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GENERALIZED JENSEN'S EQUATIONS IN BANACH MODULES OVER A C^*-ALGEBRA AND ITS UNITARY GROUP

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Abstract. We prove the generalized Hyers-Ulam-Rassias stability of generalized Jensen's equations in Banach modules over a unital C^*-algebra associated with its unitary group. It is applied to show the stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra associated with its unitary group.

1. GENERALIZED JENSEN'S EQUATIONS

Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [5] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in E_1$.

Lemma A. *Let V, W be vector spaces, and let r, s, t be positive integers. A mapping $f : V \to W$ with $f(0) = 0$ is a solution of the equation*

$$\text{(A)} \qquad rf(\frac{sx + ty}{r}) = sf(x) + tf(y)$$

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for all $x, y \in V$ *if and only if the mapping* $f : V \to W$ *satisfies the additive Cauchy equation* $f(x+y) = f(x) + f(y)$ *for all* $x, y \in V$.

Proof. Assume that $f : V \to W$ satisfies the equation (A). Then

$$rf(\frac{s}{r}x) = rf(\frac{sx + t \cdot 0}{r}) = sf(x) + tf(0) = sf(x),$$
$$rf(\frac{t}{r}x) = rf(\frac{s \cdot 0 + tx}{r}) = sf(0) + tf(x) = tf(x)$$

for all $x \in V$. So

$$f(\frac{s}{r}x) = \frac{s}{r}f(x) \quad \& \quad f(\frac{t}{r}x) = \frac{t}{r}f(x)$$

for all $x \in V$. And

$$f(x) = f(\frac{s}{r} \cdot \frac{r}{s}x) = \frac{s}{r}f(\frac{r}{s}x),$$
$$f(x) = f(\frac{t}{r} \cdot \frac{r}{t}x) = \frac{t}{r}f(\frac{r}{t}x)$$

for all $x \in V$. So

$$f(\frac{r}{s}x) = \frac{r}{s}f(x) \quad \& \quad f(\frac{r}{t}x) = \frac{r}{t}f(x)$$

for all $x \in V$. Thus

$$\begin{aligned} f(x+y) &= \frac{1}{r} \cdot rf(\frac{s}{r} \cdot \frac{r}{s}x + \frac{t}{r} \cdot \frac{r}{t}y) = \frac{1}{r}(sf(\frac{r}{s}x) + tf(\frac{r}{t}y)) \\ &= \frac{1}{r}(s \cdot \frac{r}{s}f(x) + t \cdot \frac{r}{t}f(y)) = f(x) + f(y) \end{aligned}$$

for all $x, y \in V$.

The converse is obvious. ■

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$ and $\mathcal{U}(A)$ the unitary group of A. Let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively, and ${}_A\mathcal{H}$ a left Hilbert A-module with norm $\|\cdot\|$. Let s, t be different positive integers, r a positive integer, and d an integer greater than 1.

In this paper, we prove the generalized Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^*-algebra associated with its unitary group.

2. Stability of Generalized Jensen's Equations in Banach Modules Over a C^*-Algebra

We are going to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^*-algebra for the case $s \neq t$.

Theorem 1. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ such that*

$$\text{(i)} \qquad \begin{aligned} &\widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} (\frac{t}{s})^{2k} \varphi((\frac{s}{t})^{2k} x, \left(\frac{s}{t}\right)^{2k} y) < \infty \\ &\|ruf(\frac{sx + ty}{r}) - sf(ux) - tf(uy)\| \leq \varphi(x, y) \end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\text{(ii)} \qquad \|f(x) - T(x)\| \leq \frac{1}{s} \widetilde{\varphi}(x, -\frac{s}{t} x) + \frac{t}{s^2} \widetilde{\varphi}(-\frac{s}{t} x, \left(\frac{s}{t}\right)^2 x)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. For $y = -\frac{s}{t} x$,

$$\text{(1)} \qquad \|sf(x) + tf(-\frac{s}{t} x)\| \leq \varphi\Big(x, -\frac{s}{t} x\Big).$$

