TAIWANESE JOURNAL OF MATHEMATICS
Vol. 7, No. 4, pp. 665-676, December 2003
This paper is available online at http://www.math.nthu.edu.tw/tjm/

UNIFORM CONVERGENCE IN THE DUAL OF A VECTOR-VALUED SEQUENCE SPACE

Charles Swartz and Christopher Stuart

Abstract. In this note the authors establish several results concerning the uniform convergence of series in vector-valued sequence spaces. Corollaries include sufficient conditions for the weak sequential completeness of β-duals of sequence spaces, versions of the Uniform Boundedness Theorem and the Banach-Steinhaus Theorem for elements of operator-valued β-duals, and a characterization of weakly convergent sequences in β-duals. A further application establishes a vector-valued version of the Hahn-Schur Lemma.

1. Introduction

In [5] and [12] Li Ronglu and his collaborators established several interesting results concerning the uniform convergence of series generated by elements of vector-valued sequence spaces and their operator β-duals. It was shown in [5] that one of these uniform convergence results implies the Hahn-Schur Lemma on weakly convergent sequences in l^1. In this note we continue this theme on the uniform convergence of series in vector-valued sequence spaces. As a consequence of our results for β-duals, we establish a result of Stuart on the weak sequential completeness of β-duals, versions of the Uniform Boundedness Principle and the Banach-Steinhaus Theorem for elements of operator-valued β-duals, and give a characterization of weakly convergent sequences in β-duals. We also define another operator-valued dual using unordered convergent series and establish results on uniform unordered convergence in such duals. As an application of one of these uniform convergence results, we establish a vector-valued version of the Hahn-Schur Lemma.

We begin by fixing the notation and terminology which will be used. Let X and Y be Hausdorff topological vector spaces and let $L(X, Y)$ be the space of all

Received November 6, 2001.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 46A45, 40A05.
Key words and phrases: Sequence spaces, β-Duals, Weakly sequentially complete, Gliding hump property, Uniform convergence.

continuous linear operators from X into Y. Let E be a vector space of X-valued sequences which contains the space $c_{00}(X)$ of all X valued sequences which are eventually zero. If $x \in E$, the kth coordinate of x is denoted by x_k so $x = \{x_k\}$. The β-dual of E with respect to Y, $E^{\beta Y}$, is defined to be all sequences $\{y_k\} \subset L(X, Y)$ such that the series $\sum_{k=1}^{\infty} y_k x_k$ converges in Y for every $\{x_k\} \in E$; if Y is the scalar field we write $E^{\beta Y} = E^{\beta}$. If $y = \{y_k\} \in E^{\beta Y}$ and $x = \{x_k\} \in E$, we write $y \cdot x = \sum_{k=1}^{\infty} y_k x_k$. If $F \subset E^{\beta Y}$, we let $w(F, E)$ be the weakest topology on F such that each map $y \to y \cdot x$ is continuous for each $x \in E$. We consider the uniform convergence of series generated by subsets of E and $E^{\beta Y}$.

An interval in $\mathbf{N}$ is a set of the form $[m, n] = \{k \in \mathbf{N} : m \leq k \leq n\}$ where $m, n \in \mathbf{N}$ and $m \leq n$. If $\sigma \subset \mathbf{N}$, χ_σ will denote the characteristic function of σ, and if $x \in E$, $\chi_\sigma x$ will denote the coordinatewise product of χ_σ and x. A sequence $\{I_k\}$ of finite subsets of $\mathbf{N}$ is increasing if $\max I_k < \min I_{k+1}$; $\{s_k\}$ is a sequence of signs if $s_k = \pm 1$ for every k. The space E has the signed weak gliding hump property (signed WGHP) if whenever $x \in E$ and $\{I_k\}$ is an increasing sequence of intervals, there exist a sequence of signs $\{s_k\}$ and an increasing sequence $\{n_k\}$ such that the coordinatewise sum of the series $\sum_{k=1}^{\infty} s_k \chi_{I_{n_k}} x$ belongs to E ([7], [8]). E has the weak gliding hump property (WGHP) if the signs above are all equal to 1 ([6], [10]). Any monotone space has WGHP and bs, the sequence space of bounded series, is an example of a space with signed WGHP but not WGHP ([7], [8]). One of the important consequences of the weak gliding hump properties is the weak sequential completeness of β-duals ([6], [7], [8], [10]). We show that our first uniform convergence theorem implies this weak sequential completeness result as a corollary.

Theorem 1. *Assume that E has the signed WGHP. If $\{y_k\}$ is $w(E^{\beta Y}, E)$ Cauchy and $x \in E$, then the series $\sum_{j=1}^{\infty} y_j^k x_j$ converge uniformly for $k \in \mathbf{N}$.*

Proof. If the conclusion fails,

(∗) there exists a neighborhood, U, of 0 in Y such that for every n there exist k_n, $n_n > m_n > n$ such that $\sum_{j=m_n}^{n_n} y_j^{k_n} x_j \notin U$.

