 3. The following six grid-blocks form a resolvable $P_{2\times2}(K_8)$.

∞_0	0_0
0_1	∞_1

∞_0	1_0
1_1	∞_1

∞_0	2_0
2_1	∞_1

1_0	2_0
2_1	1_1

2_0	0_0
0_1	2_1

0_0	1_0
1_1	0_1

Example 4. The following six grid-blocks form a resolvable $P_{3\times3}(K_{18})$.

0	1	2
3	4	5
6	7	8

0	4	8
5	6	9
7	10	14

0	9	13
10	17	8
12	7	3

9	10	11
12	13	14
15	16	17

1	3	15
12	16	11
17	2	13

1	5	11
6	15	2
16	14	4

For a packing $P_{r\times c}(K_v)$, let t_i be the number of grid-blocks containing a point i. Then,

$$t_i \leq \left\lfloor \frac{v-1}{r+c-2} \right\rfloor$$

holds. If a packing is resolvable, then v is divided by rc, t_i is constant($= t$) and the number of grid-blocks is

$$b = t\frac{v}{rc} \leq \frac{v}{rc}\left\lfloor \frac{v-1}{r+c-2} \right\rfloor.$$

A resolvable packing $P_{r\times c}(K_v)$ attaining this bound is said to be *optimal*. In Example 3, the resolvable $P_{2\times2}(K_8)$ is optimal. However, the resolvable $P_{3\times3}(K_{18})$

is not optimal in Example 4 since the upper bound of the number of resolution classes are 4.

Next, we define the notions of the sum, the scalar multiplication and the product over additive groups for lists to utilize in Section 4 and 6. For a finite set V, a formal sum $L = \sum_{x \in V} m_x\{x\}$ is called a *list*, where the nonnegative integer m_x is the multiplicity of x in the list L. Also we use the notation $L = (a_i | i \in I)$ to indicate the list of a_i's, where I is an indexing set. We identify a subset Y of V with a list whose multiplicities m_x are 1 or 0 depending on whether x belongs to Y or not.

We define the addition and the scalar multiplication for lists $L = \sum_{x \in V} l_x\{x\}$ and $M = \sum_{x \in V} m_x\{x\}$ by $L + M = \sum_{x \in V}(l_x + m_x)\{x\}$ and $\lambda L = \sum_{x \in V} \lambda l_x\{x\}$ for a nonnegative integer λ. Moreover, if $l_x \leq m_x$ holds for each $x \in V$, then we write $L \leq M$.

Moreover, V is an additive group with order v. The product of two lists $L = \sum_{x \in V} l_x\{x\}$ and $M = \sum_{x \in V} m_x\{x\}$ is defined by

$$L \circ M = \sum_{z \in V} \left(\sum_{\substack{x,y \\ xy=z}} l_x m_x \right) \{z\}.$$

List multiplication is commutative, associative and distributive over the addition of lists. For any subset S of V and for any element y of V, let $S + y = \{s + y | s \in S\}$ and $Sy = \{sy | s \in S\}$.

In Section 2, we will give a brief survey of its application to DNA library screening and will explain the relationship to a grid-block packing. In Section 3, we will give a brief survey of constructions of grid-block designs. And in Section 4, 5 and 6, we will give some constructions of resolvable $r \times c$ grid-block packings and designs.

2. DNA Library Screening

The notion of $r \times c$ grid-block designs was introduced by Fu, Hwang, Jimbo, Mutoh and Shiue [8]. The special case of $rc = v$ is called a *lattice square design* if $r = c$, or a *lattice rectangle design* otherwise, which was introduced by Yates [16] and Harshbarger [9], respectively. These designs are obviously resolvable. A construction of lattice square designs for $\sqrt{v}$ odd prime powers was given by Raghavarao [14]. And its use for the DNA library screening was proposed by Hwang [10].

In DNA library screening, there are many oligonucleotides (clones) to be tested whether they are positive or negative. An oligonucleotide is a short string of nucleotides A, T, G and C. The goal of a DNA library screening is to identify all

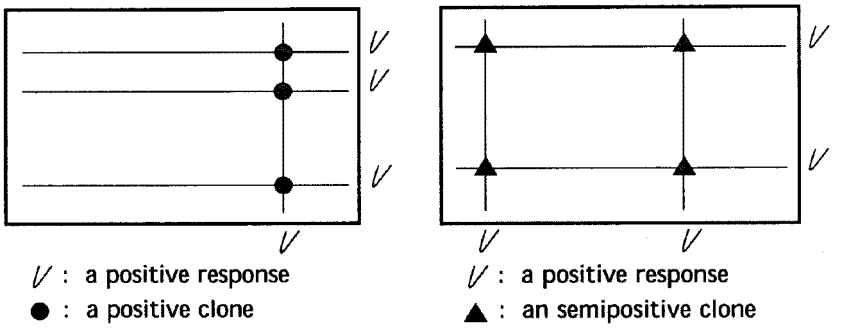
positive clones. Economy of time and costs requires that the clones be assayed in groups. Each group is called a *pool*. If a pool gives a negative outcome, all clones contained in it are found to be negative. In this case, we can save numbers of tests. On the other hand, if the pool is positive, at the second stage we test each clone individually. This screening method is called a two-stage test, which is a popular group testing.

In such screening, a microtiter plate, which is an array with size 8×12 or 16×24, etc. is utilized and different clones are settled in each spot, called *well*, of the plate.

In this method, every row and every column in a microtiter plate is tested at the same time as a pool in the first stage, and each clone with positive response is tested individually in the second stage. This method is called the *basic matrix method* (BMM). In this method each clone is tested twice. If the array contains only a single positive clone, or more generally, if there is only one row (or column) of positive then we can determine the positive clones without individual tests. However, it does not always occur, that is, arrays often contain several positive clones. For example if two rows and two columns are positive as we see in Fig. 1(b), we can not determine whether the four clones settled at the crossing spots of positives are really positive or not.

Thus, if it is allowed to test more than twice for each clone, then, it is desired that every two distinct clones occur at most once in the same row or the same column, which is called the *unique collinearity condition*. The efficiency of the unique collinearity condition was shown by Barillot, Lacroix and Cohen [1] by simulation and was also proved theoretically by Berger, Mandell and Subrahmanya [2].

We consider the case when there is a single positive clone within the set of v



(a) One column is positive. (b) Two rows and columns are positive.

Fig. 1. Results of the first stage group tests in DNA library screening.

clones and we place those clones on t $r \times c$ microtiter plates at random allowing repetition, where $n = trc \geq v$ holds. Then, the expectation of the number of different clones, which occur in the all microtiter plates, is

$$\frac{1}{v^n}\sum_{k=1}^{v} k \begin{pmatrix} v \\ k \end{pmatrix} \sum_{i=1}^{k} (-1)^{k-i} \begin{pmatrix} k \\ i \end{pmatrix} i^n = v - \frac{(v-1)^n}{v^{n-1}} = v - \left(1 - \frac{1}{v}\right)^n v.$$

In this case, the expectation of the number of individual tests we need is at least $(1-\frac{1}{v})^n v$. However, if $n = v$ and each clone is settled exactly once on the microtiter plates, then we can decide the positive/negative only by the first stage group tests and we can reduce about $(1 - \frac{1}{v})^v v$ tests. We can also show it by an example shown in Figure 2. In this figure, there are $v = 1000$ clones and p is the probability of positives. The vertical line is the number of tests for (i) the case of the same replication number and (ii) the case when the replication numbers are not constant. By Figure 2, we can see that, in case of the constant replications, the number of tests can be reduced comparing with the case of non-constant replications.

It is a favorite property that the number of replications for each clone should be almost the same in the first stage. This condition is called *the equal replication number of tests*.

An $r \times c$ grid-block packing defined by Section 1 satisfies 'the unique collinearity condition', moreover, a resolvable $r \times c$ grid-block packing satisfies also 'the equal replication number of tests'.

Berger *et al.* [2] gave the optimal size of the array and the optimal replication number according as the probability (ratio) of positive clones under the implicit condition of the equal replication number of tests. Though they utilized the terminology of 'n-dimensional array', it implies that the replication numbers are all equal ($= n$). Knill, Bruno and Torney [11] considered non-adaptive group testing problems with some errors.