Replacing x by $-\frac{s}{t} x$ and y by $(\frac{s}{t})^2 x$, one can obtain

$$\text{(2)} \qquad \|sf(-\frac{s}{t} x) + tf((\frac{s}{t})^2 x\Big)\| \leq \varphi(-\frac{s}{t} x, (\frac{s}{t})^2 x)$$

for all $x \in {}_A\mathcal{B}$. From (1) and (2), we get

$$\|f(x) - (\frac{t}{s})^2 f((\frac{s}{t})^2 x)\| \leq \frac{1}{s} \varphi(x, -\frac{s}{t} x) + \frac{t}{s^2} \varphi(-\frac{s}{t} x, (\frac{s}{t})^2 x)$$

for all $x \in {}_A\mathcal{B}$. So

$$\text{(3)} \qquad \begin{aligned} \|f(x) - (\frac{t}{s})^{2n} f((\frac{s}{t})^{2n} x)\| \leq &\sum_{k=0}^{n-1} (\frac{1}{s} (\frac{t}{s})^{2k} \varphi((\frac{s}{t})^{2k} x, -(\frac{s}{t})^{2k+1} x) \\ &+ \frac{1}{s} (\frac{t}{s})^{2k+1} \varphi(-(\frac{s}{t})^{2k+1} x, (\frac{s}{t})^{2k+2} x)) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$.

We claim that the sequence $\{(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)\}$ is a Cauchy sequence. Indeed, for $n > m$, we have

$$\begin{aligned}
&\|(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2m} f((\frac{s}{t})^{2m}x)\| \\
&\le \sum_{k=m}^{n-1} \|(\frac{t}{s})^{2k+2} f((\frac{s}{t})^{2k+2}x) - (\frac{t}{s})^{2k} f((\frac{s}{t})^{2k}x)\| \\
&\le \sum_{k=m}^{n-1} \big(\frac{1}{s}(\frac{t}{s})^{2k}\varphi\big((\frac{s}{t})^{2k}x, -\big(\frac{s}{t}\big)^{2k+1}x\big) + \frac{1}{s}\big(\frac{t}{s}\big)^{2k+1}\varphi\big(-\big(\frac{s}{t}\big)^{2k+1}x, \big(\frac{s}{t}\big)^{2k+2}x\big)\big)
\end{aligned}$$

for all $x \in {}_A\mathcal{B}$. It follows from (i) that

$$\lim_{m\to\infty}\sum_{k=m}^{n-1}(\frac{1}{s}(\frac{t}{s})^{2k}\varphi((\frac{s}{t})^{2k}x, -(\frac{s}{t})^{2k+1}x)+\frac{1}{s}(\frac{t}{s})^{2k+1}\varphi(-(\frac{s}{t})^{2k+1}x, (\frac{s}{t})^{2k+2}x)) = 0$$

for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is a Banach space, the sequence $\{(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)\}$ converges. Define

$$T(x) = \lim_{n\to\infty}(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$$

for all $x \in {}_A\mathcal{B}$. Taking the limit in (3) as $n \to \infty$, we obtain

$$\|f(x) - T(x)\| \le \frac{1}{s}\widetilde{\varphi}(x, -\frac{s}{t}x) + \frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x, (\frac{s}{t})^2x)$$

for all $x \in {}_A\mathcal{B}$, which is the inequality (ii). From the definition of T, we get

$$(\frac{s}{t})^{2n}T(x) = T((\frac{s}{t})^{2n}x) \text{ and } T(0) = 0. \tag{4}$$

By (i) and the definition of T,

$$\begin{aligned}
\|rT(\frac{sx+ty}{r}) &- sT(x) - tT(y)\| \\
&= \lim_{n\to\infty}(\frac{t}{s})^{2n}\|rf((\frac{s}{t})^{2n}\frac{sx+ty}{r}) - sf((\frac{s}{t})^{2n}x) - tf((\frac{s}{t})^{2n}y)\| \\
&\le \lim_{n\to\infty}(\frac{t}{s})^{2n}\varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}y) = 0
\end{aligned}$$

for all $x, y \in {}_A\mathcal{B}$. So

$$rT(\frac{sx+ty}{r}) = sT(x) + tT(y)$$

for all $x, y \in {}_A\mathcal{B}$. By Lemma A, T is additive.