By (∗), for $n = 1$ there exist k_1, $m_1 < n_1$ such that $\sum_{j=m_1}^{n_1} y_j^{k_1} x_j \notin U$. There exists $m' > n_1$ such that $\sum_{j=m}^{n} y_j^k x_j \in U$ for $n > m > m'$, $1 \leq k \leq k_1$. By (∗) there exist k_2, $n_2 > m_2 > m'$ such that $\sum_{j=m_2}^{n_2} y_j^{k_1} x_j \notin U$. Hence, $k_2 > k_1$. Continuing this construction produces increasing sequences k_i, m_i, n_i with $m_i < n_i < m_{i+1}$ and

$$(\#) \quad y^{k_i} \cdot \chi_{I_i} x \notin U \quad \text{where} \quad I_i = [m_i, n_i].$$

Define the matrix $M = [y^{k_i} \cdot \chi_{I_j} x]$. We claim that M is a signed K-matrix ([7], [8], [10, 2.2.4]). First, the columns of M converge since $\{y^k\}$ is $w(E^{\beta Y}, E)$

Cauchy. Second, given any increasing sequence of integers, there is a subsequence $\{p_k\}$ and a sequence of signs $\{s_k\}$ such that $z = \sum_{k=1}^{\infty} s_k \chi_{I_{p_k}} x \in E$. Then $\sum_{k=1}^{\infty} y^{k_i} \cdot \chi_{I_{p_k}} x = y^{k_i} \cdot z$ and $\lim_i y^{k_i} \cdot z$ exists. Hence, M is a signed K-matrix so the diagonal of M converges to 0 ([7], [8], [10, 2.2.4]), contradicting $(\#)$.

Example 2. Theorem 1 fails if the signed WGHP assumption is dropped.

Take $E = c$ so $E^{\beta} = l^1$. Let e^k be the canonical unit vector with 1 in the kth position and 0 otherwise. Then $\{e^k\}$ is $w(l^1, c)$ Cauchy and if x is the constant sequence $\{1\}$, the series $\sum_{j=1}^{\infty} e_j^k x_j$ do not converge uniformly.

We next observe that Theorem 1 implies the result in [7] and [8] on the weak sequential completeness of β-duals. The pair (X, Y) is said to have the Banach-Steinhaus property if whenever $\{T_k\} \subset L(X, Y)$ is pointwise convergent, $\lim T_k x = Tx$ exists for each $x \in X$, then $T \in L(X, Y)$. If X is barrelled or a complete metric space, then (X, Y) has the Banach-Steinhaus property ([9], [11]).

Corollary 3. ([7], [8]) *If E has the signed WGHP and (X, Y) has the Banach-Steinhaus property, then $w(E^{\beta Y}, E)$ is sequentially complete.*

Proof. Let $\{y^k\}$ be $w(E^{\beta Y}, E)$ Cauchy. For each j and $z \in X$,

$$\lim_k y^k \cdot (e^j \otimes z) = y_j(z) \text{exists},$$

where $e^j \otimes z$ denotes the sequence with z in the j^{th} position and 0 otherwise. By the Banach-Steinhaus property, $y_j \in L(X, Y)$; put $y = \{y_j\}$.

We claim that $y \in E^{\beta Y}$ and $y^k \to y$ in $w(E^{\beta Y}, E)$. Let U be a balanced neighborhood of 0 in Y and pick a balanced neighborhood V such that $V + V + V \subset U$. By Theorem 1, there exists p such that $\sum_{j=n}^{\infty} y_j^k x_j \in V$ for $k \in \mathbb{N}$, $n \geq p$. Fix $n \geq p$. Pick $k_n = k$ such that $\sum_{j=1}^{\infty} y_j^k x_j - u \in V$ and $\sum_{j=1}^{n} \left(y_j^k - y_j \right) x_j \in V$. Then $\sum_{j=1}^{n} y_j x_j - u = \left(\sum_{j=1}^{\infty} y_j^k x_j - u \right) + \sum_{j=1}^{n} \left(y_j - y_j^k \right) x_j - \sum_{j=n+1}^{\infty} y_j^k x_j \in V + V + V$ and the result follows.

A subset F of $E^{\beta Y}$ is said to be conditionally $w(E^{\beta Y}, E)$ sequentially compact if every sequence $\{y^k\} \subset F$ has a subsequence $\{y^{n_k}\}$ which is $w(E^{\beta Y}, E)$ Cauchy ([5]). From Theorem 1 we have an analogue of Theorem 2 of [12].

Corollary 4. *Assume the E has signed WGHP. If $F \subset E^{\beta Y}$ is conditionally $w(E^{\beta Y}, E)$ sequentially compact and $x \in E$, then the series $\sum_{j=1}^{\infty} y_j x_j$ converge uniformly for $y \in F$.*

We next use Theorem 1 to give a characterization of $w(E^{\beta Y}, E)$ convergent series in spaces with signed WGHP.

