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THE POWER INDICES FOR MULTI-CHOICE MULTI-VALUED GAMES

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Abstract. We extend the *simple multi-choice game* to a *multi-valued multi-choice game.* We prove that the Shapley value for *multi-valued multi-choice games* is unique and coincides with the Shapley value for multi-choice cooperative games.

1. Introduction

The set of all traditional simple games is not close under addition, therefore when we restrict the domain of the Shapley value to the set of all simple games, the Shapley value becomes very complicated. It is well-known that Dubey solved the problem in [2] (1975). In that paper, Dubey showed that the Shapley value for simple games is unique and coincides with the Shapley value for cooperative games.

In chapter 2, 3, 4 in [3] (1991), we extended the traditional Shapley value to the *Shapley value for multi-choice cooperative games*. For abbreviation, we call the *Shapley value for multi-choice cooperative games* the *multi-choice Shapley value*.

In chapter 8 in [3] (1991), we extended Dubey's result to *simple multi-choice games.* In that chapter, we restricted the domain of the multi-choice Shapley value to the set of all simple multi-choice games, and got a result analogous to Dubey's result in [2] (1975) i.e. we showed that the Shapley value for simple multi-choice games is unique and coincides with the multi-choice Shapley value.

In this article, we will extended Dubey's result to an even more complicated game called *the multi-valued multi-choice game*. We will show that the Shapley value for *multi-valued multi-choice games* is unique and coincides with the multi-choice Shapley value. Therefore, we may use the multi-choice Shapley value as the power indices for the players in a multi-valued multi-choice game.

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2. Definitions and Notations

Following [5] (1993), we have the following definitions and notations.

Let $N = \{1, 2, \dots, n\}$ be the set of players. We allow each player to have $(m+1)$ actions, say $\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_m$, where k is the level of action σ_k.

Let $\beta = \{0, 1, \dots, m\}$. The action space of N is defined by $\beta^n = \{(x_1, \dots, x_n) \mid x_i \in \beta,\ \forall i \in N\}$. Thus $(x_1, \dots, x_n)$ is called an action vector of N, and $x_i = k$ if and only if player i takes action σ_k.

Definition 2.1. A *multi-choice cooperative game* in characteristic function form is the pair (β^n, V) defined by $V : \beta^n \to \mathbb{R}$, such that $V(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, \dots, 0)$.

We can identify the set of all multi-choice cooperative games by: $G \simeq \mathbb{R}^{(m+1)^n - 1}$.

As we allow players to have more than two choices, we should expect some differences due to actions. Since we do not assume that action σ_2 is say, twice as powerful as action σ_1, and since we do not assume that the difference between σ_{k-1} and σ_k is the same as the difference between σ_k and σ_{k+1}, etc., giving weights (discrimination) to actions is necessary.

Let $w : \beta \to \mathbb{R}_+$ be a non-negative function such that $w(0) = 0$, $w(0) < w(1) \le w(2) \le \dots \le w(m)$; then w is called a **weight function** and $w(i)$ is said to be a **weight** of σ_i.

Remark 2.1. In the real world, people used to estimate the value of an action before the players execute the action in a game. Therefore we regard $w(i)$ as a *prior value*, or say *prior power index* of action σ_i, without discrimination on the players.

We define the value, or say power index for multi-choice cooperative games by $\phi^w : G \to M_{m \times n}$ such that

$$\phi^w(V) = \begin{pmatrix} \phi^w_{11}(V) & \cdots & \phi^w_{1n}(V) \\ \phi^w_{21}(V) & & \phi^w_{2n}(V) \\ \vdots & & \vdots \\ \phi^w_{m1}(V) & \cdots & \phi^w_{mn}(V) \end{pmatrix}.$$

Here $\phi^w_{ji}(V)$ is the power index or the value of player i when he takes action σ_j in game V.

Remark 2.2. Player i has the value $\phi^w_{ji}(V)$ only after he executes σ_j in game V. Therefore, we regard $\phi^w_{ji}(V)$ as the *posterior value*, or say *posterior power index* of action σ_j for player i in game V.

In [5] (1993), we showed that when w is given, there exists a unique ϕ^w satisfying five axioms analogous to the axioms of the traditional Shapley value.