If $F : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is another additive mapping satisfying (ii), then it follows from (ii), (4) and the proof of Lemma A that

$$\begin{aligned}
&\|T(x) - F(x)\| = \|(\frac{t}{s})^{2n}T((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}F((\frac{s}{t})^{2n}x)\| \\
&\le \|(\frac{t}{s})^{2n}T((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)\| + \|(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}F((\frac{s}{t})^{2n}x)\| \\
&\le 2(\frac{t}{s})^{2n}(\frac{1}{s}\widetilde{\varphi}((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}(-\frac{s}{t})x) + \frac{t}{s^2}\widetilde{\varphi}((\frac{s}{t})^{2n}(-\frac{s}{t})x), (\frac{s}{t})^{2n}(\frac{s}{t})^2x)),
\end{aligned}$$

which tends to zero as $n \to \infty$ by (i). Thus we conclude that

$$T(x) = F(x)$$

for all $x \in {}_A\mathcal{B}$. This completes the uniqueness of T.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\begin{aligned}
&\|ruf(\frac{s+t}{r}(\frac{s}{t})^{2n}x) - (s+t)f((\frac{s}{t})^{2n}ux)\| \le \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x), \\
&\|rf(\frac{s+t}{r}(\frac{s}{t})^{2n}ux) - (s+t)f((\frac{s}{t})^{2n}ux)\| \le \varphi((\frac{s}{t})^{2n}ux, (\frac{s}{t})^{2n}ux)
\end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}
&\|rf(\frac{s+t}{r}(\frac{s}{t})^{2n}ux) - ruf(\frac{s+t}{r}(\frac{s}{t})^{2n}x)\| \\
&\quad \le \|rf(\frac{s+t}{r}(\frac{s}{t})^{2n}ux) - (s+t)f((\frac{s}{t})^{2n}ux)\| \\
&\qquad + \|ruf(\frac{s+t}{r}(\frac{s}{t})^{2n}x) - (s+t)f((\frac{s}{t})^{2n}ux)\| \\
&\quad \le \varphi((\frac{s}{t})^{2n}ux, (\frac{s}{t})^{2n}ux) + \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x)
\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Thus

$$(\frac{t}{s})^{2n}\|rf(\frac{s+t}{r}(\frac{s}{t})^{2n}ux) - ruf(\frac{s+t}{r}(\frac{s}{t})^{2n}x)\| \to 0$$

as $n \to \infty$ for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned}
rT(\frac{s+t}{r}ux) &= \lim_{n\to\infty}(\frac{t}{s})^{2n}rf(\frac{s+t}{r}(\frac{s}{t})^{2n}ux) = \lim_{n\to\infty} ruf(\frac{s+t}{r}(\frac{s}{t})^{2n}x) \\
&= ruT(\frac{s+t}{r}x)
\end{aligned}$$

for all $u \in \mathcal{U}(A)$. So

$$T(ux) = \frac{r}{s+t} T(\frac{s+t}{r} ux) = \frac{r}{s+t} uT(\frac{s+t}{r} x) = uT(x)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

Now let $a \in A$ ($a \neq 0$) and M an integer greater than $4|a|$. Then

$$|\frac{a}{M}| = \frac{1}{M}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [3, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. And $T(x) = T(3 \cdot \frac{1}{3}x) = 3T(\frac{1}{3}x)$ for all $x \in {}_A\mathcal{B}$. So $T(\frac{1}{3}x) = \frac{1}{3}T(x)$ for all $x \in {}_A\mathcal{B}$. Thus

$$\begin{aligned} T(ax) &= T(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot T(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3} T(3\frac{a}{M}x) \\ &= \frac{M}{3} T(u_1 x + u_2 x + u_3 x) = \frac{M}{3}(T(u_1 x) + T(u_2 x) + T(u_3 x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3) T(x) = \frac{M}{3} \cdot 3\frac{a}{M} T(x) \\ &= aT(x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. Obviously, $T(0x) = 0T(x)$ for all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$. So the unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is an A-linear mapping, as desired. ■

Corollary 2. *Let* $0 < p < 1$ *and* $t < s$. *Let* $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *be a mapping with* $f(0) = 0$ *such that*

$$\|ruf(\frac{sx+ty}{r}) - sf(ux) - tf(uy)\| \leq \|x\|^p + \|y\|^p$$

for all $u \in \mathcal{U}(A)$ *and all* $x, y \in {}_A\mathcal{B}$. *Then there exists a unique* A*-linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *such that*

$$\|f(x) - T(x)\| \leq \frac{s^{2(1-p)}}{s^{2(1-p)} - t^{2(1-p)}} (\frac{1}{s} + \frac{1}{t^p s^{1-p}} + \frac{t^{1-p}}{s^{2-p}} + \frac{t^{1-2p}}{s^{2-2p}}) ||x||^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [\prime, \infty)$ by $\varphi(x, y) = \|x\|^p + \|y\|^p$, and apply Theorem 1. ■