Proposition 5. *Let $\{y^k\} \subset E^{\beta Y}$.*

(1) *If* $y^k \to 0$ $w(E^{\beta Y}, E)$, *then* (a) $\lim_k y_j^k = 0$ *pointwise on* X *for each* j.

(2) *If* (a) *holds and for every* $x \in E$ *the series* $\sum_{j=1}^{\infty} y_j^k x_j$ *converges uniformly for* $k \in \mathbf{N}$, *then* $y^k \to 0$ $w(E^{\beta Y}, E)$.

(3) *If* E *has signed WGHP*, *then the converse of* (1) *holds.*

Proof. (1) follows by considering $e^j \otimes z$ for $z \in X$. For (2) let $x \in E$ and consider

$$(+) \quad y^k \cdot x = \sum_{j=1}^{n} y_j^k x_j + \sum_{j=n+1}^{\infty} y_j^k x_j .$$

By hypothesis there exists n such that the last term in $(+)$ belongs to V for all k, where we continue the notation from Corollary 3. By (a) there exists p such that the first term on the right hand side of $(+)$ belongs to V if $k \geq p$. Hence, if $k \geq p$, then $y^k \cdot x \in V + V \subset U$ and (2) holds.

In Theorem 1 and Corollary 4 we considered the uniform convergence of series generated by subsets of $E^{\beta Y}$ and a single element of E. We now consider uniform convergence of series generated by subsets of both $E^{\beta Y}$ and E. For this we need an additional assumption on E. Assume that E has a Hausdorff vector topology under which E is a K-space, that is, the coordinate maps $x \to x_k$ from E into X are continuous for each k. E has the zero gliding hump property (0-GHP) if whenever $x^k \to 0$ and $\{I_k\}$ is an increasing sequence of intervals, there exists a subsequence $\{n_k\}$ such that the coordinatewise sum of the series $\sum_{k=1}^{\infty} \chi_{I_{n_k}} x^{n_k}$ belongs to E. This property was introduced by Lee ([4]); examples of spaces with 0-GHP are given in ([10], 12.5).

Lemma 6. *Assume that* E *has 0-GHP. If* $y \in E^{\beta Y}$ *and* $x^k \to 0$ *in* E, *then* $\sum_{j=1}^{\infty} y_j x_j^k$ *converges uniformly for* $k \in \mathbf{N}$.

See [12], Theorem 2.

Without the 0-GHP assumption the conclusion of Lemma 6 can fail.

Example 7. Let $X = c_{00}$ be the space of all scalar sequences which are eventually zero with the sup-norm. Then $X^{\beta} = s$, the space of all scalar sequences. Let y be the constant sequence $\{1\}$ and $x^k = \frac{\sum_{j=1}^{k} e^j}{k}$. Then $x^k \to 0$ but $\sum_{j=1}^{\infty} x_j^k y_j$ does not converge uniformly for $k \in \mathbf{N}$.

Corollary 8. ([12] *Corollary* 4) *Assume that* E *has* 0-*GHP. Each* $y \in E^{\beta Y}$ *is sequentially continuous.*

Theorem 9. *Assume that* E *has signed WGHP and* 0-*GHP. If* $\{y^k\}$ *is* $w(E^{\beta Y}, E)$ *Cauchy and* $x^l \to 0$, *then the series* $\sum_{j=1}^{\infty} y_j^k x_j^l$ *converge uniformly for* $k, l \in \mathbf{N}$.

Proof. If the conclusion fails,

$(**)$ there is a neighborhood U of 0 in Y such that for every n there exist k_n, l_n, $n_n > m_n > n$ such that $\sum_{j=m_n}^{n_n} y_j^{k_n} x_j^{l_n} \notin U$.

By $(**)$ for $n = 1$ there exist k_1, l_1, $n_1 > m_1$ such that $\sum_{j=m_1}^{n_1} y_j^{k_1} x_j^{l_1} \notin U$. By Theorem 1 and Lemma 6, there exist $m' > n_1$ such that $\sum_{j=m}^{p} y_j^{k} x_j^{l} \in U$ for $p > m \geq m'$ and for $1 \leq l \leq l_1$, $k \in \mathbf{N}$ or $1 \leq k \leq k_1$, $l \in \mathbf{N}$. By $(**)$ there exist k_2, l_2, $n_2 > m_2 > m'$ such that $\sum_{j=m_2}^{n_2} y_j^{k_2} x_j^{l_2} \notin U$. Hence, $k_2 > k_1$ and $l_2 > l_1$. Continuing this construction produces increasing sequences k_i, l_i, m_i, n_i with $m_i < n_i < m_{i+1}$ and

$$(\#\#) \quad y^{k_i} \cdot \chi_{I_i} x^{l_i} \notin U, \text{ where } I_i = [m_i,\ n_i].$$

Define the matrix $M = [y^{k_i} \cdot \chi_{I_j} x^{l_j}]$. By an argument like that in Theorem 1 it is easily checked that M is a K-matrix ([10], 2.2.2) so the diagonal of M should converge to 0 ([10], 2.2.2). But this contradicts $(\#\#)$.