Definition 2.2. Given $x \in \beta^n$, let

$$V^x(y) = \begin{cases} 1 & \text{if } y \geq x \\ 0 & \text{otherwise;} \end{cases}$$

then V^x is called a multi-choice unanimity game.

Axiom 1. Given an action space β^n and the weights $w(0), w(1), \ldots, w(m)$, for each multi-choice unanimity game

$$V^x(y) = \begin{cases} 1 & \text{if } y \geq x \\ 0 & \text{otherwise;} \end{cases}$$

the value $\phi^w_{x_i,i}(V^x)$ is proportional to $w(x_i)$.

The above Axiom is similar to Shapley's original Axiom of the weighted Shapley value for unanimity games. However, we give weights to the actions rather than the players. Therefore, our multi-choice Shapley value is symmetric in columns (players) and asymmetric in rows(actions), please see [3] for detail.

Given two action vectors $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, we define

$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\})$$

and

$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}).$$

Definition 2.3. An action vector $x^* \in \beta^n$ is called a **carrier** of V, if $V(x^* \wedge x) = V(x)$ for all $x \in \beta^n$. We call x^0 a *minimal carrier* of V if $\sum x_i^0 = \min\{\sum x_i \mid x \text{ is a carrier of } V\}$.

The following is a version of the usual efficiency axiom.

Axiom 2. If x^* is a carrier of V then, for $m = (m, m, \ldots, m)$ we have

$$\sum_{x_i^* \in x^*} \phi^w_{x_i^*,i}(V) = V(m).$$

By $x_i^* \in x^*$ we mean x_i^* is the i-th component of x^*.

Axiom 3. $\phi^w(V^1 + V^2) = \phi^w(V^1) + \phi^w(V^2)$, where $(V^1 + V^2)(x) = V^1(x) + V^2(x)$.

Axiom 4. Given $x^0 \in \beta^n$ if $V(x) = 0$, whenever $x \not\geq x^0$, then for each $i \in N$, $\phi^w_{k,i}(V) = 0$, for all $k < x_i^0$.

Remark 2.3. Given a multi-choice cooperative game $(\beta^n; V)$, suppose there exist two different action vectors $x^0; y^0 \in \beta^n$ such that

$$V(x) = 0; \text{ whenever } x \not\geq x^0$$

and

$$V(x) = 0; \text{ whenever } x \not\geq y^0;$$

then it is trivial that

$$V(x) = 0; \text{ whenever } x \not\geq x^0 \vee y^0$$

Hence, Axiom 4 states that in games that stipulate a minimal exertion from players, those who fail to meet this minimal level cannot be rewarded.

Let $(x \mid x_i = k)$ denote a vector with $x_i = k$.

Definition 2.4. Player i is said to be a **dummy** player if $V((x \mid x_i = k)) = V((x \mid x_i = 0))$ for all $x \in \beta^n$ and for all $k \in \beta$.

An action σ_k with $k \neq 0$ is said to be a **dummy action** for player i if $V((x \mid x_i = k)) = V((x \mid x_i = k-1))$ for all $x \in \beta^n$.

Axiom 5. Given $(\beta^n; V)$, if player i is a dummy player, then $\varphi^w_{k;i}(V) = 0$, for all $\sigma_k \in \beta$.

Definition 2.5. Given $S \subseteq N$, let $b(S)$ be the binary vector with components $b_i(S)$ satisfying

$$b_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Let $|S|$ be the number of elements of S.

Definition 2.6. Given β^n and $w(0) = 0, w(1); \ldots; w(m)$, for any $x \in \beta^n$, we define $\|x\|_w = \sum_{r=1}^{n} w(x_r)$.

Definition 2.7. Given $x \in \beta^n$ and $j \in N = \{1; 2; \ldots; n\}$, we define $M_j(x) = \{i \mid x_i \neq m; \ i \neq j\}$.