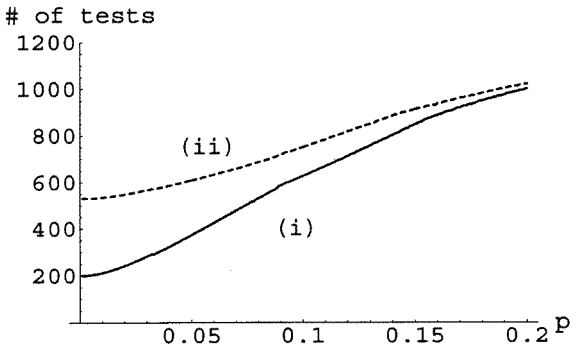


Fig. 2. A simulation result of a comparison between constant replications and random replications.

3. Known Results

In this section, we will list some known results for $r \times c$ grid-block designs. Firstly, we give necessary conditions of a $D_{r\times c}(K_v)$ obtained by Fu *et al.* [8].

Proposition 1. *Necessary conditions for the existence of a* $D_{r\times c}(K_v)$ *are*

(i) $v - 1 \equiv 0 \pmod{r + c - 2}$, *and*

(ii) $v(v-1) \equiv 0 \pmod{rc(r + c - 2)}$.

These conditions are not sufficient in general. It may be difficult to determine those parameters for which $D_{r\times c}(K_v)$s exist. Among them, the existence of a $D_{2\times 2}(K_v)$ is known in terms of a '4-cycle system'.

Proposition 2. *The necessary condition* $v \equiv 1 \pmod 8$ *for the existence of a* $D_{2\times 2}(K_v)$ *is sufficient.*

And the existence of a $D_{2\times 3}(K_v)$ was shown by Carter [5] by decomposing K_v into cubic graphs.

Proposition 3. ([5]) *The necessary condition* $v \equiv 1 \pmod 9$ *for the existence of a* $D_{2\times 3}(K_v)$ *is sufficient.*

Meanwhile, Fu *et al.* [8] showed the existence of a $D_{3\times 3}(K_v)$ by some direct constructions and gave some general constructions.

Proposition 4. ([8]) *The necessary condition* $v \equiv 1, 9 \pmod{36}$ *for the existence of a* $D_{3\times 3}(K_v)$ *is sufficient.*

Mutoh, Morihara, Jimbo and Fu [13] showed the existence of a $D_{2\times 4}(K_v)$ by utilizing some direct and recursive constructions.

Proposition 5. ([13]) *The necessary condition* $v \equiv 1 \pmod{32}$ *for the existence of a* $D_{2\times 4}(K_v)$ *is sufficient.*

Next, we define some notions. For a set of positive integers K, let V be a set of v *points* (*vertices*) and let $\mathcal{B}$ be a collection of k-subsets (called *blocks*) of V for $k \in K$. If every two distinct points occur exactly λ times in blocks, then a pair $(V, \mathcal{B})$ is called a *pairwise balanced design*, denoted by $B(v, K, \lambda)$. Especially, a $B(v, \{k\}, \lambda)$ is written by $B(v, k, \lambda)$ for simplicity of the notation and it is a BIB design.

For sets of positive integers K and M, let V be a set of v points, $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ be a partition of V such that each G_i has m points for $m \in M$

and $\mathcal{B}$ be a collection of k-subsets (blocks) of V for $k \in K$. A triple $(V, \mathcal{G}, \mathcal{B})$ is called a *group divisible design*, denoted by $GD(v, K, \lambda, M)$, if every two distinct points contained in the different groups occur in exactly λ blocks and if every two distinct points contained in the same group do not occur together in any blocks. Especially, a $GD(v, \{k\}, \lambda, \{m\})$ is simply written by $GD(v, k, \lambda, m)$.

Suppose that the set of st vertices are partitioned into s subsets of size t each. Let $K_s(t)$ be the complete multipartite graph such that the vertex set is divided into t groups each of which has s vertices and that $\{i, j\}$ is an edge if i and j are not in the same subset. Let V be the set of vertices in $K_s(t)$ and let $\mathcal{A}$ be a collection of $r \times c$ grid-blocks with elements in V. For any edge $\{i, j\}$ in $K_s(t)$, if there is a single grid-block containing i and j in the same row or in the same column, then a pair $(V, \mathcal{A})$ is called a *group divisible grid-block design*, denoted by $D_{r\times c}(K_s(t))$. It is easy to see that the following proposition holds:

Proposition 6. ([8]) *Necessary conditions for a $D_{r\times c}(K_s(t))$ to exist are*

(i) $(s-1)t \equiv 0 \pmod{r+c-2}$, *and*

(ii) $(s-1)st^2 \equiv 0 \pmod{rc(r+c-2)}$.

We list some recursive constructions obtained by Fu *et al.* [8] and Mutoh *et al.* [13].

Proposition 7. ([8]) *A $D_{r\times c}(K_{st+1})$ exists if a $D_{r\times c}(K_{t+1})$ and a $D_{r\times c}(K_s(t))$ exist.*

Proposition 8. ([8]) *A $D_{r\times c}(K_v(t))$ exists if a $B(v, K, 1)$ and $D_{r\times c}(K_s(t))$'s for $s \in K$ exists. Especially, a $D_{r\times c}(K_v(t))$ exists if a $B(v, s, 1)$ and a $D_{r\times c}(K_s(t))$ exist.*

Corollary 9. ([8]) *A $D_{r\times c}(K_{vt+1})$ exists if a $B(v, K, 1)$, a $D_{r\times c}(K_{t+1})$ and $D_{r\times c}(K_s(t))$'s for $s \in K$ exist.*

Proposition 10. ([8]) *A $D_{r\times c}(K_{(v-1)t+1})$ exists if a $B(v, s, 1)$, a $D_{r\times c}(K_{t+1})$, a $D_{r\times c}(K_{(s-1)t+1})$ and a $D_{r\times c}(K_s(t))$ exist.*

Proposition 11. ([8]) *A $D_{r\times c}(K_{(v+i)t+1})$ exists if a resolvable $B(v, s, 1)$ with at least i resolution classes, a $D_{r\times c}(K_{t+1})$, a $D_{r\times c}(K_{it+1})$, a $D_{r\times c}(K_s(t))$ and a $D_{r\times c}(K_{s+1}(t))$ exist.*

Proposition 12. ([13]) *A $D_{r\times c}(K_{vt+1})$ exists if a $GD(v, K, 1, M)$ exists and if a $D_{r\times c}(K_k(t))$ and a $D_{r\times c}(K_{mt+1})$ exist for any $k \in K$ and for any $m \in M$.*

Proposition 13. ([8]) *Suppose a $D_{r\times c}(K_s(t))$ and $s-2$ mutually orthogonal Latin squares of order m exist for $s \geq 3$. Then a $D_{r\times c}(K_s(mt))$ exist.*

Moreover, we list two direct constructions.

Proposition 14. ([8]) *For an even integer n and an odd prime power q, there exists a $D_{q\times q}(K_{q^n})$.*

Proposition 15. ([8]) *Let p be an odd prime and $v \equiv p \pmod{2p(p-1)}$. If there exists a cyclic $B(v, p, 1)$, then there exists a $D_{p\times p}(K_{pv})$.*

4. Some Optimal Resolvable $r \times c$ Grid-block Packings

In this section, we will construct an optimal resolvable 2×2 grid-block packing and a $q \times q$ grid-block packing for a prime power q. Firstly, we will give the following theorem by constructing directly.

Theorem 16. *There exists an optimal resolvable $P_{2\times 2}(K_v)$ for any $v \equiv 0 \pmod 4$.*

Before we prove Theorem 16, we give some notations. For an additive group V with v elements, we introduce the list of differences of a k-set $B = \{b_i\}$ with elements in V by $\Delta B = (b_i - b_j | 1 \leq i, j, \leq k,\ i \neq j)$. For a family of k-sets $\boldsymbol{B} = (B_i | i \in I)$ with elements in V, we define $\Delta \boldsymbol{B} = \sum_{i\in I} \Delta B_i$. If $\Delta \boldsymbol{B} = V \backslash \{0\}$ holds, then $\boldsymbol{B}$ is a $(v, k, 1)$-*difference family* in V, denoted by $DF(v, k, 1)$ in V.