We next give examples which show that neither assumption on the space E can be dropped from Theorem 9.

Example 10. Example 2 shows that signed WGHP cannot be dropped even when 0-GHP holds.

Example 11. Let E be as in Example 7. Let $y^k = \sum_{j=1}^{k} e^j$ in $s = c_{00}^{\beta}$ and $x^l = \sum_{j=1}^{l} \frac{e^j}{l}$ so $\{y^k\}$ is $w(s, c_{00})$ Cauchy and $x^l \to 0$. Then $\sum_{j=N}^{\infty} y_j^k x_j^l = \frac{k-l}{l}$ if $k > l \geq N$ so the series do not converge uniformly for $k, l \in \mathbf{N}$.

Corollary 12. *Assume that E has signed WGHP and 0-GHP. If $F \subset E^{\beta Y}$ is conditionally $w(E^{\beta Y}, E)$ sequentially compact and $x^l \to 0$ in E, then $\sum_{j=1}^{\infty} y_j x_j^l$ converges uniformly for $y \in F, l \in \mathbf{N}$.*

Remark 13. The conclusion in Corollary 12 is the same as in Theorem 4 of [5] but their hypothesis is that E is an AK-space with 0-GHP. The following proposition shows their hypothesis implies the hypothesis in Corollary 12.

Proposition 14. *0-GHP and AK imply WGHP.*

Proof. Let $x \in E$ and $\{I_k\}$ be an increasing sequence of intervals. Since E has AK, $\chi_{I_k} x \to 0$. By 0-GHP there is a subsequence n_k such that $\sum_{k=1}^{\infty} \chi_{I_{n_k}} x \in E$ so WGHP holds.

The following example shows that Corollary 12 is stronger than Theorem 4 of [5].

Example 15. Let $E = l^{\infty}$ with the sup-norm. Then E has WGHP and 0-GHP but not AK.

We note that in general there is no comparison between WGHP and AK.

Eexample 16. Any scalar sequence space λ is AK for $w(\lambda, \lambda^{\beta})$ so $(c, w(c, l^1))$ has AK but does not have WGHP.

We next consider weakly convergent sequences in β-duals and show that such sequences converge in a stronger topology if E has signed WGHP and 0-GHP.

Notation. If B is a subset of E or $E^{\beta Y}$, set $B_j = \{x_j : x \in B\}$ for $j \in \mathbf{N}$.

Lemma 17. *Let* $\{y^k\} \subset E^{\beta Y}$. *If* (I) $\forall j \ \lim_k y_j^k x = 0$ *uniformly for* $x \in B_j$ *and* (II) $\sum_{j=1}^{\infty} y_j^k x_j$ *converges uniformly for* $k \in \mathbf{N}$, $x \in B$, *then* $\lim_k y^k \cdot x = 0$ *uniformly for* $x \in B$.

Proof. The proof uses (+) and proceeds as in the proof of (2) in Proposition 5.

Corollary 18. *Assume that* E *has signed WGHP and* 0*-GHP and that* X *is barrelled. If* $y^k \to 0$ $w(E^{\beta Y}, E)$ *and* $x^l \to 0$ *in* E, *then* $\lim_k y^k \cdot x^l = 0$ *uniformly for* $l \in \mathbf{N}$.

Proof. Put $B = \{x^l : l \in \mathbf{N}\}$. Since X is barrelled, $\lim_k y^k \cdot x_j^l = 0$ uniformly for $l \in \mathbf{N}$ ([9], [11]) so (I) of Lemma 17 holds by Theorem 9. The result then follows from Lemma 17.

Garnir, DeWilde, and Schmets have compared uniform convergence on null sequences as in Corollary 18 with uniform convergence on precompact subsets ([2] III.II.19). In particular, if E in Corollary 18 is a metrizable locally convex space, then whenever $y^k \to 0$ in $w(E^{\beta Y}, E)$ $y^k \to 0$ uniformly on precompact subsets of E ([2] III.II. 19b).

We next consider uniform convergence on subsets of β-duals for null sequences in E.

Lemma 19. *If* $F \subset E^{\beta Y}$ *is such that* F_j *is sequentially equicontinuous for every* j, $x^l \to 0$ *and the series* $\sum_{j=1}^{\infty} y_j x_j^l$ *converge uniformly for* $y \in F$, $l \in \mathbf{N}$, *then* $\lim y \cdot x^l = 0$ *uniformly for* $y \in F$.

Proof. Let U be a neighborhood of 0 in Y and pick a balanced neighborhood of 0, V, such that $V + V \subset U$. There exists p such that $\sum_{j=p+1}^{\infty} y_j x_j^l \in V$ for $y \in F$, $l \in \mathbf{N}$. Since each F_j is sequentially equicontinuous, there exists q such that $l \geq q$ implies $\sum_{j=1}^{p} y_j x_j^l \in V$ for $y \in F$. Hence, if $l \geq q$, then $y \cdot x^l = \sum_{j=1}^{p} y_j x_j^l + \sum_{j=p+1}^{\infty} y_j x_j^l \in V + V \subset U$ for all $y \in F$.