From Theorem 2 in [5] (1993), we have an explicit formula for the Shapley value as follows.

$$\varphi^w_{ij}(V) = \sum_{k=1}^{m} \sum_{\substack{x_j = k \\ x \neq 0 \\ x \in \beta^n}} \left[\sum_{T \subseteq M_j(x)} (-1)^{|T|} \frac{w(x_j)}{\|x\|_w + [w(x_r+1) - w(x_r)]} \right] \tag{2.1}$$

$$\cdot [V(x) - V(x - b(\{j\}))].$$

Definition 2.8. A *simple multi-choice game* in characteristic function form is the pair $(\beta^n; V)$ defined by $V : \beta^n \to \{0; 1\}$, such that $V(\mathbf{0}) = 0$, where $\mathbf{0} = (0; 0; \dots; 0)$.

In a voting game, if each player has three choices namely, vote NO or vote NO COMMENT or vote YES, then this is a typical example of simple multi-choice game.

Definition 2.9. A multi-choice cooperative game is called a non-decreasing multi-choice cooperative game if $x \geq y \Rightarrow V(x) \geq V(y)$.

Let $NSMG$ denote the set of all non-decreasing simple multi-choice games. Since $NSMG$ is not closed under addition, if we restrict G to $NSMG$ then, we encounter a complicated problem as Dubey did in [2] (1975).

The following axiom is a well-known axiom proposed by Dubey in [2] (1975), however, it is also a well-known concept in Reliability Theory.

Axiom 3** When the weight function w is given then

$$\phi^w(V^1 \vee V^2) + \phi^w(V^1 \wedge V^2) = \phi^w(V^1) + \phi^w(V^2);$$

where

$$(V^1 \vee V^2)(x) = \max\left\{V^1(x); V^2(x)\right\};$$

and

$$(V^1 \wedge V^2)(x) = \min\left\{V^1(x); V^2(x)\right\};$$

for all $x \in \beta^n$.

In chapter 8 in [3] (1991), we showed that if w is given,then there exists a unique value or say power index ψ^w defined on $NSMG$ such that ψ^w satisfies axioms 1, 2, 3^{**},4 and 5, and ψ^w coincides with the multi-choice Shapley value ϕ^w given by formula (2.1). In this article, we will extend the above result to an even more complicated game called the *multi-valued multi-choice game.*

3. Multi-valued Multi-choice Games

Given a natural number r, let $E = \{v_0; v_1; v_2; \dots; v_r\}$ be a set of $(r + 1)$ real numbers, such that $0 = v_0 < v_1 \leq v_2 \leq \dots \leq v_r$.

Definition 3.1. A *multi-valued multi-choice game* in characteristic function form is the pair $(\beta^n; V)$ defined by $V : \beta^n \to E$, such that $V(\mathbf{0}) = 0$, where $\mathbf{0} = (0; 0; \dots; 0)$.

we can identify the set of all multi-valued multi-choice games by: $MMG \simeq E^{(m+1)^n - 1}$. Let $NMMG$ be the set of all non-decreasing multi-valued multi-choice games, since a multi-valued multi-choice game is not necessarily non-decreasing, we will consider MMG and $NMMG$ respectively.

Given Γ^n, for any $x \in \Gamma^n$, we define $|x| = \sum_{r=1}^{n} x_r$.

Given a $(r+1)$-valued $(m+1)$-choice game V and a constant c, we define

$$(c \cdot V)(x) = c \cdot V(x); \text{ for all } x \in \Gamma^n:$$

Lemma 3.1. *Suppose* $w(0); w(1); \ldots; w(m)$ *are given. If* V *is of the form*

$$V(y) = \begin{cases} c > 0 & \textit{if } y \geq x \\ 0 & \textit{otherwise}; \end{cases}$$

then $\phi^w(V)$ *is uniquely determined by axioms 1, 2, 4, and 5.*

Proof. Trivial!

The set of all multi-valued multi-choice games MMG is not closed under addition. We consider the following to make this article self-contained.

If we replace axiom 3 by

Axiom 3*:**

$$\phi^w(V^1 + V^2) = \phi^w(V^1) + \phi^w(V^2) \text{ for all } V^1; V^2 \in MMG$$

such that $V^1 + V^2 \in MMG$. Then we have the following Theorem.

Theorem 3.1. *When* w *is given, there exists a unique* ϕ^w *defined on* MMG *such that* ϕ^w *satisfies axioms 1, 2,* 3^**, 4 and 5. Moreover,* ϕ^w *is also given by* $(2{:}1)$.

Proof. By lemma 3.1, the proof of this theorem is similar to the proof of theorem 8.2 in [3] (1991).