Corollary 3. *Let $p > 1$ and $t > s$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|ruf(\frac{sx+ty}{r}) - sf(ux) - tf(uy)\| \le \|x\|^p + \|y\|^p$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \le \frac{t^{2(p-1)}}{t^{2(p-1)} - s^{2(p-1)}}(\frac{1}{s} + \frac{s^{p-1}}{t^p} + \frac{s^{p-1}}{t^{p-1}} + \frac{s^{2-2p}}{t^{2p-1}})||x||^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = \|x\|^p + \|y\|^p$, and apply Theorem 1. ∎

Theorem 4. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ satisfying* (i) *such that*

$$\|ruf(\frac{sx+ty}{r}) - sf(ux) - tf(uy)\| \le \varphi(x, y)$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $\{(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n} x)\}$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (ii).

Proof. By the same reasoning as the proof of Theorem 1, there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (ii). By the continuity of f, the uniform convergence and the definition of T, the A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is continuous, as desired. ∎

Theorem 5. *Let $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be a continuous mapping with $h(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{H} \times {}_A\mathcal{H} \to [0, \infty)$ satisfying* (i) *such that*

$$\|ruh(\frac{sx+ty}{r}) - sh(ux) - th(uy)\| \le \varphi(x, y)$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{H}$. Assume that $h((\frac{s}{t})^{2n} x) = (\frac{s}{t})^{2n} h(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$. Then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a bounded A-linear operator. Furthermore,

(1) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|h(x) - h^*(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a self-adjoint operator,

(2) *if the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequality*

$$\|h \circ h^*(x) - h^* \circ h(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a normal operator*,

(3) *if the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequalities*

$$\|h \circ h^*(x) - x\| \le \varphi(x, x),$$
$$\|h^* \circ h(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a unitary operator*, *and*

(4) *if the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *satisfies the inequalities*

$$\|h \circ h(x) - h(x)\| \le \varphi(x, x),$$
$$\|h^*(x) - h(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, *then the mapping* $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ *is a projection.*

Proof. The sequence $\{(\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x)\}$ converges uniformly on ${}_A\mathcal{H}$. By Theorem 4, there exists a unique continuous A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (ii). By the assumption,

$$T(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} (\frac{s}{t})^{2n} h(x) = h(x)$$

for all $x \in {}_A\mathcal{H}$, where the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Theorem 1. Hence the A-linear operator T is the mapping h. So the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a continuous A-linear operator. Thus the A-linear operator $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is bounded (see [1, Proposition II.1.1).

(1) By the assumption,

$$\|h((\frac{s}{t})^{2n} x) - h^*((\frac{s}{t})^{2n} x)\| \le \varphi((\frac{s}{t})^{2n} x, (\frac{s}{t})^{2n} x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$(\frac{t}{s})^{2n} \|h((\frac{s}{t})^{2n} x) - h^*((\frac{s}{t})^{2n} x)\| \to 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$h(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} h^*((\frac{s}{t})^{2n} x) = h^*(x)$$

for all $x \in {}_A\mathcal{H}$. So the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a self-adjoint operator.

The proofs of the others are similar to the proof of (1). ■

Now we are going to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^*-algebra for the case $s = t = 1$ and $r = d$.

Theorem 6. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ such that*

$$\text{(iii)} \qquad \begin{aligned} \widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} \frac{1}{(d-1)^k} \varphi((d-1)^k x, (d-1)^k y) < \infty \\ \|duf(\frac{x+y}{d}) - f(ux) - f(uy)\| \le \varphi(x, y) \end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x, y \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\text{(iv)} \qquad \|f(x) - T(x)\| \le \frac{1}{d-1} \widetilde{\varphi}(x, (d-1)x)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Replacing y by $(d-1)x$, one can obtain

$$\|(d-1)f(x) - f((d-1)x)\| \le \varphi(x, (d-1)x)$$

for all $x \in {}_A\mathcal{B}$. So

$$\|f(x) - \frac{f((d-1)x)}{d-1}\| \le \frac{1}{d-1} \varphi(x, (d-1)x),$$

and hence

$$\text{(5)} \quad \|f(x) - \frac{f((d-1)^n x)}{(d-1)^n}\| \le \sum_{k=0}^{n-1} \frac{1}{(d-1)^{k+1}} \varphi((d-1)^k x, (d-1) \cdot (d-1)^k x)$$

for all $x \in {}_A\mathcal{B}$.