Corollary 20. *Assume that E has signed WGHP and 0-GHP and that X is barrelled. If $F \subset E^{\beta Y}$ is either conditionally $w(E^{\beta Y}, E)$ sequentially compact or the range of a $w(E^{\beta Y}, E)$ Cauchy sequence and $x^l \to 0$ in E, then $\lim_l y \cdot x^l = 0$ uniformly for $y \in F$.*

Proof. Each F_j is pointwise bounded on X and is, therefore, equicontinuous since X is barrelled. The result now follows from Theorem 9 or Corollary 12 and Lemma 19.

We can easily obtain a uniform boundedness result from the sequential convergence result above.

Corollary 21. (*Uniform Boundedness*) *Assume that E has signed WGHP and 0-GHP and X is barrelled. If $B \subset E^{\beta Y}$ is $w(E^{\beta Y}, E)$ bounded and $A \subset E$ is bounded, then $\{y \cdot x : y \in B,\ x \in A\}$ is bounded.*

Proof. Let $\{y^k\} \subset B$, $\{x^k\} \subset A$. It suffices to show $\frac{1}{k} y^k \cdot x^k \to 0$. Since $\frac{1}{\sqrt{k}} y^k \to 0$ $w(E^{\beta Y}, E)$ and $\frac{1}{\sqrt{k}} x^k \to 0$ in E, this follows from Corollary 18.

We now show that the uniform convergence result of Theorem 9 can be strengthened if we make a stronger gliding hump assumption on E. If E is a K-space, then E has the strong gliding hump property (SGHP) if $\{x^k\}$ is a bounded sequence in E and $\{I_k\}$ is an increasing sequence of intervals, then there is a subsequence $\{n_k\}$ such that the coordinatewise sum of the series $\sum_{k=1}^{\infty} \chi_{I_{n_k}} x^{n_k}$ belongs to E ([6]). For example, l^∞ has SGHP; see [6] for other examples.

Lemma 22. *Assume that E has SGHP. If $y \in E^{\beta Y}$ and $A \subset E$ is bounded, then $\sum_{j=1}^{\infty} y_j x_j$ converges uniformly for $x \in A$.*

Proof. If the conclusion fails,

$(***)$ there is a neighborhood U of 0 such that for every n there exist $x^n \in A, n_n > m_n > n$ such that $\sum_{j=m_n}^{n_n} y_j x_j^n \notin U$.

By $(***)$ for $n = 1$, there exist $x^1 \in A$, $n_1 > m_1$ such that $\sum_{j=m_1}^{n_1} y_j x_j^1 \notin U$. By $(***)$ again, there exist $x^2 \in A$, $n_2 > m_2 > n_1$ such that $\sum_{j=m_2}^{n_2} y_j x_j^2 \notin U$. Continuing produces increasing sequences n_k, m_k, $n_{k+1} > m_{k+1} > n_k$ and $\{x^k\} \subset A$ such that

$$(\maltese) \quad y \cdot \chi_{I_k} x^k \notin U, \text{ where } I_k = [m_k, n_k].$$

By SGHP, there exists $\{p_k\}$ such that $x = \sum_{k=1}^{\infty} \chi_{I_{p_k}} x^k \in E$. But $(\maltese)$ implies that the series $\sum_{j=1}^{\infty} y_j x_j$ does not converge or $y \notin E^{\beta Y}$.

Example 23. The SGHP assumption in Lemma 22 cannot be replaced by WGHP. Let A be the unit ball of l^1 and let $y \in (l^1)^\beta = l^\infty$ be the constant sequence $\{1\}$.

Example 24. The SGHP assumption in Lemma 22 is only a sufficient condition for the uniform convergence conclusion of Lemma 22. If $A \subset l^2$ is bounded and $y \in l^2$, then $\left|\sum_{j=N}^\infty y_j x_j\right|^2 \le \sum_{j=N}^\infty |y_j|^2 \sum_{j=N}^\infty |x_j|^2$ so the conclusion of Lemma 22 holds but l^2 does not have SGHP.

Theorem 25. *Assume that E has SGHP. If* $\{y^k\} \subset E^{\beta Y}$ *is* $w(E^{\beta Y}, E)$ *Cauchy and* $A \subset E$ *is bounded, then* $\sum_{j=1}^\infty y_j^k x_j$ *converges uniformly for* $x \in A$, $k \in \mathbf{N}$.

Proof. Using Lemma 22 and Theorem 1, the proof proceeds as the proof of Theorem 9 where SGHP is used to show that M is a K-matrix.

Corollary 26. ([5] *Theorem* 1) *Assume that E has SGHP. If* $F \subset E^{\beta Y}$ *is conditionally* $w(E^{\beta Y}, E)$ *sequentially compact and* $A \subset E$ *is bounded, then* $\sum_{j=1}^\infty y_j x_j$ *converges uniformly for* $y \in F$, $x \in A$.