Since $NMMG$ is not closed under addition, if we restrict G to $NMMG$, then we encounter a much more complicated problem than Dubey did in [3] (1975) . Now, we will solve the problem by some concepts in Reliability Theory.

Here, we give an example to justify the importance of solving the problem.

Example 1. In a mathematics competition, a team has 3 students (players), each student individually may win a golden medal, a sliver medal, a coper medal, or nothing. The award for the whole team is the worst award among the three players' awards.

We may model the game as follows. The players may have 4 options, say, σ_0: win nothing, σ_1: win a coper medal, σ_2: win a silver medal, σ_3: win a golden medal. Note: If a player win nothing, then σ_1, σ_2, σ_3 are dummy actions for the player. Furthermore, if there is an action σ_k which player i can not make it, then σ_k is a dummy action for player i.

The characteristic function is $V((x_1; x_2; x_3)) = \min\{x_1; x_2; x_3\}$. People may ask what is the power index of the first student when he win a medal? Without a well-known property in Reliability Theory, which is the Dubey's axiom $3''$in Game Theory, the multi-choice Shapley value does not work for the above question.

Mathematically, it is trivial that $a + b = \max\{a; b\} + \min\{a; b\}$ for any real numbers a, b. Here, a, b are not necessarily zero or one.

Even when V^1 and V^2 are both multi-valued multi-choice games, it is still trivial that

$$(V^1 \vee V^2)(x) + (V^1 \wedge V^2)(x) = V^1(x) + V^2(x);$$

for all $x \in \beta^n$.

The above arguments assure that we may use Dubey's axiom even if the range of the characteristic function is not $\{0; 1\}$.

Definition 3.2. Given two action vectors x, $y \in \beta^n$, we say that y is strictly less than x, denoted by $y < x$, if and only if $y \leq x$ and $y_i < x_i$ for some $i \in N$.

Definition 3.3. Given a non-decreasing $(r + 1)$-valued $(m + 1)$-choice game $(\beta^n; V)$ an action vector $x \in \beta^n$ is called a critical v_k-valued action vector if $V(x) = v_k$ and $V(y) < v_k$ whenever $y < x$.

Please note that we define a critical v_k-valued action vector only for a non-decreasing multi-valued multi-choice game.

Definition 3.4. Given $x \in \beta^n$, and $v_k \in E$, define

$$W_k^x(y) = \begin{cases} v_k & \text{if } y \geq x \\ 0 & \text{otherwise;} \end{cases}$$

then W_k^x is called a v_k-valued basic game.

Remark 3.1. Of course, x is the only critical v_k-valued action vector of the v_k-valued basic game W_k^x. Furthermore, for each $k = 1; 2; :::; r$ and each $x \in \beta^n$, $\phi^w(W_k^x)$ is uniquely determined by axioms 1, 2, and 4.

The following Lemmas are easy to see.

Lemma 3.2. *Given a $V \in NMMG$, let x and y be two distinct critical v_k-valued action vectors of V, then $x \not< y$ and $y \not< x$.*

Lemma 3.3. *Given a* $V \in NMMG$ *and an action vector* $x \in \mathbb{R}^n$*, suppose* $V(x) = v_k \in E$*, then* x *is either a critical* v_k*-valued action vector or greater than a critical* v_k*-valued action vector.*

Lemma 3.4. *Given a* $V \in NMMG$ *and a* $v_k \in E$*, suppose* V *has a finite number of distinct critical* v_k*-valued action vectors, say,* $x^{(k,1)}, x^{(k,2)}, \ldots, x^{(k,s_k)}$*, then*

$$V(y) = v_k \text{ if } y = x^{(k,i)} \text{ for some } i \in \{1, 2, \ldots, s_k\}$$

and

$$V(y) < v_k \text{ if } y < x^{(k,i)} \text{ for some } i \in \{1, 2, \ldots, s_k\}.$$

Theorem 3.2. *If* w *is given, then there exists a unique* ϕ^w *defined on* $NMMG$ *such that* ϕ^w *satisfies axioms 1, 2,* $3''$*, 4 and 5. Moreover,* ϕ^w *is also given by* (2.1).