We claim that the sequence $\{\frac{f((d-1)^n x)}{(d-1)^n}\}$ is a Cauchy sequence. Indeed, for $n > m$, we have

$$\begin{aligned} \|\frac{f((d-1)^n x)}{(d-1)^n} - \frac{f((d-1)^m x)}{(d-1)^m}\| &\le \sum_{k=m}^{n-1} \|\frac{f((d-1)^{k+1} x)}{(d-1)^{k+1}} - \frac{f((d-1)^k x)}{(d-1)^k}\| \\ &\le \sum_{k=m}^{n-1} \frac{1}{(d-1)^{k+1}} \varphi((d-1)^k x, (d-1) \cdot (d-1)^k x) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. It follows from (iii) that

$$\lim_{m\to\infty} \sum_{k=m}^{n-1} \frac{1}{(d-1)^{k+1}} \varphi((d-1)^k x, (d-1)\cdot(d-1)^k x) = 0$$

for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is a Banach space, the sequence $\{\frac{f((d-1)^n x)}{(d-1)^n}\}$ converges. Define

$$T(x) = \lim_{n\to\infty} \frac{f((d-1)^n x)}{(d-1)^n}$$

for all $x \in {}_A\mathcal{B}$. Taking the limit in (5) as $n \to \infty$, we obtain

$$\|f(x) - T(x)\| \leq \frac{1}{d-1} \widetilde{\varphi}(x, (d-1)x)$$

for all $x \in {}_A\mathcal{B}$, which is the inequality (iv). From the definition of T, we get

$$(d-1)^n T(x) = T((d-1)^n x) \ \text{ and } \ T(0) = 0. \tag{6}$$

By (iii) and the definition of T,

$$\begin{aligned}
&\|dT(\frac{x+y}{d}) - T(x) - T(y)\| \\
&= \lim_{n\to\infty} \frac{1}{(d-1)^n} \|df(\frac{(d-1)^n(x+y)}{d}) - f((d-1)^n x) - f((d-1)^n y)\| \\
&\leq \lim_{n\to\infty} \frac{1}{(d-1)^n} \varphi((d-1)^n x, (d-1)^n y) = 0
\end{aligned}$$

for all $x, y \in {}_A\mathcal{B}$. So

$$dT(\frac{x+y}{d}) = T(x) + T(y)$$

for all $x, y \in {}_A\mathcal{B}$. By Lemma A, T is additive.

If $F : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is another additive mapping satisfying (iv), then it follows from (iv), (6) and the proof of Lemma A that

$$\begin{aligned}
\|T(x) - F(x)\| &= \|\frac{T((d-1)^n x)}{(d-1)^n} - \frac{F((d-1)^n x)}{(d-1)^n}\| \\
&\leq \|\frac{T((d-1)^n x)}{(d-1)^n} - \frac{f((d-1)^n x)}{(d-1)^n}\| \\
&\quad + \|\frac{f((d-1)^n x)}{(d-1)^n} - \frac{F((d-1)^n x)}{(d-1)^n}\| \\
&\leq 2\frac{1}{(d-1)^{n+1}} \widetilde{\varphi}((d-1)^n x, (d-1)\cdot(d-1)^n x),
\end{aligned}$$

which tends to zero as $n \to \infty$ by (iii). Thus we conclude that

$$T(x) = F(x)$$

for all $x \in {}_A\mathcal{B}$. This completes the uniqueness of T.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\|duf(\frac{2(d-1)^n}{d}x) - 2f((d-1)^n ux)\| \le \varphi((d-1)^n x, (d-1)^n x),$$

$$\|df(\frac{2(d-1)^n}{d}ux) - 2f((d-1)^n ux)\| \le \varphi((d-1)^n ux, (d-1)^n ux)$$

for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}
&\|df(\frac{2(d-1)^n}{d}ux) - duf(\frac{2(d-1)^n}{d}x)\| \\
&\le \|df(\frac{2(d-1)^n}{d}ux) - 2f((d-1)^n ux)\| \\
&\quad + \|duf(\frac{2(d-1)^n}{d}x) - 2f((d-1)^n ux)\| \\
&\le \varphi((d-1)^n ux, (d-1)^n ux) + \varphi((d-1)^n x, (d-1)^n x)
\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Thus