We next establish a stronger conclusion than that in Corollary 18 under the stronger assumption of SGHP. The pair (X, Y) is said to have the strong Banach-Steinhaus property if whenever $\{T_k\} \subset L(X, Y)$ is such that $\lim_k T_k x = Tx$ exists for each $x \in X$, then $T \in L(X, Y)$ and the convergence is uniform for x belonging to precompact subsets of X (i.e., the conclusion of the classical Banach-Steinhaus Theorem holds ([9], [11]).

Corollary 27. ([10] 12.5.10) *Assume that E has SGHP and* (X, Y) *has the strong Banach-Steinhaus property. If* $y^k \to 0$ $w(E^{\beta Y}, E)$ *and* $A \subset E$ *is bounded with precompact coordinates, then* $\lim_k y^k \cdot x = 0$ *uniformly for* $x \in A$.

Proof. Let U be a neighborhood of 0 and pick a balanced neighborhood V such that $V + V \subset U$. By Theorem 25 there exists p such that $\sum_{j=p+1}^\infty y_j^k x_j \in V$ for $k \in \mathbf{N}$, $x \in A$. For each j $\lim_k y_j^k = 0$ pointwise on X so the convergence on A_j is uniform since (X, Y) has the strong Banach-Steinhaus property. Therefore, there exists q such that $k \ge q$ implies $\sum_{j=1}^p y_j^k x_j \in V$ for $x \in A$. Hence, if $k \ge q$, $y^k \cdot x = \sum_{j=1}^p y_j^k x_j + \sum_{j=p+1}^\infty y_j^k x_j \in V + V \subset U$ for $x \in A$.

Finally, we define a new dual for a vector-valued sequence space using unordered convergent series, establish an analogue of Theorem 1 and show that the result has as a corollary a vector version of the Hahn-Schur Lemma. Let F be the finite subsets of $\mathbf{N}$. Then F is a net if F is directed by set inclusion. A series $\sum y_j$ in

Y is unordered convergent if the net $\lim_{\sigma \in F} \sum_{j \in \sigma} y_j$ converges ([1]). We define the unordered dual of E, E^{uY}, to be

$$\left\{ \begin{array}{c} \{y_j\} : y_j \in L(X,Y) \text{ and the series } \sum_j y_j x_j \text{ is unordered convergent} \\ \text{for every } \ x \in E \end{array} \right\}.$$

If Y is sequentially complete, then unordered convergence and subseries convergence are equivalent ([1]) so in this case E^{uY} coincides with the σ-dual, $E^{\sigma Y}$. ([10]).

Obviously, we have $E^{uY} \subset E^{\beta Y}$, but, in general, the containment is proper.

Example 28. Let Y be a non-trivial Banach space with $\sum y_j$ convergent in Y but not subseries convergent $[\sum \frac{(-1)^j}{j} y,\ y \neq 0]$. Set $E = c_{00} \oplus span\{1\}$. Then $\{y_j\} \in E^{\beta Y} \backslash E^{uY}$.

Definition 29. E has the signed F-weak gliding hump property (signed F-WGHP) if whenever $x \in E$ and $\{\sigma_k\}$ is an increasing sequence of subsets of F, there exist a sequence of signs $\{s_k\}$ and a subsequence $\{n_k\}$ such that the coordinatewise sum of the series $\sum_{k=1}^{\infty} s_k \chi_{\sigma_{n_k}} x$ belongs to E. If all the signs s_k are equal to 1, E is said to have the F-WGHP.

Of course, the difference between the signed WGHP and the property defined above is the use of the increasing sequences of arbitrary finite subsets of $\mathbf{N}$ instead of intervals. Any monotone space obviously has signed F-WGHP. We give an example of a non-monotone space with F-WGHP.

Example 30. Haydon has shown the existence of an algebra of subsets of $\mathbf{N}$, H, which contains F and has the properties that for every pairwise disjoint sequence $\{A_j\} \subset F$ there is a subsequence $\{n_j\}$ such that $\cup_{j=1}^{\infty} A_{n_j} \in H$ and for no infinite subset $A \in H$ do we have that $\{A \cap B : B \in H\}$ equals the power set A ([3]). Now let $S(H)$ be the vector space of all real-valued H simple functions. If $\varphi \in S(H)$, then φ has a representation as $\sum_{k=1}^{n} a_k \chi_{A_k}$, where $a_k \in \mathbf{R}$ and $\{A_k : 1 \leq k \leq n\}$ is a pairwise disjoint collection from H. Let $\{\sigma_j\}$ be an increasing sequence from F. There is a subsequence $\{n_j\}$ such that $B_k = \cup_{j=1}^{\infty} A_k \cap \sigma_{n_j} \in H$ for every k. Then $\sum_{j=1}^{\infty} \chi_{\sigma_{n_j}} \varphi = \sum_{k=1}^{n} a_k \sum_{j=1}^{\infty} \chi_{\sigma_{n_j} \cap A_k} = \sum_{k=1}^{n} a_k \chi_{B_k} \in S(H)$. Hence, $S(H)$ has F-WGHP. However, $S(H)$ is not monotone by the second property of the Haydon algebra H.