Proof. Given $V \in NMMG$, for each $v_k \in E$, V has a finite number of critical v_k-valued action vectors, say, $x^{(k,1)}, x^{(k,2)}, \ldots, x^{(k,s_k)}$, then by Lemmas 3.2, 3.3, and 3.4, we can easily see that

$$\begin{aligned} V = & \left(W_1^{x^{(1,1)}} \vee W_1^{x^{(1,2)}} \vee \cdots \vee W_1^{x^{(1,s_1)}} \right) \vee \\ & \left(W_2^{x^{(2,1)}} \vee W_2^{x^{(2,2)}} \vee \cdots \vee W_2^{x^{(2,s_2)}} \right) \vee \\ & \left(W_3^{x^{(3,1)}} \vee W_3^{x^{(3,2)}} \vee \cdots \vee W_3^{x^{(3,s_3)}} \right) \vee \ldots \vee \ldots \\ & \ldots \vee \ldots \vee \left(W_r^{x^{(r,1)}} \vee W_r^{x^{(r,2)}} \vee \cdots \vee W_r^{x^{(r,s_r)}} \right) \end{aligned}$$

where the right hand side is defined associatively.
Define $C(V) = \max\{k \mid k \in Z^+$ such that V has at least one v_k-valued action vector $\}$.
We will prove this Theorem by mathematical induction on $C(V)$.

Case 1. Suppose $C(V) = 1$, let $n^{(1,1)}(V) = \min\{p \in Z^+$: there exists a critical v_1-valued action vector y of V such that $|y| = p\}$; and let

$$n^{(1,2)}(V) = \text{ the number of critical } v_1\text{-valued action vector } y \text{ of } V$$

such that $\sum y_i = n^{(1,1)}(V)$.

For $n^{(1;1)}(V) = m \times n$, $V = W_1^m$, in which case $\phi^w(V)$ is obviously unique.

Step 1.1. Suppose $\phi^w(V)$ has been shown to be unique for all V such that $n^{(1;1)}(V) = k+1, k+2; \ldots ; m \times n$, we should claim that $\phi^w(V)$ is unique when $n^{(1;1)}(V) = k$ and $n^{(1;2)}(V) = 1$.

Let x be the unique critical v_1-valued action vector with $|x| = k$. If x is the only critical v_1-valued action vector of V then $V = W_1^x$ and $\phi^w(V)$ is unique. Otherwise let $x^{(1;1)}; x^{(1;2)}; \ldots x^{(1;t)}$ denote all the critical v_1-valued action vectors of V apart from x.

NOTE: $|x^{(1;j)}| > k$ for $1 \leq j \leq t$ since $n^{(1;2)}(V) = 1$.

Now $(W_1^{x^{(1;1)}} \vee W_1^{x^{(1;2)}} \vee \ldots \vee W_1^{x^{(1;t)}}) \vee W_1^x = V$, say $V^* \vee W_1^x = V$. It follows that $n^{(1;1)}(V^*) > k$. Therefore $\phi^w(V^*)$ is unique by the inductive assumption. Further,
$n^{(1;1)}(W_1^x \wedge V^*) > k$. This is obvious from the definition of $\wedge$. Therefore $\phi^w(W_1^x \wedge V^*)$ is also unique by the inductive assumption. Invoke axiom 3^{**}. Then

$$\phi^w(V) = \phi^w(V^* \vee W_1^x) = \phi^w(V^*) + \phi^w(W_1^x) - \phi^w(V^* \wedge W_1^x):$$

Since all the three matrices on the right hand side are unique, so is $\phi^w(V)$.

Step 1.2. Suppose $\phi^w(V)$ has been shown to be unique for all V such that either

(1.1.1) $$n^{(1;1)}(V) = k+1; k+2; \ldots ; m \times n$$

or

(1.1.2) $$n^{(1;1)}(V) = k \text{ and } n^{(1;2)}(V) = 1; 2; \ldots ; j:$$

We should claim that $\phi^w(V)$ is unique when $n^{(1;1)}(V) = k$ and $n^{(1;2)}(V) = j+1$.

Let $x^{(1;1)}; x^{(1;2)}; \ldots ; x^{(1;j+1)}$ be the critical v_1-valued action vectors of V with $|x^{(1;i)}| = k$ for all $i = 1; 2; \ldots ; j+1$. And let $y^{(1;1)}; \ldots ; y^{(1;t)}$ be all the other critical v_1-valued action vectors of V.