We generalize such notion to an $r \times c$ grid-block. We introduce the list of differences of an $r \times c$ grid-block $A = (a_{ij})$ with elements of V as follows:

$$\begin{aligned}\partial A = &(a_{jl} - a_{il} | 1 \leq i \neq j \leq r,\ 1 \leq l \leq c)\\ &+(a_{lj} - a_{li} | 1 \leq l \leq r,\ 1 \leq i \neq j \leq c).\end{aligned}$$

For a family of $r \times c$ arrays $\boldsymbol{A} = (A_i | i \in I)$ with elements in V, we define $\partial \boldsymbol{A} = \sum_{i\in I} \partial A_i$. If $\partial \boldsymbol{A} \leq V \setminus \{0\}$ holds, then $\boldsymbol{A}$ is called a *grid-block difference packing*, denoted by $DP(v, K_r \times K_c, 1)$. Especially, if $\partial \boldsymbol{A} = V \setminus \{0\}$ holds, then $\boldsymbol{A}$ is called a *grid-block difference family*, denoted by $DF(v, K_r \times K_c, 1)$. In fact, the development of $\boldsymbol{A}$ is defined by $\mathcal{A} = \{A_s + g | g \in V\}$. Then a pair $(V, \mathcal{A})$ is an $r \times c$ grid-block packing if and only if $\partial \boldsymbol{A} \leq V \setminus \{0\}$ holds and an $r \times c$ grid-block design if and only if $\partial \boldsymbol{A} = V \setminus \{0\}$.

Next, we give a notion of *the method of mixed differences* introduced by Bose [4]. For an additive group G and an index set $M = \{0, 1, \ldots, t-1\}$, let $V = G \times M$

be a set of points. For $g \in G$ and $B = \{(b_1, l_1), (b_2, l_2), \ldots, (b_k, l_k)\} \subseteq V$, we define the addition $B + g$ by

$$B + g = \{(b_1 + g, l_1), (b_2 + g, l_2), \ldots, (b_k + g, l_k)\}.$$

For a k-subset $B = \{(b_1, l_1), (b_2, l_2), \ldots, (b_k, l_k)\}$, let $\Delta_{ij}B$ be the list of differences $b_{n'} - b_n$ such that $((b_n, i))$ and $((b_{n'}, j))$ occur in B, that is,

$$\Delta_{ij}B = (b_{n'} - b_n | 1 \leq n \neq n' \leq k, l_n = i, l_{n'} = j).$$

Note that if $i \neq j$ the difference $b_{n'} - b_n = 0$ can occur, but not for $i = j$. Obviously, $\Delta_{ij}B = -\Delta_{ji}B$. For a family of k-subsets $\boldsymbol{B} = \{B_s | s \in S\}$ with elements in G, we define $\Delta_{ij}\boldsymbol{B} = \sum_{s \in S} \Delta_{ij}B_s$. $\Delta_{ii}\boldsymbol{B}$ is called the *i-th list of pure differences*. In case of $i \neq j$ the list $\Delta_{ij}\boldsymbol{B}$ is called the *list of mixed differences* for the index pair (i, j). Obviously, $\Delta_{ij}\boldsymbol{B} = -\Delta_{ji}\boldsymbol{B}$ holds. Note that the difference 0 is allowed in $\Delta_{ij}\boldsymbol{B}$ if and only if $i \neq j$ holds.

The development of $\boldsymbol{B}$ is defined by $\mathcal{B} = \{B_s + g | s \in S, g \in G\}$. A pair $(V, \mathcal{B})$ is a balanced incomplete block design with parameter $\lambda = 1$ if and only if

(i) $\Delta_{ii}\boldsymbol{B} = G \setminus \{0\}$ *holds for every* $i \in M$ *and*

(ii) $\Delta_{ij}\boldsymbol{B} = G$ *holds for each* $i, j \in M$.

We generalize the notion of the mixed (or pure) differences to an $r \times c$ grid-block packing. Let G be an additive group and $M = \{0, 1, \ldots, t-1\}$ be an index set. For an $r \times c$ array $A = ((a_{m,n}, l_{m,n}))$ with elements in $V = G \times M$, let $\partial_{ij}A$ be the list of differences $a_{m',n'} - a_{m,n}$ such that $((a_{m,n}, i))$ and $((a_{m',n'}, j))$ occur together in the same row or the same column of A, that is,

$$\begin{aligned}\partial_{ij}A = &(a_{m',u} - a_{m,u} | 1 \leq m \neq m' \leq r, 1 \leq u \leq c, l_{m,u} = i, l_{m',u} = j) \\ &+ (a_{u,n'} - a_{u,n} | 1 \leq u \leq r, 1 \leq n \neq n' \leq c, l_{u,n'} = i, , l_{u,n} = j).\end{aligned}$$

For a family of $r \times c$ arrays $\boldsymbol{A} = \{A_s | s \in S\}$ with elements in G, we define $\partial_{ij}\boldsymbol{A} = \sum_{s \in S} \partial_{ij}A_s$. Obviously, $\partial_{ij}\boldsymbol{A} = -\partial_{ji}\boldsymbol{A}$ holds. Note that the difference 0 is allowed in $\partial_{ij}\boldsymbol{A}$ if and only if $i \neq j$ holds.

The development of $\boldsymbol{A}$ is defined by $\mathcal{A} = \{A_s + g | g \in G\}$. Then a pair $(V, \mathcal{A})$ is an $r \times c$ grid-block packing if and only if

(i) $\partial_{ii}\boldsymbol{A} \leq G$ *0 holds for every* $i \in M$ *and*

(ii) $\partial_{ij}\boldsymbol{A} \leq G$ *holds for each* $i, j \in M$.

If $\partial_{ii}\boldsymbol{A} = G \setminus \{0\}$ and $\partial_{ij}\boldsymbol{A} = G$ hold for each $i, j \in M$, a pair $(V, \mathcal{A})$ is an $r \times c$ grid-block design. Now we are ready to prove Theorem 16.

Proof of Theorem 16. Let $V = (\boldsymbol{Z}_{2t-1} \cup \{\infty\}) \times \{0,1\}$ for any $t \geq 1$. We define base grid-blocks

$$A_\infty = \boxed{\begin{array}{cc} \infty_0 & 0_0 \\ 0_1 & \infty_1 \end{array}}$$

and

$$A_l = \boxed{\begin{array}{cc} l_0 & (2t-l-1)_0 \\ (2t-l-1)_1 & l_1 \end{array}},$$

for $l = 1, 2, \ldots, t-1$. Also we define the family $\boldsymbol{A}$ of base grid-blocks by

$$\boldsymbol{A} = \{A_x | x = \infty, 1, 2, \ldots, t-1\}.$$

Moreover we define a map

$$\tau_g : (x, j) \mapsto (x + g, j)$$

for $x \in \boldsymbol{Z}_{2t-1} \cup \{\infty\}$, $g \in \boldsymbol{Z}_{2t-1}$ and $j = 0, 1$. Note that $\infty + g = \infty$. Let $\mathcal{A} = \{\tau_g(A_x) | x = \infty, 1, 2, \ldots, t-1, g \in \boldsymbol{Z}_{2t-1}\}$. Then $(V, \mathcal{A})$ is a $K_2 \times K_2$ packing since

$$\partial_{00}\boldsymbol{A} = \partial_{01}\boldsymbol{A} = \partial_{10}\boldsymbol{A} = \partial_{11}\boldsymbol{A} = \boldsymbol{Z}_{2t-1} \setminus \{0\}$$

hold and ∞_i and g_j occurs exactly once in the same row or in the same column of a grid-block for each $i, j = \{0, 1\}$ and each $g \in \boldsymbol{Z}_{2t-1}$.

Also, $\tau_g(\boldsymbol{A})$ is obviously a resolution class for $g \in \boldsymbol{Z}_{2t-1}$. The number of resolution classes is

$$2t - 1 = \left\lfloor \frac{4t-1}{2} \right\rfloor,$$

which implies that the resolvable grid-block packing is optimal. ■

Next, for a prime power q, let $AG(n, q)$ be the n-dimensional affine geometry over $GF(q)$. Here, an m-flat means an m-dimensional linear subspace or its coset in $AG(n, q)$. Then, we obtain the following theorem by generalizing Proposition 14.