Obviously, signed F-WGHP implies signed WGHP but the converse implication does not hold.

Example 31. The space bs has signed WGHP ([7], [8]) but we show that it does not have signed F-WGHP. Let $x = \left\{(-1)^{k+1}\right\} \in bs$. Set $\sigma_0 = \{1\}$,

$\sigma_1 = \{3,5,7\}$, ... where σ_k consists of 3^k consecutive odd integers. We show that

$$\lim_n \left| \sum_{k=0}^{n} s_k \sum_{j\in\sigma_k} x_j \right| = \infty$$

for any choice of signs $\{s_k\}$, which implies that $\sum s_k \chi_{\sigma_k} x \notin bs$. We have

$$\left| \sum_{k=0}^{n} s_k \sum_{j\in\sigma_k} x_j \right| \geq 3^n - \sum_{j=0}^{n-1} 3^j = 3^n - \frac{1-3^n}{1-3} = \frac{3^n}{2} + \frac{1}{2}.$$

A similar argument shows $\sum_{k=0}^{n} s_k \sum_{j\in\sigma_k} x_j \notin bs$ for any subsequence $\{n_k\}$ so bs does not have signed F-WGHP

We now establish the analogue of Theorem 1 for unordered convergent series. A family of unordered convergent series $\sum_j y_j^a$, $a \in A$, is uniformly unordered convergent for $a \in A$ if for every neighborhood of 0, U, in Y there exists $\sigma_0 \in F$ such that $\sum_{j\in\sigma} y_j^a - \sum_{j=1}^{\infty} y_j^a \in U$ for $\sigma \supset \sigma_0$ and $a \in A$.

Ttheorem 32. *Assume that E has signed F-WGHP. If $\{y^k\}$ is $w\left(E^{uY}, E\right)$ Cauchy and $x \in E$, then the series $\sum_{j=1}^{\infty} y_j^k x_j$ are uniformly unordered convergent for $k \in \mathbf{N}$.*

Proof. If the conclusion fails,

$(****)$ there exists a neighborhood of 0, U, such that for every n there exist $k = k_n, \sigma = \sigma_n \subset [n,\infty), \sigma \in F$, such that $\sum_{j\in\sigma_n} y_j^{k_n} x_j \notin U$.

By $(****)$ for $k = 1$, there exist $k_1, \sigma_1 \in F$ such that $\sum_{j\in\sigma_1} y_j^{k_1} x_j \notin U$. There exists $m > \max \sigma_1$ such that $\sigma \in F$, $\sigma \subset [m,\infty)$ implies $\sum_{j\in\sigma} y_j^k x_j \in U$ for $1 \leq k \leq k_1$. There exist $k_2, \sigma_2 \in F, \sigma_2 \subset [m,\infty)$ such that $\sum_{j\in\sigma_2} y_j^{k_2} x_j \notin U$. Therefore $k_2 > k_1$ and $\max \sigma_1 < \min \sigma_2$. Continuing produces increasing sequences $\{k_j\}$ and $\{\sigma_j\} \subset F$ such that $y^{k_j} \cdot \chi_{\sigma_j} x \notin U$. Define the matrix $M = \left[y^{k_i} \cdot \chi_{\sigma_j} x\right]$. Using the signed F-WGHP, it is easily checked as in the proof of Theorem 1 that M is a signed K-matrix, and the desired contradiction is obtained.

We also have the analogue of Corollary 3 for unordered duals.

Corollary 33. *Assume that E has signed F-WGHP and that (X,Y) is a Banach-Steinhaus pair. Then $w\left(E^{uY}, E\right)$ is sequentially complete.*

Proof. Let $\{y^k\}$ be $w\left(E^{uY}, E\right)$ Cauchy. For each j and $z \in X$, $\lim_k y_j^k(z) = \lim_k y^k \cdot \left(e^j \otimes z\right) = y_j(z)$ exists and $y_j \in L(X,Y)$ by the Banach-Steinhaus property. If $x \in E$, then $\lim y^k \cdot x = \sum_{j=1}^{\infty} y_j x_j$ by Corollary 3. It suffices to show that $\sum_{j=1}^{\infty} y_j x_j$ is unordered convergent. Let U be a neighborhood of 0 and pick a closed, balanced neighborhood V with $V+V \subset U$. By Theorem 32 there exists $\sigma \in F$ such that $\sigma \supset \sigma_0$ implies $\sum_{j\in\sigma\setminus\sigma_0} y_j^k x_j \in V$ for all k. Hence, $\sum_{j\in\sigma\setminus\sigma_0} y_j x_j \in V$ for $\sigma \supset \sigma_0$. Let $p = \max \sigma_0$. By Corollary 3, there exists $q \geq p$ such that $\sum_{j=n}^{\infty} y_j x_j \in V$ for $n \geq q$. If $\sigma \supset [1,q]$, we have $\sum_{j=1}^{\infty} y_j x_j - \sum_{j\in\sigma} y_j x_j = \left(\sum_{j\in\sigma\setminus[1,q]} y_j x_j\right) + \sum_{j=q+1}^{\infty} y_j x_j \in V + V \subset U$. Hence, $\lim_{\sigma\in F} \sum_{j\in\sigma} y_j x_j = \sum_{j=1}^{\infty} y_j x_j$.