By the conditions on $n^{(1;1)}(V)$ and $n^{(1;2)}(V)$ it is clear that $|y^{(1;i)}| > k$ for $1 \leq i \leq t$

Now

$$(W_1^{y^{(1;1)}} \vee W_1^{y^{(1;2)}} \vee \ldots \vee W_1^{y^{(1;t)}} \vee W_1^{x^{(1;1)}} \vee \ldots \vee W_1^{x^{(1;j)}}) \vee W_1^{x^{(1;j+1)}} = V$$

Say $V^{**} \vee W_1^{x^{(1;j+1)}} = V$.

Clearly $V^{¤¤}$ satisfies (1.1.2) and $V^{¤¤} \wedge W_1^{x^{(1;j+1)}}$ satisfies (1.1.1). Therefore $Á^w(V^{¤¤})$ and $Á^w(V^{¤¤} \wedge W_1^{x^{j+1}})$ are both unique by the inductive assumption.

By Axiom $3^{¤¤}$,

$$Á^w(V) = Á^w(V^{¤¤} \vee W_1^{x^{(1;j+1)}}) = Á^w(V^{¤¤}) + Á^w(W_1^{x^{(1;j+1)}}) - Á^w(V^{¤¤} \wedge W_1^{x^{(1;j+1)}})$$

which proves the uniqueness of $Á^w(V)$.

Putting together **step 1.1** and **step 1.2** we get the uniqueness of $Á^w(V)$ for any feasible number $n^{(1;1)}(V)$ and $n^{(1;2)}(V)$, i.e. $Á^w(V)$ is unique for all $V \in NMMG$ with $C(V) = 1$.

Case 2. Suppose $C(V) = 2$, let $n^{(2;1)}(V) = \min\{p \in Z^+$: there exists a critical v_2-valued action vector y of V such that $|y| = p\}$; and let

$$n^{(2;2)}(V) = \text{ the number of critical } v_2\text{-valued action vector } y \text{ of } V$$

such that $\sum y_i = n^{(2;1)}(V)$.

If $n^{(2;1)}(V) = m \times n$, then m is the unique critical v_2-valued action vector for V. Then there exists $W \in NMMG$ with $C(W) \le 1$ such that $V = W \vee W_2^m$. It is clear that $C(W \wedge W_2^m) \le 1$, hence $Á^w(W_2^m)$, $Á^w(W)$ and $Á^w(W \wedge W_2^m)$ are unique. Invoke Axiom $3^{¤¤}$. Then $Á^w(V) = Á^w(W \vee W_2^m)$=$Á^w(W) + Á^w(W_2^m) - Á^w(W \wedge W_2^m)$, therefore $Á^w(V)$ is unique.

Step 2.1. Suppose $Á^w(V)$ has been shown to be unique for all V such that $n^{(2;1)}(V) = k+1, k+2; \ldots; m \times n$, we should claim that $Á^w(V)$ is unique when $n^{(2;1)}(V) = k$ and $n^{(2;2)}(V) = 1$.

Let x be the unique critical v_2-valued action vector with $|x| = k$. If x is the only critical v_2-valued action vector of V Then there exists $W \in NMMG$ with $C(W) \le 1$ such that $V = W \vee W_2^x$. It is clear that $C(W \wedge W_2^x) \le 1$, hence $Á^w(W_2^x)$, $Á^w(W)$ and $Á^w(W \wedge W_2^x)$ are unique. Therefore, by Axiom $3^{¤¤}$, $Á^w(V)$ is also unique.

If V has more than one critical v_2-valued action vector, let $x^{(2;1)}; x^{(2;2)}; \ldots; x^{(2;t)}$ denote all the critical v_2-valued action vectors of V apart from x.

NOTE: $|x^{(2;j)}| > k$ for $1 \le j \le t$ since $n^{(2;2)}(V) = 1$.