Theorem 17. *An optimal resolvable grid-block packing $P_{q\times q}(K_{q^n})$ exists for a prime power q and an integer n. Moreover, when n is even and q is odd, the optimal resolvable grid-block packing is a resolvable grid-block design $D_{q\times q}(K_{q^n})$.*

Proof. Let α be a primitive element of $GF(q^n)$. Then each point of $AG(n, q)$ is represented by α^i. For convenience, let $\alpha^\infty = 0$. We define a base $q \times q$

grid-block G_0 as follows:

$$\begin{array}{|cccccc|}
\alpha^{\infty} & \alpha^{0} & \alpha^{2u} & \cdots & \alpha^{(2q-4)u} \\
\alpha^{u} & \alpha^{0}+\alpha^{u} & \alpha^{2u}+\alpha^{u} & \cdots & \alpha^{(2q-4)u}+\alpha^{u} \\
\alpha^{3u} & \alpha^{0}+\alpha^{3u} & \alpha^{2u}+\alpha^{3u} & \cdots & \alpha^{(2q-4)u}+\alpha^{3u} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha^{(2q-3)u} & \alpha^{0}+\alpha^{(2q-3)u} & \alpha^{2u}+\alpha^{(2q-3)u} & \cdots & \alpha^{(2q-4)u}+\alpha^{(2q-3)u}
\end{array},$$

where

$$u = \left\lfloor \frac{q^n - 1}{2(q-1)} \right\rfloor.$$

Then G_0 is a 2-flat in $AG(n, q)$ and rows and columns of G_0 are lines (1-flats) in $AG(n, q)$. Thus, G_0 generates a $P_{q\times q}(K_{q^n})$ together with its cyclic shifts $\alpha^i G_0$ for $i = 0, 1, \ldots, u-1$ and with the parallel 2-flats of them.

In fact, let α^i and α^j be two points in $AG(n, q)$. To count the number of rows and columns of $q \times q$ grid-blocks containing α^i and α^j simultaneously, we have only to count the number of rows and columns such that $0\ (= \alpha^{\infty})$ and $\alpha^i - \alpha^j$ occur together. We can represent $\alpha^i - \alpha^j = \alpha^l$ for some integer l. When n is even and q is odd, $u = \frac{q^n-1}{2(q-1)}$ holds and there is exactly one line passing through the origin 0 and α^l. Otherwise, there is at most one line passing through the origin 0 and α^l.

Furthermore, we partition the arrays into resolution classes. We define a class $\boldsymbol{R}_0$ as a set of G_0 and its parallel 2-flats. Its cyclic shifts $\alpha^i \boldsymbol{R}_0$ for $i = 0, 1, \ldots, u-1$ are obviously resolution classes, and it is obvious that there are u resolution classes, which implies that the packing is optimal. Thus the theorem is proved. ■

5. Recursive Constructions by Orthogonal Arrays

In this section, we will give constructions of resolvable $r \times c$ grid-block packings by utilizing orthogonal arrays and latin squares. Firstly, we define a notation. For $S = \{0, 1, \ldots, s-1\}$, an *orthogonal array of order* s, *degree* k *and index* λ, denoted by $OA(s, k, \lambda)$, is an $(s^2\lambda \times m)$-matrix with entries from S such that each $(s^2\lambda \times 2)$-submatrix contains every ordered pair of S precisely λ times.

We will give a recursive construction of a resolvable $r \times c$ grid-block packing by utilizing an orthogonal array.

Theorem 18. *Assume that $r \leq c$. If there exist a resolvable $P_{r\times c}(K_v)$ with t resolution classes and an $OA(s, c+1, 1)$, then there exists a resolvable $P_{r\times c}(K_{sv})$ with st resolution classes.*

Proof. For a v-set V, let a pair $(V, \mathcal{A})$ be a resolvable $P_{r\times c}(K_v)$, where $\mathcal{A} = \{A_1, A_2, \ldots, A_b\}$ is a collection of arrays. Let $\mathcal{R} = \{\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_t\}$ be a resolution of the resolvable $P_{r\times c}(K_v)$. By applying a permutation to rows of M, we assume the each element of $(c+1)$-th column as follows without loss of generality:

$$\begin{array}{llll} \rho_{0,c} = 0, & \rho_{1,c} = 0, & \ldots, & \rho_{s-1,c} = 0, \\ \rho_{s,c} = 1, & \rho_{s+1,c} = 1, & \ldots, & \rho_{2s-1,c} = 1, \\ \vdots & \vdots & & \vdots \\ \rho_{(s-1)s,c} = s-1, & \rho_{(s-1)s+1,c} = s-1, & \ldots, & \rho_{s^2-1,c} = s-1. \end{array}$$

Then, each $s\times 1$-column vector $(\rho_{us,j}, \rho_{us+1,j}, \ldots, \rho_{us+s-1,j})'$ for $u = 0, 1, \ldots, s-1$ and $j = 0, 1, \ldots, c-1$ contains every element of S precisely once.

For $V^* = V \times S = \{(a, \rho) | a \in V, \rho \in S\}$ and for each $r \times c$ grid-block $A_i = (a^i_{x,y})$ of $(V, \mathcal{A})$, we define

$$A^*_{ij} = \Big((a^i_{x,y}, \rho_{j,x+y})\Big)$$

$$= \boxed{\begin{array}{llll} (a^i_{00}, \rho_{j,0}) & (a^i_{01}, \rho_{j,1}) & \ldots & (a^i_{0,c-1}, \rho_{j,c-1}) \\ (a^i_{10}, \rho_{j,1}) & (a^i_{11}, \rho_{j,2}) & \ldots & (a^i_{1,c-1}, \rho_{j,0}) \\ (a^i_{20}, \rho_{j,2}) & (a^i_{21}, \rho_{j,3}) & \ldots & (a^i_{2,c-1}, \rho_{j,1}) \\ \vdots & \vdots & \ddots & \vdots \\ (a^i_{r-1,0}, \rho_{j,r-1}) & (a^i_{r-1,1}, \rho_{j,r}) & \ldots & (a^i_{r-1,c-1}, \rho_{j,r+c-2}) \end{array}}$$

for $i = 1, 2, \ldots, b$ and $j = 0, 1, \ldots, s^2 - 1$. Note that in the second subscript of ρ, $x + y$ means $x + y \pmod c$. We define the set $\mathcal{A}^*$ as $\{A^*_{ij} | i = 1, 2, \ldots, b, j = 0, 1, \ldots, s^2 - 1\}$.

If two distinct elements a_1 and a_2 in V occur together at most once in A_i and the pair (ρ_1, ρ_2) occur exactly once in the $OA(s, c + 1, 1)$, then each pair (a_1, ρ_1) and (a_2, ρ_2) occurs at most once in the same row or in the same column of an array in $\mathcal{A}^*$ for any $\rho_1, \rho_2 \in S$. That is, the pair $(V^*, \mathcal{A}^*)$ is an $r \times c$ grid-block packing. It remains to show that $(V^*, \mathcal{A}^*)$ is resolvable.

We partition the arrays into st resolution classes. Let $\boldsymbol{R}^*_{uw}$ for $w = 1, 2, \ldots, t$ and $u = 0, 1, \ldots, s - 1$ be as follows:

$$\boldsymbol{R}^*_{uw} = \{A^*_{ij} | j = us, us + 1, \ldots, us + s - 1 \text{ and for any } i \text{ such that } A_i \in \boldsymbol{R}_w\}.$$

Note that, each $(\rho_{us,j}, \rho_{us+1,j}, \ldots, \rho_{us+s-1,j})'$ contains every element of S precisely once for $j = 0, 1, \ldots, c-1$. That is, each $\boldsymbol{R}^*_{uw}$ is a resolution class and the number of resolution classes is st. ∎

Moreover, by utilizing a resolvable $D_{r\times c}(K_m(n))$ instead of $P_{r\times c}(K_{mn})$ in Theorem 18 and a resolvable $D_{r\times c}(K_{sn})$, we will give a recursive construction of a resolvable $r \times c$ grid-block design.

Theorem 19. *Assume that $r \leq c$. If there exist a resolvable $D_{r\times c}(K_m(n))$, an $OA(s, c+1, 1)$ and a resolvable $D_{r\times c}(K_{sn})$, then there exists a resolvable $D_{r\times c}(K_{smn})$.*

Proof. For an nm-set V, let $(V, \mathcal{A}, \mathcal{G})$ be a resolvable $D_{r\times c}(K_m(n))$, where $\mathcal{A} = \{A_1, A_2, \ldots, A_b\}$ is a collection of grid-blocks and $\mathcal{G} = \{G_1, G_2, \ldots, G_m\}$ is a family of n-partite sets, that is a partition of V. The number b of the grid-blocks is $n^2m(m-1)/rc(r+c-2)$. Let $\mathcal{R} = \{\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_t\}$ be a resolution of the resolvable $D_{r\times c}(K_m(n))$, the number t of the resolution classes is $n(m-1)/(r+c-2)$.

Similarly, for an sn-set W, let $(W, \mathcal{F})$ be a resolvable $D_{r\times c}(K_{sn})$, where $\mathcal{F} = \{F_1, F_2, \ldots, F_{b'}\}$ is a collection of grid-blocks. The number b' of the grid-blocks is $sn(sn-1)/rc(r+c-2)$. Let $\mathcal{Q} = \{\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_{t'}\}$ be a resolution of the resolvable $D_{r\times c}(K_{sn})$, the number t' of the resolution classes is $(sn-1)/(r+c-2)$.