Corollary 34. *Assume that E has signed WGHP and that (X,Y) is a Banach-Steinhaus pair. If $\{y^k\}$ is $w(E^{uY},E)$ Cauchy, then there exists $y \in E^{uY}$ such that for all $x \in E$ $\lim_k \sum_{j\in\sigma} y_j^k x_j = \sum_{j\in\sigma} y_j x_j$ uniformly for $\sigma \in F$.*

Proof. By Corollary 33 we may assume that $y^k \to 0$ in $w\left(E^{\beta Y},E\right)$. By Theorem 32 there exists p such that $\sigma \subset [p,\infty)$ implies $\sum_{j\in\sigma} y_j^k x_j \in V$ for all k, where we continue the notation from Corollary 33. Since $\lim_k y_j^k(z) = 0$ for each j and $z \in X$, there exists q such that $\sum_{j\in\sigma} y_j^k x_j \in V$ for $k \geq q$, $\sigma \subset [1,p]$. Therefore, $\sum_{j\in\sigma} y_j^k = \sum_{j\in\sigma\cap[1,p]} y_j^k x_j + \sum_{j\in\sigma\cap[p+1,\infty)} y_j^k x_j \in V + V \subset U$.

Remark 35. If Y is sequentially complete in Corollary 34, then unordered convergent series are subseries convergent ([1]) so the conclusion in Corollary 34 can be improved to read: $\lim_k \sum_{j\in\sigma} y_j^k x_j = \sum_{j\in\sigma} y_j^k x_j$ uniformly $\sigma \subset \mathbf{N}$.

We observe that Corollary 34 implies the scalar version of the classical Hahn-Schur Lemma ([9] 16.14, [11], 1.3.2, 14.4.7) and can, therefore, be legitimately viewed as a vector version of the lemma. Let m_0 be the space of all scalar-valued sequences with finite range. Since m_0 is monotone, m_0 has F-WGHP and also $m_0^u = m_0^\beta = l^1$.

Corollary 36. (*Hahn-Schur*) *Let $\{y^k\} \subset l^1$ be $w(l^1,m_0)$ Cauchy. Then there exists $y \in l^1$ such that* $\lim_k \left\|y^k - y\right\|_1 = 0$.

Proof. Let x be the constant sequence $\{1\}$ in m_0, and $\varepsilon > 0$. From Corollary 34 there exist k_0 such that $\left|\sum_{j\in\sigma} y_j^k - \sum_{j\in\sigma} y_j\right| < \varepsilon$ for $k \geq k_0$. Then

$$\sum_{j=1}^{\infty} \left|y_j^k - y_j\right| = \|y^k - y\|_1 \leq 4\varepsilon \text{ for } k \geq k_0 \text{ ([9],9.5.1).}$$

Other versions of the classical Hahn-Schur Lemma can be found in ([10]).

References

1. M. Day, *Normed Linear Spaces*, Springer-Verlag, N.Y., 1962.
2. H. G. Garnier, M. DeWilde, and J. Schmets, *Analyse Fonctionnelle*, Birkhauser, Basel, 1968.
3. R. Haydon, A non-reflexive Grothendieck space that does not contain l^∞, *Israel Journal of Math*., **40** (1981), 65-73.
4. Lee Peng Yee, Sequence spaces and the gliding hump property, *Southeast Asia Bull. Math*., Special Issue (1993), 65-72.
5. Li Ronglu and Min-Hyung Cho, On the gliding humps property, *Taiwan Journal of Math*., **3** (1999), 115-122.
6. D. Noll, Sequential completeness and spaces with the gliding humps property, *Manuscripta Math.*, **66** (1990), 237-252.
7. C. Stuart, *Weak Sequential Completeness in Sequence Spaces*, Ph.D. dissertation, New Mexico State University, 1993.
8. C. Stuart, Weak sequential completeness of β-duals, *Rocky Mountain Math. Journal*, **26** (1996), 1559-1568.
9. C. Swartz, *Introduction to Functional Analysis*, Marcel Dekker, N.Y., 1992.
10. C. Swartz, *Infinite Matrices and the Gliding Hump*, World Scientific Publications, Singapore, 1996.
11. A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, N.Y., 1978.
12. Wu Junde, Li Ronglu, and C. Swartz, Continuity and boundedness for operator-valued matrix mappings, *Taiwan Journal of Math*., **2** (1998), 447-455.

Charles Swartz
Department of Mathematical Sciences
New Mexico State University
Las Cruces, New Mexico 88003
USA

Christopher Stuart
Department of Mathematical Sciences
New Mexico State University
Las Cruces, New Mexico 88003
USA