$$\frac{1}{(d-1)^n}\|df(\frac{2(d-1)^n}{d}ux) - duf(\frac{2(d-1)^n}{d}x)\| \to 0$$

as $n \to \infty$ for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$dT(\frac{2}{d}ux) = \lim_{n\to\infty} \frac{df(\frac{2(d-1)^n}{d}ux)}{(d-1)^n} = \lim_{n\to\infty} \frac{duf(\frac{2(d-1)^n}{d}x)}{(d-1)^n} = duT(\frac{2}{d}x)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$T(ux) = \frac{d}{2}T(\frac{2}{d}ux) = \frac{d}{2}uT(\frac{2}{d}x) = uT(x)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 1. ■

Corollary 7. *Let d be an integer greater than 2 and $0 < p < 1$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|duf(\frac{x+y}{d}) - f(ux) - f(uy)\| \le ||x||^p + ||y||^p$$

for all $u \in \mathcal{U}(A)$ *and all* $x, y \in {}_A\mathcal{B}$. *Then there exists a unique* A*-linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *such that*

$$\|f(x) - T(x)\| \le \frac{1 + (d-1)^p}{d - 1 - (d-1)^p}\|x\|^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = \|x\|^p + \|y\|^p$, and apply Theorem 6. ■

Theorem 8. *Let* $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *be a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ *such that*

$$\text{(v)} \qquad \begin{aligned} &\widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty}(d-1)^k\varphi(\frac{1}{(d-1)^k}x, \frac{1}{(d-1)^k}y) < \infty \\ &\|duf(\frac{x+y}{d}) - f(ux) - f(uy)\| \le \varphi(x, y) \end{aligned}$$

for all $u \in \mathcal{U}(A)$ *and all* $x, y \in {}_A\mathcal{B}$. *Then there exists a unique* A*-linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *such that*

$$\text{(vi)} \qquad \|f(x) - T(x)\| \le \widetilde{\varphi}(\frac{1}{d-1}x, x)$$

for all $x \in {}_A\mathcal{B}$.

Proof. Put $u = 1 \in \mathcal{U}(A)$. Replacing x by $\frac{x}{d-1}$ and y by x, one can obtain

$$\|(d-1)f(\frac{x}{d-1}) - f(x)\| \le \varphi(\frac{x}{d-1}, x)$$

for all $x \in {}_A\mathcal{B}$. So

$$\text{(7)} \qquad \|(d-1)^n f(\frac{x}{(d-1)^n}) - f(x)\| \le \sum_{k=0}^{n-1}(d-1)^k\varphi(\frac{x}{(d-1)^{k+1}}, \frac{x}{(d-1)^k})$$

for all $x \in {}_A\mathcal{B}$.

We claim that the sequence $\{(d-1)^n f(\frac{x}{(d-1)^n})\}$ is a Cauchy sequence. Indeed, for $n > m$, we have

$$\begin{aligned} &\|(d-1)^n f(\frac{x}{(d-1)^n}) - (d-1)^m f(\frac{x}{(d-1)^m})\| \\ &\qquad \le \sum_{k=m}^{n-1}\|(d-1)^{k+1}f(\frac{x}{(d-1)^{k+1}}) - (d-1)^k f(\frac{x}{(d-1)^k})\| \\ &\qquad \le \sum_{k=m}^{n-1}(d-1)^k\varphi(\frac{x}{(d-1)^{k+1}}, \frac{x}{(d-1)^k}) \end{aligned}$$

for all $x \in {}_A\mathcal{B}$. It follows from (v) that

$$\lim_{m\to\infty}\sum_{k=m}^{n-1}(d-1)^k\varphi(\frac{x}{(d-1)^{k+1}},\frac{x}{(d-1)^k})=0$$

for all $x \in {}_A\mathcal{B}$. Since ${}_A\mathcal{C}$ is a Banach space, the sequence $\{(d-1)^n f(\frac{x}{(d-1)^n})\}$ converges. Define

$$T(x)=\lim_{n\to\infty}(d-1)^n f(\frac{x}{(d-1)^n})$$

for all $x \in {}_A\mathcal{B}$. Taking the limit in (7) as $n\to\infty$, we obtain

$$\|T(x)-f(x)\|\le\widetilde{\varphi}(\frac{x}{d-1},x)$$

for all $x \in {}_A\mathcal{B}$, which is the inequality (vi). From the definition of T, we get