For $V^* = V \times S = \{(a, \rho) | a \in V, \rho \in S\}$ and for each $r \times c$ grid-block $A_i = (a^i_{x,y})$ of $(V, \mathcal{A}, \mathcal{G})$, we define $A^*_{ij} = ((a^i_{x,y}, \rho_{j,x+y}))$. And let $\mathcal{A}^*_1 = \{A^*_{ij} | i = 1, 2, \ldots, b, j = 0, 1, \ldots, s^2-1\}$. Up to now we have $s^2b = s^2n^2m(m-1)/rc(r+c-2)$ grid-blocks, but we need $smn(smn-1)/rc(r+c-2)$ grid-blocks in total.

In order to get further grid-blocks we consider a one-to-one map from W to the set $W_j = G_j \times S$, and let $(W_j, \mathcal{F}_j)$ be a resolvable $D_{r\times c}(K_{sn})$ on the point set W_j. Let $\mathcal{A}^*_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_m$. Then we obtain more $mb_2 = msn(sn-1)/rc(r+c-2)$ additional grid-blocks, in total

$$s^2b + mb_2 = \frac{smn(smn-1)}{rc(r+c-2)}$$

grid-blocks A^*_{ij} and $F^{(j)}_i \in \mathcal{F}_j$ are obtained as desired. Now let $\mathcal{A}^* = \mathcal{A}^*_1 \cup \mathcal{A}^*_2$.

By Theorem 18, if two distinct elements a_1 and a_2 in V are not contained in the same partite set G_j, then each pair (a_1, ρ_1) and (a_2, ρ_2) occurs exactly once in the same row or in the same column of a grid-block in $\mathcal{A}^*_1$ and does not occur in $\mathcal{A}^*_2$. On the other hand, in the case when two elements a_1 and a_2 in V are contained in the same partite set G_j including the case of $a_1 = a_2$, each pair (a_1, ρ_1) and (a_2, ρ_2) occurs exactly once in the same row or in the same column of a grid-block in $(W_j, \mathcal{F}_j)$ and does not occur in $\mathcal{A}^*_1$. That is, $(V^*, \mathcal{A}^*)$ is an $r \times c$ grid-block design. It remains to show that $(V^*, \mathcal{A}^*)$ is resolvable.

We partition the grid-blocks into $t^* = (smn-1)/(r+c-2)$ resolution classes. At first, let

$$\boldsymbol{R}^*_{uw} = \{A^*_{ij} | j = us, us+1, \ldots, us+s-1 \text{ and for any } i \text{ such that } A_i \in \boldsymbol{R}_w\}$$

for $w = 1, 2, \ldots, t$ and $u = 0, 1, \ldots, s-1$. Note that, each $s \times 1$ column vector $(\rho_{us,j}, \rho_{us+1,j}, \ldots, \rho_{us+s-1,j})'$ for $j = 0, 1, \ldots, c-1$ contains every element of S precisely once. That is, each $\boldsymbol{R}^*_{uw}$ is a resolution class.

For resolution classes $\boldsymbol{Q}^{(j)}_w$ in $(W_j, \mathcal{F}_j)$, let

$$\boldsymbol{Q}^*_w = \bigcup_{j=1}^{m} \boldsymbol{Q}^{(j)}_w.$$

Obviously, $\boldsymbol{Q}^*_w$ is a resolution class. The total number of resolution classes $\boldsymbol{R}^*_{uw}$ and $\boldsymbol{Q}^*_w$ is $ts + t' = (smn - 1)(r + c - 2)$ as desired. ∎

If each partite set has a single point, then we obtain the following corollary.

Corollary 20. *Assume that $r \leq c$. If there exist a resolvable $D_{r\times c}(K_m)$, an $OA(s, c+1, 1)$ and a resolvable $D_{r\times c}(K_s)$, then there exists a resolvable $D_{r\times c}(K_{sm})$.*

Moreover, we will give a construction of a resolvable $r \times c$ grid-block packing by utilizing a resolvable packing.

Theorem 21. *Assume that $r \leq c$. If there exists a resolvable $PA(v, c, 1)$ with t resolution classes, then there exists a resolvable $P_{r\times c}(K_{rv})$ with t resolution classes.*

Proof. For a v-set V, let a pair $(V, \mathcal{B})$ be a resolvable $PA(v, c, 1)$ with t resolution classes, where $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ is a collection of blocks. Let $\mathcal{R} = \{\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots \boldsymbol{R}_t\}$ be a resolution of the resolvable packing $(V, \mathcal{B})$.

For $S = \{0, 1, \ldots, r-1\}$, let $V^* = V \times S$ and let

$$A^*_i = \Big((b^i_{x+y}, y)\Big)$$

$$= \begin{array}{|llll|} \hline (b^i_0, 0) & (b^i_1, 0) & \ldots & (b^i_{c-1}, 0) \\ (b^i_1, 1) & (b^i_2, 1) & \ldots & (b^i_0, 1) \\ (b^i_2, 2) & (b^i_3, 2) & \ldots & (b^i_1, 2) \\ \vdots & \vdots & \ddots & \vdots \\ (b^i_{r-1}, r-1) & (b^i_r, r-1) & \ldots & (b^i_{r+c-2}, r-1) \\ \hline \end{array}$$

for each block $B_i = (b^i_j)$ of $(V, \mathcal{B})$ and for $i = 1, 2, \ldots, b$. Note that in the first subscript of b, $x + y$ means $x + y \pmod c$. We define the set $\mathcal{A}^*$ as $\{A^*_i | i = 1, 2, \ldots, b\}$.

Since two distinct elements b_1 and b_2 in V occur together at most once in a block of the $PA(v, c, 1)$, each pair (b_1, i) and (b_2, i) occurs at most once in the

same row of an array in $\mathcal{A}^*$ for any $i \in S$. And each pair (b_1, i) and (b_2, j) occurs at most once in the same column of an array in for any $i \neq j \in S$. Hence, the pair $(V, \mathcal{A}^*)$ is an $r \times c$ grid-block packing. Moreover, let

$$\boldsymbol{R}_w^* = \{A_i^* | \text{ for any } B_i \in \boldsymbol{R}_w\}$$

for $w = 1, 2, \ldots, t$. Obviously, each $\boldsymbol{R}_w^*$ is a resolution class. Thus, the theorem is proved. ■

Theorem 22. *Assume that $r \leq c$. If there exist a resolvable $PA(v, c, 1)$ with t resolution classes and a resolvable $P_{r\times c}(K_{rc})$ with $s+1$ grid-blocks, then there exists a resolvable $P_{r\times c}(K_{rv})$ with $st+1$ resolution classes.*

Proof. Let M_0, M_1, ..., M_s be $r \times c$ grid-blocks of a resolvable $P_{r\times c}(K_{rc})$ on the point set $R \times C$, where $R = \{0, 1, \ldots, r-1\}$ and $C = \{0, 1, \ldots, c-1\}$. We rename the (i, j)-th element of $M_0 = (m_{ij}^0)$ simply by (i, j). Then we have

$$M_0 = \begin{array}{|cccc|} \hline (0,0) & (0,1) & \ldots & (0,c-1) \\ (1,0) & (1,1) & \ldots & (1,c-1) \\ \vdots & \vdots & \ddots & \vdots \\ (r-1,0) & (r-1,1) & \ldots & (r-1,c-1) \\ \hline \end{array} \tag{5.1}$$

and by this renaming, M_1, M_2, ..., M_s are also represented as grid-blocks with elements in $R \times C$. It is obvious that each element in $R \times C$ occurs exactly once in every M_l. Moreover, (i, j) and (i, j') do not occur in the same row and column of M_1, M_2, ..., M_s. Similarly, (i, j) and (i', j) do not occur, too.