Now $(W_2^{x^{(2;1)}} \vee W_2^{x^{(2;2)}} \vee \ldots \vee W_2^{x^{(2;t)}}) \vee (W \vee W_2^x) = V$, say $V^{¤} \vee (W \vee W_2^x) = V$. It follows that $n^{(2;1)}(V^{¤}) > k$. Therefore $Á^w(V^{¤})$ is unique by the inductive assumption. Further, $n^{(2;1)}((W \vee W_2^x) \wedge V^{¤}) > k$. This is obvious from the definition of $\wedge$. Therefore $Á^w((W \vee W_2^x) \wedge V^{¤})$ is also unique by the inductive assumption. Invoke axiom $3^{¤¤}$. Then

$$Á^w(V) = Á^w(V^{¤} \vee (W \vee W_2^x)) = Á^w(V^{¤}) + Á^w(W \vee W_2^x) - Á^w((W \vee V^{¤}) \wedge W_2^x):$$

Since all the three matrices on the right hand side are unique, so is $Á^w(V)$.

Step 2.2. Suppose $Á^w(V)$ has been shown to be unique for all V such that either

(1.2.1) $$n^{(2;1)}(V) = k+1; k+2; :::; m \times n$$

or

(1.2.2) $$n^{(2;1)}(V) = k \text{ and } n^{(2;2)}(V) = 1; 2; :::; j:$$

We should claim that $Á^w(V)$ is unique when $n^{(2;1)}(V) = k$ and $n^{(2;2)}(V) = j+1$.

Let $x^{(2;1)}; x^{(2;2)}; :::; x^{(2;j+1)}$ be the critical v_2-valued action vectors of V with $|x^{(2;i)}| = k$ for all $i = 1; 2; :::; j+1$. And let $y^{(2;1)}; :::; y^{(2;t)}$ be all the other critical v_2-valued action vectors of V.

By the conditions on $n^{(2;1)}(V)$ and $n^{(2;2)}(V)$ it is clear that $|y^{(2;i)}| > k$ for $1 \le i \le t$

Now there exists $W \in NMMG$ such that $C(W) \le 1$ and

$$(W \vee W_2^{y^{(2;1)}} \vee W_2^{y^{(2;2)}} \vee ::: \vee W_2^{y^{(2;t)}} \vee W_2^{x^{(2;1)}} \vee ::: \vee W_2^{x^{(2;j)}}) \vee W_2^{x^{(2;j+1)}} = V$$

Say $V^{¤¤} \vee W_2^{x^{(2;j+1)}} = V$.

Clearly $V^{¤¤}$ satisfies (1.2.2) and $V^{¤¤} \wedge W_2^{x^{(2;j+1)}}$ satisfies (1.2.1). Therefore $Á^w(V^{¤¤})$ and $Á^w(V^{¤¤} \wedge W_2^{x^{(2;j+1)}})$ are both unique by the inductive assumption.

By $3^{¤¤}$,

$$Á^w(V) = Á^w(V^{¤¤} \vee W_2^{x^{(2;j+1)}}) = Á^w(V^{¤¤}) + Á^w(W_2^{x^{(2;j+1)}}) - Á^w(V^{¤¤} \wedge W_2^{x^{(2;j+1)}})$$

which proves the uniqueness of $Á^w(V)$.

Putting together **step 2.1** and **step 2.2** we get the uniqueness of $Á^w(V)$ for any feasible number $n^{(2;1)}(V)$ and $n^{(2;2)}(V)$, i.e. $Á^w(V)$ is unique for all $V \in NMMG$ with $C(V) = 2$.

Continuing in this way, we can prove **Case 1**, **Case 2**,...., **Case r**. Therefore $Á^w(V)$ is unique for all $V \in NMMG$.

It is clear that the multi-choice Shapley value $Á^w$ on G satisfies axiom 1, 2, $3^{¤¤}$, and 4 when it is restricted to $NMMG$. Indeed $V^1 + V^2 = (V^1 \vee V^2) + (V^1 \wedge V^2)$ where we regard the $+$ as taking place in the vector space G. Hence by axiom 3

$$Á^w(V^1) + Á^w(V^2) = Á^w(V^1 \vee V^2) + Á^w(V^1 \wedge V^2):$$

Therefore the multi-choice Shapley value is the unique $Á^w$ on $NMMG$. ■

Remark 3.2. By Theorem 3.2, we may use the multi-choice Shapley value as the power indices for the players in Example 1.

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