$$\frac{1}{(d-1)^n}T(x)=T(\frac{x}{(d-1)^n})\ \text{ and }\ T(0)=0. \tag{8}$$

By (v) and the definition of T,

$$\begin{aligned}&\|dT(\frac{x+y}{d})-T(x)-T(y)\|\\&=\lim_{n\to\infty}(d-1)^n\|df(\frac{x+y}{d(d-1)^n})-f(\frac{x}{(d-1)^n})-f(\frac{y}{(d-1)^n})\|\\&\le\lim_{n\to\infty}(d-1)^n\varphi(\frac{x}{(d-1)^n},\frac{y}{(d-1)^n})=0\end{aligned}$$

for all $x,y \in {}_A\mathcal{B}$. So

$$dT(\frac{x+y}{d})=T(x)+T(y)$$

for all $x,y \in {}_A\mathcal{B}$. By Lemma A, T is additive.

If $F : {}_A\mathcal{B}\to{}_A\mathcal{C}$ is another additive mapping satisfying (vi), then it follows from (vi), (8) and the proof of Lemma A that

$$\begin{aligned}\|T(x)-F(x)\|&=\|(d-1)^nT(\frac{x}{(d-1)^n})-(d-1)^nF(\frac{x}{(d-1)^n})\|\\&\le\|(d-1)^nT(\frac{x}{(d-1)^n})-(d-1)^nf(\frac{x}{(d-1)^n})\|\\&\quad+\|(d-1)^nf(\frac{x}{(d-1)^n})-(d-1)^nF(\frac{x}{(d-1)^n})\|\\&\le 2(d-1)^n\widetilde{\varphi}(\frac{x}{(d-1)^{n+1}},\frac{x}{(d-1)^n}),\end{aligned}$$

which tends to zero as $n \to \infty$ by (v). Thus we conclude that

$$T(x) = F(x)$$

for all $x \in {}_A\mathcal{B}$. This completes the uniqueness of T.

By the assumption, for each $u \in \mathcal{U}(A)$,

$$\|duf(\frac{2x}{d(d-1)^n}) - 2f(\frac{ux}{(d-1)^n})\| \le \varphi(\frac{x}{(d-1)^n}, \frac{x}{(d-1)^n}),$$
$$\|df(\frac{2ux}{d(d-1)^n}) - 2f(\frac{ux}{(d-1)^n})\| \le \varphi(\frac{ux}{(d-1)^n}, \frac{ux}{(d-1)^n})$$

for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}
&\|df(\frac{2ux}{d(d-1)^n}) - duf(\frac{2x}{d(d-1)^n})\| \\
&\le \|df(\frac{2ux}{d(d-1)^n}) - 2f(\frac{ux}{(d-1)^n})\| + \|duf(\frac{2x}{(d-1)^n}) - 2f(\frac{ux}{(d-1)^n})\| \\
&\le \varphi(\frac{ux}{(d-1)^n}, \frac{ux}{(d-1)^n}) + \varphi(\frac{x}{(d-1)^n}, \frac{x}{(d-1)^n})
\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Thus

$$(d-1)^n \|df(\frac{2ux}{d(d-1)^n}) - duf(\frac{2x}{d(d-1)^n})\| \to 0$$

as $n \to \infty$ for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned}
dT(\frac{2}{d}ux) &= \lim_{n\to\infty} (d-1)^n df(\frac{2ux}{d(d-1)^n}) \\
&= \lim_{n\to\infty} (d-1)^n duf(\tfrac{2x}{d(d-1)^n}) \\
&= duT(\frac{2}{d}x)
\end{aligned}$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$. So

$$T(ux) = \frac{d}{2}T(\frac{2}{d}ux) = \frac{d}{2}uT(\frac{2}{d}x) = uT(x)$$

for all $u \in \mathcal{U}(A)$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 1. ■

Corollary 9. *Let d be an integer greater than 2 and $p > 1$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|duf(\frac{x+y}{d}) - f(ux) - f(uy)\| \le ||x||^p + ||y||^p$$

for all $u \in \mathcal{U}(A)$ *and* $x, y \in {}_A\mathcal{B}$. *Then there exists a unique* A*-linear mapping* $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ *such that*

$$\|f(x) - T(x)\| \leq \frac{(d-1)^p + 1}{(d-1)^p + 1 - d}\|x\|^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = \|x\|^p + ||y||^p$, and apply Theorem 8. ■

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