Now, let $(V, \mathcal{B})$ be a resolvable $PA(v, c, 1)$ with t resolution classes $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, ..., $\boldsymbol{R}_t$. For each block $B = (b_0, b_1, \ldots, b_{c-1})$ in $\mathcal{B}$ and for each $M_l = \left((\sigma_{xy}^l, \tau_{xy}^l)\right)$, let

$$A_B^l = \left((b_{\tau_{xy}^l}, \sigma_{xy}^l)\right)$$

be an $r \times c$ grid-block with element in $V \times R$. Let $\mathcal{A}_w^l = \{A_B^l | B \in \boldsymbol{R}_w\}$, then $\mathcal{A}_w^l$ is a resolution class in $V \times R$, since every element in $B \times R$ occurs exactly once in A_B^l. Here, let

$$\mathcal{A} = \mathcal{A}_1^0 \cup \Big\{ \bigcup_{l=1}^{s} \bigcup_{w=1}^{t} \mathcal{A}_w^l \Big\}.$$

Then, for any two distinct elements (b, i) and (b', i') in $V \times R$,

(i) *in case of $b = b'$, (b, i) and (b, i'), $i \neq i'$, occur exactly once in a column of a grid-block in $\mathcal{A}_1^0$,*

(ii) *in case of $b \neq b'$,*

(a) *if there is no block in* $(V, \mathcal{B})$ *containing* b *and* b' *simultaneously, then* (b, i) *and* (b', i') *do not occur in the same grid-block of* $\mathcal{A}$,

(b) *if there is a block* B *containing* b *and* b', *there is at most one row or one column of a grid-block in* $\mathcal{A}_w^l$ *for* $l = 1, 2, \ldots, s$ *which contains* (b, i) *and* (b', i'), $i \neq i'$, *simultaneously*,

(c) *if there is a block* B *containing* b *and* b' *and* B *belongs to* $\boldsymbol{R}_1$, *there is exactly one row of a grid-block in* $\mathcal{A}_1^0$ *which contains* (b, i) *and* (b', i), $i = 0, 1, \ldots, r-1$, *simultaneously, by the definition of a resolvable* $P_{r\times c}(K_{rc})$.

Thus, $(V \times R, \mathcal{A})$ is a resolvable $P_{r\times c}(K_{rv})$ with $st+1$ resolution classes $\mathcal{A}_w^l$. ■

By coupling two mutually orthogonal $r \times r$ latin squares, we can obtain an Euler square. Thus together with M_0 in (5.1) for $r = c$, we obtain a $P_{r\times r}(K_{r^2})$ with two grid-blocks. Since there are two mutually orthogonal latin squares except for $r = 6$, we obtain the following corollary.

Corollary 23. *For a positive integer* $r \neq 6$, *if there exist a resolvable* $B(v, r, 1)$, *then there exists a resolvable* $P_{r\times r}(K_{rv})$ *with* $(v-1)/(r-1)+1$ *resolution classes.*

Moreover, when r is an odd prime power, it was shown that there exists a resolvable $D_{r\times r}(K_{r^2})$ with $(r+1)/2$ grid-blocks by Raghavarao [14]. Thus, we obtain the following corollary.

Corollary 24. *For an odd prime power, if there exist a resolvable* $PA(v, r, 1)$ *with* t *resolution classes, then there exists a resolvable* $P_{r\times r}(K_{rv})$ *with* $t(r-1)/2+1$ *resolution classes.*

For example, in the case of $r = c = 3$, it is well known that there is a resolvable $B(6t+3, 3, 1)$ for any positive integer t. By Corollary 23, we obtain a resolvable $P_{3\times 3}(K_{18t+9})$ with $3t+2$ resolution classes for any t. The number of pairs which occur in the same row or in the same column in a grid-block is $18(3t+2)(2t+1)$ and the total number of the pairs of two distinct points is $(18t+9)(18t+8)/2$. That is, more than $2/3$ of the pairs occur in the same row or in the same column in the grid-block packing.

In addition, it is known that there is a resolvable $PA(6t, 3, 1)$ with $3t-1$ resolution classes for any $t \geq 2$ (see Colbourn and Dinitz [7, pp. 413]). That is, we obtain a resolvable $P_{3\times 3}(K_{18t})$ with $3t$ resolution classes. Similarly, in this case, about $2/3$ of the pairs occur in the same row or in the same column in the grid-block packing.

6. Constructions by the Method of Difference and an Asymptotic Existence Theorem

In this section, we will give constructions of $r \times c$ grid-block designs and resolvable $r \times c$ grid-block designs by utilizing the *method of difference*.

For an integer e such that $e|(q-1)$, we define H_i^e as $\{\alpha^t | t \equiv i \pmod{e}\}$, where α is a primitive element of $GF(q)$. Clearly H_0^e is a subgroup of $GF(q) \setminus \{0\}$, which is denoted by H^e. We select an element s_m from each H_m^e for $m = 0, 1, \ldots, e-1$ and call the set $S_e = \{s_0, s_1, \ldots, s_{e-1}\}$ a *system of representatives* for cosets modulo H^e. Then $GF(q) \setminus \{0\} = H^e \circ S_e$ holds. We define $\mathcal{H}^e$ as the class of cosets $\{H_0^e, H_1^e, \ldots, H_{e-1}^e\}$.

We note that if q is even, i.e., if q is a power of 2, then $-1 = 1$ is always an e-th root of unity. If q is odd, then $-1 \in H^e$ if and only if $2e|(q-1)$, since $-1 = \alpha^{(q-1)/2}$ is an e-th power if and only if $(q-1)/2 \equiv 0 \pmod{e}$.

We will give constructions by utilizing the method of difference over finite fields.

Theorem 25. *If there exists a $DF(q, K_r \times K_c, 1)$ in $G(q)$, then there exists a $DF(q^n, K_r \times K_c, 1)$ in $G(q^n)$, hence a $D_{r\times c}(K_{q^n})$ exists for $n \geq 1$.*

Proof. Let $\boldsymbol{A} = (A_i | i \in I)$ be a $DF(q, K_r \times K_c, 1)$ in $G(q)$. Then $\partial \boldsymbol{A} = \sum_{i\in I} \partial A_i = G(q) \setminus \{0\}$. If we consider $GF(q)$ as a subfield of $GF(q^n)$, then $GF(q) \setminus \{0\}$ is the group H^e of the e-th powers of a primitive element in $GF(q^n)$ where $e = (q^n - 1)/(q-1)$.

Let S be a system of representatives for the cosets $\mathcal{H}^e$ modulo H^e in $GF(q^n)$. Thus S is a set of e field elements such that $S \circ H^e = G(q^n) \setminus \{0\}$. Consider the family $\boldsymbol{A}^* = (sA_i | i \in I,\ s \in S)$. Noting that the list of differences from the set sA_i is $(s) \circ \partial A_i$, we have

$$\partial \boldsymbol{A}^* = \sum_{s\in S} \sum_{i\in I} (s) \circ \partial A_i = S \circ \partial \boldsymbol{A} = S \circ H^e = G(q^n) \setminus \{0\}.$$

That is, $\boldsymbol{A}^*$ is a $DF(q^n, K_r \times K_c, 1)$ and there exists a $D_{r\times c}(K_{q^n})$. ∎

Theorem 26. *For a prime power $q \equiv 1 \pmod{rc(r+c-2)}$, if there exists an $r \times c$ array $A = (a_{ij})$ over $GF(q)$ such that two differences in ∂A lie in each coset modulo $H^{\frac{1}{2}rc(r+c-2)}$, or equivalently, such that*

$$\begin{aligned}\overrightarrow{\partial} A = &\{a_{jl} - a_{il} | 1 \leq i < j \leq r,\ 1 \leq l \leq c\} \\ &\cup \{a_{lj} - a_{li} | 1 \leq l \leq r,\ 1 \leq i < j \leq c\}.\end{aligned}$$

is a system of representatives for the cosets $\mathcal{H}^{\frac{1}{2}rc(r+c-2)}$, then there exists a $DF(q, K_r \times K_c, 1)$ in $G(q)$, hence a $D_{r\times c}(K_q)$ exists.

Proof. Let $e = \frac{1}{2}rc(r+c-2)$. Then $-1 \in H^e$ holds, since $2e|(q-1)$. By the assumption, $\overrightarrow{\partial} A$ must have precisely one entry in each coset H^e_m for $0 \leq m \leq e-1$, and $\partial A = (1,-1) \circ \overrightarrow{\partial} A$ holds. Let S be a system of representatives for the cosets of the quotient group $H^e/\{1,-1\}$, so that $H^e = S \circ (1,-1)$. Let $\boldsymbol{A}$ be the family $(sA|s \in S)$. Then,

$$\partial \boldsymbol{A} = S \circ \partial A = S \circ (1,-1) \circ \overrightarrow{\partial} A = H^e \circ \overrightarrow{\partial} A = GF(q) \setminus \{0\},$$

i.e., $\boldsymbol{A}$ is a $DF(q, K_r \times K_c, 1)$ in $G(q)$ and there exists a $D_{r\times c}(K_q)$. ∎

The following construction for a resolvable BIB design is obtained by Ray-Chaudhuri and Wilson [6] (see also Beth *et al.* [3, pp. 356-358]).

Proposition 27. *Theorem 1.1. Lemma 2.1. For a prime power q, if there is a mutually disjoint $DF(q,k,1)$ in $G(q)$, then there exists a resolvable $B(kq,k,1)$.*

By utilizing Proposition 27, we obtain the following corollary.

Corollary 28. *Let q be a prime power, if there exist a mutually disjoint $DF(q,rc,1)$ in $G(q)$ and a $D_{r\times c}(K_{rc})$, then a resolvable $D_{r\times c}(K_{rcq})$ exists.*

Proof. By assumption, there is a resolvable $B(rcq, rc, 1)$. For each block in the resolvable $B(v, rc, 1)$, we construct $(rc-1)/(r+c-2)$ grid-blocks of a $D_{r\times c}(K_{rc})$ whose elements set is the block. Thus we obtain a resolvable $D_{r\times c}(K_{rcq})$. ∎

Here, we define a mutually disjoint $DF(q, K_r \times K_c, 1)$ similar to the case of $DF(q,k,1)$. Then we obtain the following theorem.

Theorem 29. *For a prime power q, assume that $rc(r+c-2)|(q-1)$ holds. If there exist a mutually disjoint $DF(q, K_r \times K_c, 1)$ in $G(q)$ and a $D_{r\times c}(K_{rc})$, then there exists a resolvable $D_{r\times c}(K_{rcq})$.*

Proof. Let $E_0, E_1, \ldots, E_{b_{rc}-1}$ be $r \times c$ grid-blocks of a $D_{r\times c}(K_{rc})$, where $b_{rc} = (rc-1)/(r+c-2)$. For an rc-set B, we define $E_h(B)$ as the $r \times c$ grid-block E_h whose elements are labelled by the points in B.

Let $\boldsymbol{A} = \{A_1, A_2, \ldots, A_s\}$ be a mutually disjoint $DF(q, K_r \times K_c, 1)$, where the number of base grid-blocks A_i is $s = (q-1)/rc(r+c-1)$. Hence,

$$\sum_{i=1}^{s} |A_i| = s \cdot rc < q,$$

and without loss of generality, we assume that $0 \notin A_i$ for $i = 1, 2, \ldots, s$. For $M = \{0, 1, \ldots, rc-1\}$, let $V = G(q) \times M$. For an rc-set let $B_0 = \{0\} \times$

$M = \{(0,0),(0,1),\ldots,(0,rc-1)\}$, let $E_h(B_0)$ be a grid-block labelled by B_0 for $h = 0,1,\ldots,b_{rc}-1$ and

$$B_i^j = A_i \times \{j\} = \Big((a_{m,n}^i, j)\Big)$$

for $i = 1,2,\ldots,s$ and $j \in M$. Up to now we have $b_{rc}+rcs = (rc+q-2)/(r+c-2)$ base grid-blocks. In order to get further base grid-blocks we choose rc distinct elements $u_0 = 1, u_1, \ldots, u_{rc-1}$ of $GF(q) \setminus \{0\}$, and let

$$C_x = \{(u_0 x, 0), (u_1 x, 1), \ldots, (u_{rc-1}x, rc-1)\}$$

for $x \in GF(q) \setminus \{0\}$. We have $b_{rc}(q-1)$ new base grid-blocks which are $E_h(C_x)$ for $h = 0,1,\ldots,b_{rc}-1$ and for $x \in GF(q) \setminus \{0\}$, in total

$$(rc+q-2)/(r+c-2) + b_{rc}(q-1) = (rcq-1)/(r+c-2)$$

base gird-blocks $E_h(B_0)$, B_i^j and $E_h(C_x)$ are obtained. Now we replace the base grid-blocks B_i^j by $u_j B_i^j$ to satisfy the condition of resolvability and we define the set of $\boldsymbol{A}^*$ of new base grid-blocks by

$$\begin{aligned}\boldsymbol{A}^* = &\{E_h(B_0) | h = 0,1,\ldots,b_{rc}-1\}\\ &\cup \{u_j B_i^j | i = 1,2,\ldots,s,\ j \in M\}\\ &\cup \{E_h(C_x) | x \in GF(q)\setminus\{0\},\ h = 0,1,\ldots,b_{rc}-1\}.\end{aligned}$$

The pure differences arise from the $u_j B_i^j$, and the mixed differences come from the $E_h(C_x)$ and $E_h(B_0)$. Since $\boldsymbol{A}$ is a $DF(q, K_r \times K_c, 1)$,

$$\sum_i \partial_{jj}(u_j B_i^j) = \sum_i u_j \partial_{jj} B_i^j = u_j(G(q)\setminus\{0\}) = G(q)\setminus\{0\}$$

holds. Furthermore, for $i < j$,

$$\sum_{h=0}^{b_{rc}-1} \partial_{ij} E_h(B_0) = \Delta_{ij} B_0 = \{0\}$$

and

$$\begin{aligned}\sum_{x\in G(q)\setminus\{0\}} \left(\sum_{h=0}^{b_{rc}-1} \partial_{ij} E_h(C_x)\right) &= \sum_{x\in G(q)\setminus\{0\}} \Delta_{ij} C_x\\ &= (u_j - u_i)(G(q)\setminus\{0\})\\ &= G(q)\setminus\{0\}\end{aligned}$$

hold.

Hence $\partial_{jj}\boldsymbol{A}^* = G(q)\backslash\{0\}$ and $\partial_{ij}\boldsymbol{A}^* = G(q)$ hold for $i \neq j$, which implies that the pair $(V, \mathcal{A}^*)$ is an $r \times c$ grid-block design for $\mathcal{A}^* = \{A+g | A \in \boldsymbol{A}^*,\ g \in G(q)\}$. It remains to show that $\mathcal{A}^*$ is resolvable.

We partition the grid-blocks into $t = (rcq-1)/(r+c-2)$ resolution classes. For $A_i = (a^i_{m,n})$, we identify the set $\{a^i_{m,n} | m = 1, 2, \ldots, t,\ n = 1, 2, \ldots, c\}$ with the grid-block A. Let $\boldsymbol{P}_0$ be as follows:

$$\begin{aligned}\boldsymbol{P}_0 = &\{E_0(B_0)\} \cup \{u_j B^j_i | i = 1, 2, \ldots, s,\ j \in M\} \\ &\cup \{E_0(C_x) | x \notin A_i \text{ for any } i\}.\end{aligned}$$

Then the number of grid-blocks in $\boldsymbol{P}_0$ is $1 + rcs + (q - 1 - rcs) = q$. The points in these grid-blocks are

$(0,0), (0,1), \ldots, (0, rc-1)$,

$(u_j a^i_{11}, j), \ldots, (u_j a^i_{21}, j), \ldots, (u_j a^i_{r,c}, j)$ for $i = 1, 2, \ldots, s$ and $j \in M$, and

$(u_0 x, 0), (u_1 x, 1), \ldots, (u_{rc-1} x, rc-1)$ for all i, j, n, m and for $0 \neq x \neq a^i_{m,n}$.

Obviously every point of $V = G(q) \times M$ occurs exactly once, i.e. $\boldsymbol{P}_0$ is a resolution class.

We define a map $\tau_g : (x, j) \mapsto (x + g, j)$ for $g \in G(q)$ and $\boldsymbol{P}_g = \{\tau_g(A) | A \in \boldsymbol{P}_0\}$. Then it is obvious that $\boldsymbol{P}_g$'s are resolution classes. It is easy to see that $\boldsymbol{Q}_x = \{\tau_g E_0(C_x) : g \in G(q)\}$ is a resolution class for each $x \in A_1 \cup A_2 \cup \ldots \cup A_s$. Similarly, we construct more classes $\boldsymbol{R}^h_x = \{\tau_g E_h(C_x) : g \in G(q)\}$ and $\boldsymbol{R}^h_0 = \{\tau_g E_h(B_0) : g \in G(q)\}$ for $h = 1, 2, \ldots, b_{rc} - 1$. $\boldsymbol{R}^h_x$ is also a resolution class. The total number of resolution classes $\boldsymbol{P}_g$, $\boldsymbol{Q}_x$ and $\boldsymbol{R}^h_x$ is

$$q + rcs + (b_{rc} - 1)q = \frac{rcq - 1}{r + c - 2} = t.$$

Hence the theorem is proved ■

From Corollary 28, we obtain a resolvable $D_{r\times c}(K_{rc(rc(rc-1)n+1)})$ when $rc(rc-1)n+1$ is a prime power. But from Theorem 29, we obtain a resolvable $D_{r\times c}(K_{rc(rc(r+c-2)n+1)})$ when $rc(r+c-2)n+1$ is a prime power. By the existence of a $D_{r\times c}(K_{rc})$, $(r+c-2)|(rc-1)$ holds. That is, Corollary 28 is included in Theorem 29. For example, in case of $r = c = 3$ and $q = 37$, there exists a cyclic $D_{3\times 3}(K_{37})$ but a $B(37, 9, 1)$ does not exist. Hence we can not find the existence of a resolvable $D_{3\times 3}(K_{333})$ by Corollary 28, while we can claim the existence by Theorem 29.

Moreover, by utilizing a $DP(q, K_r \times K_c, 1)$ with s base grid-blocks in $G(q)$, we obtain the following corollary.

Corollary 30. *For a prime power q, if there exists a $DP(q, K_r \times K_c, 1)$ with $s \geq 0$ base grid-blocks in $G(q)$, then there exists a resolvable $P_{r\times c}(K_{rcq})$ with $q+rcs$ resolution classes. Moreover, if there exists a $D_{r\times c}(K_{rc})$, then there exists a resolvable $P_{r\times c}(K_{rcq})$ with $q(rc-1)/(r+c-2)+rcs$ resolution classes.*

Next, by utilizing Wilson's choice function, we will show the existence of a $D_{r\times c}(K_q)$ and a resolvable $D_{r\times c}(K_{rcq})$ for a sufficiently large prime power q. For a prime power $q = ef + 1$ and an integer $k \geq 2$, let P_k be the set of ordered pairs $\{(i,j)|1 \leq i < j \leq k\}$. We define a *choice* to be any map $C : P_k \to \mathcal{H}^e$, assigning each pair $(i,j) \in P_k$ to a coset $C(i,j)$ modulo H^e in $GF(q)$. A k-tuple $(a_1, a_2, \ldots, a_k)$ of elements of $GF(q)$ is *consistent* with the choice C if and only if $a_j - a_i \in C(i,j)$ for all $1 \leq i < j \leq k$. Then, the following lemma is well known by Wilson [15].

Lemma 31. *Let $q = ef + 1$ be a prime power such that $q > e^{k(k-1)}$, then for any choice $C : P_k \to \mathcal{H}^e$, there exists a k-tuple $(a_1, a_2, \ldots, a_k)$ of elements of $GF(q)$ which is consistent with C.*

The following existence theorem is based on the facts of Theorem 26 and Lemma 31.

Theorem 32. *Let q be a prime power with $q > \{\frac{1}{2}rc(r+c-2)\}^{rc(rc-1)}$ and assume that $q-1 \equiv 0 \pmod{rc(r+c-2)}$ holds, then there exists a $DF(q, K_r \times K_c, 1)$ in $G(q)$, hence a $D_{r\times c}(K_q)$ exists.*

Proof. Firstly, let $M = \{(i,j)|1 \leq i \leq r,\ 1 \leq j \leq c\}$ and $P_{r\times c}$ be the set of the following ordered pairs of M:

$$P_{r\times c} = \{((i_1,j_1),(i_2,j_2))|1 \leq i_1 \leq i_2 \leq r,\ 1 \leq j_1 \leq j_2 \leq c\}\backslash$$
$$\{((i,j),(i,j))|1 \leq i \leq r,\ 1 \leq j \leq c\}.$$

We divide $P_{r\times c}$ into two subsets Q and R such that

$$Q = \{((i_1,l),(i_2,l))|1 \leq i_1 < i_2 \leq r,\ 1 \leq l \leq c\}$$
$$\cup \{((l,j_1),(l,j_2))|1 \leq l \leq r,\ 1 \leq j_1 < j_2 \leq c\} \text{ and}$$
$$R = \{((i_1,j_1),(i_2,j_2))|1 \leq i_1 < i_2 \leq r,\ 1 \leq j_1 \neq j_2 \leq c\}.$$

By considering a pair $(i,j) \in M$ as $cj+i$, the set of pairs in $P_{r\times c}$ can be identified with $P_{rc} = \{(i',j')|1 \leq i' < j' \leq rc\}$.

Let $e = \frac{1}{2}rc(r+c-2)$ and $C : P_{r\times c} \to \mathcal{H}^e$ be a choice which is an injection from Q to $\mathcal{H}^e$ and which maps R into $\mathcal{H}^e$ arbitrarily.

Since $q > \{\frac{1}{2}rc(r+c-2)\}^{rc(rc-1)}$, we can find an $r \times c$ matrix $A = (a_{ij})$ consistent with the choice C by Lemma 31. That is, C maps the differences of ∂A exactly twice to each coset in $\mathcal{H}^e$. Thus, there exists a $DF(q, K_r \times K_c, 1)$ in $G(q)$, hence a $D_{r\times c}(K_q)$ exists by Theorem 26. ■

By virtue of Theorem 29 and 32, we obtain the following theorem.

Theorem 33. *Let q be a prime power with $q > \{\frac{1}{2}rc(r+c-2)\}^{rc(rc+1)}$ and assume that $q-1 \equiv 0 \pmod{rc(r+c-2)}$ holds. Then, there exists a resolvable $D_{r\times c}(K_{rcq})$ if a $D_{r\times c}(K_{rc})$ exists.*

Proof. It is sufficient that there exists a mutually disjoint $DF(q, K_r \times K_c, 1)$ in $G(q)$. Let $M = \{(i,j)|1 \le i \le r,\ 1 \le j \le c\}$ and let $P_{r\times c+1}$ be the set of the following ordered pairs of $M \cup \{0\}$, that is,

$$\begin{aligned} P_{r\times c+1} = &\{((i_1,j_1),(i_2,j_2))|1 \le i_1 \le i_2 \le r,\ 1 \le j_1 \le j_2 \le c\}\backslash \\ &\{((i,j),(i,j))|1 \le i \le r,\ 1 \le j \le c\} \\ &\cup \{(0,(i,j))|1 \le i \le r,\ 1 \le j \le c\}. \end{aligned}$$

We divide $P_{r\times c+1}$ into three subsets as follows:

$$\begin{aligned} Q = &\{((i_1,l),(i_2,l))|1 \le i_1 < i_2 \le r,\ 1 \le l \le c\} \\ &\cup \{((l,j_1),(l,j_2))|1 \le l \le r,\ 1 \le j_1 < j_2 \le c\}, \\ R = &\{((i_1,j_1),(i_2,j_2))|1 \le i_1 < i_2 \le r,\ 1 \le j_1 \ne j_2 \le c\} \text{ and} \\ S = &\{(0,(i,j))|1 \le i \le r,\ 1 \le j \le c\}. \end{aligned}$$

By considering a pair $(i,j) \in M$ as $cj+i$, the set of pairs in $P_{r\times c+1}$ can be identified with $P_{rc+1} = \{(i',j')|1 \le i' < j' \le rc+1\}$.

Let $e = \frac{1}{2}rc(r+c-2)$ and $C : P_{r\times c+1} \to \mathcal{H}^e$ be a choice such that (i) C is an injection from Q to $\mathcal{H}^e$, (ii) it maps R into $\mathcal{H}^e$ arbitrarily and (iii) it does each ordered pair $(0,(i,j)) \in S$ into mutually distinct cosets H^e_m. Then by Lemma 31, we can find an element $a_0 \in GF(q)$ and an $r \times c$ matrix (a_{ij}) over $GF(q)$ consistent with the choice C.

Let $B = (b_{ij}) = (a_{ij} - a_0)$, then the elements of ∂B occur exactly twice in each coset of $\mathcal{H}^e$. Then $\boldsymbol{B} = \{hB|h \in H^e/\{1,-1\}\}$ is a $DF(q, K_r \times K_c, 1)$. And, b_{ij}'s lie in distinct cosets modulo H^e for $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,c$. Moreover, all points contained in $hB \in \boldsymbol{B}$ are distinct, that is, the sets hB for $h \in H^e/\{1,1\}$ are disjoint, i.e., $\boldsymbol{B}$ is also a mutually disjoint $DF(q, K_r \times K_c, 1)$, which proves the theorem by Theorem 29. ■

Acknowledgement

The authors would like to appreciate Professor Kunio Shimizu for his valuable comments. The authors would like to thank my colleague Maho Ikeda for helping us by developing a simulation program. Also the authors would like to thank the referee for his constructive comments.

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