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SOBOLEV TYPE INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION IN BANACH SPACES

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Abstract. In this paper we prove the existence of mild and strong solutions of an integrodifferential equation of Sobolev type with nonlocal condition. The results are obtained by using uniformly continuous semigroups and the Schauder fixed point theorem.

1. Introduction

The problem of existence of solutions of evolution equations with nonlocal conditions in Banach spaces has been studied first by Byszewski [8]. In that paper he established the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in (t_0, t_0 + b], \tag{1}$$

$$u(t_0) + g(t_1, t_2, ..., t_p, u(.)) = u_0 \tag{2}$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, on a Banach space X, $0 \leq t_0 < t_1 < t_2 < ... < t_p \leq t_0 + b$, $b > 0$, $u_0 \in X$ and $f : [t_0, t_0+b] \times X \to X$, $g : [t_0, t_0+b]^p \times X \to X$ are given functions. Subsequently several authors have investigated the same type of problem to different classes of abstract differential equations in Banach spaces [1-4, 9, 12, 14, 15]. Brill [7] and Showalter [17] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. Lightbourne and Rankin [13] studied a partial functional differential equation of Sobolev type in a Banach

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space. This type of equations arise in various applications such as in the flow of fluid through fissured rocks [6], the propagation of long waves of small amplitudes [5], thermodynamics [10] and shear in second order fluids [11, 18]. The purpose of this paper is to prove the existence of mild and strong solutions for an integrodifferential equation of Sobolev type with nonlocal condition of the form

$$(Bu(t))' + Au(t) = f(t, u(t), \int_0^t a(t,s)k(s,u(s))ds), \quad t \in (0,b], \tag{3}$$

$$u(0) + g(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p)) = u_0, \tag{4}$$

where B and A are linear operators with domains contained in a Banach space X and ranges contained in a Banach space Y and the nonlinear operators $f : I \times X \times X \to Y, k : I \times X \to X, g : I^p \times X^p \to X$ and the function $a : I \times I \to R$ are given. Here $u_0 \in D(B)$ and $I = [0, b]$.

2. Preliminaries

In order to prove our main theorem we assume certain conditions on the operators A and B. Let X and Y be Banach spaces with norm $|.|$ and $\|.\|$ respectively. The operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the following hypothesis:

(H_1) A and B are closed linear operators,

(H_2) $D(B) \subset D(A)$ and B is bijective,

(H_3) $B^{-1} : Y \to D(B)$ is compact.

From the above fact and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \to Y$. Further $-AB^{-1}$ generates a uniformly continuous semigroup $T(t)$, $t \geq 0$ and so $\max_{t\in I} \|T(t)\|$ is finite. We denote $M = \max_{t\in I} \|T(t)\|$, $R = \|B^{-1}\|$. Let $B_r = \{x \in X : |x| \leq r\}$.

Definition 2.1. [16] A continuous solution u of the integral equation

$$\begin{aligned} u(t) = &B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, ..., t_p, u(t_1), ..., u(t_p)) \\ &+ \int_0^t B^{-1}T(t-s)f(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau)ds, \ t \in I, \end{aligned}$$

is said to be a mild solution of problem (3)-(4) on I.

Definition 2.2. [16] A function u is said to be a strong solution of problem (3)-(4) on I if u is differentiable almost everywhere on I, $u' \in L^1(I, X)$,

$$u(0) + g(t_1, ..., t_p, u(t_1), ..., u(t_p)) = u_0,$$

and

$$(Bu(t))' + Au(t) = f(t, u(t), \int_0^t a(t,s)k(s, u(s))ds), \quad \text{a.e.on} \quad I.$$

Further assume that,

(H_4) $a : I \times I \to R$ and $k : I \times B_r \to X$ are continuous,

(H_5) $f : I \times B_r \times B_r \to Y$ is continuous in t on I and there exists a constant $L > 0$ such that

$$\|f(t, u, v)\| \leq L \quad \text{for} \quad t \in I \quad \text{and} \quad u, v \in B_r,$$

(H_6) $g : I^p \times B_r^p \to D(B) \subset X$, Bg is continuous and there exists a constant $G > 0$ such that

$$\|Bg(t_1, t_2, ..., t_p, u(t_1), ..., u(t_p))\| \leq G, \quad \text{for} \quad t_i \in I \text{ and } u(t_i) \in B_r,$$

(H_7) The set $\{u(0) : u \in C(I, X),\ \|u\| \leq r,\ u(0)+g(t_1, ..., t_p; u(t_1), ..., u(t_p)) = u_0\}$ where $r = RM(\|Bu_0\| + G + Lb)$, is precompact in X.

3. Main Results

Theorem 3.1. *If the assumptions $H_1 \sim H_7$ hold, then the problem* (3)-(4) *has a mild solution on I.*

Proof. Let $E = C(I, X)$ and $E_0 = \{u \in E : u(t) \in B_r,\ t \in I\}$. Clearly, E_0 is a bounded closed convex subset of E. We define a mapping $F : E_0 \to E_0$ by

$$\begin{aligned}(Fu)(t) =& B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, t_2, ..., t_p, u(t_1), ..., u(t_p)) \\ &+ \int_0^t B^{-1}T(t-s)f(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau)ds, \quad t \in I.\end{aligned}$$

Obviously F is continuous, since all the functions involved in the definition of the operator are continuous. Further, from our assumptions, we have

$$\begin{aligned}\|(Fu)(t)\| \leq& \|B^{-1}T(t)Bu_0\| + \|B^{-1}T(t)Bg(t_1, t_2, ..., t_p, u(t_1), ..., u(t_p))\| \\ &+ \int_0^t \|B^{-1}T(t-s)f(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau)\|ds \\ \leq& RM\|Bu_0\| + RMG + RMLb = r,\end{aligned}$$

and therefore F maps E_0 into E_0. Moreover, F maps E_0 into a precompact subset of E_0. To prove this, we first show that the set $E_0(t) = \{(Fu)(t) : u \in E_0\}$ is precompact in X, for every fixed $t \in I$. This is clear for $t = 0$ by (H_7). Let $t > 0$ be fixed. For $0 < \epsilon < t$, take

$$\begin{aligned}(F_\epsilon u)(t) =& B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, t_2, ..., t_p, u(t_1), ..., u(t_p)) \\ &+ \int_0^{t-\epsilon} B^{-1}T(t-s)f(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau)ds, \quad t \in I.\end{aligned}$$

Since B^{-1} is compact and $T(t)$ is uniformly continuous for every $t > 0$, the set $E_\epsilon(t) = \{(F_\epsilon u)(t) : u \in E_0\}$ is precompact in X for every ϵ. Furthermore, for $u \in E_0$ we have

$$\begin{aligned}&\|(Fu)(t) - (F_\epsilon u)(t)\| \\ &\leq \int_{t-\epsilon}^t \|B^{-1}T(t-s)f(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau)\|ds \\ &\leq \epsilon RML\end{aligned}$$

which implies that $E_0(t)$ is totally bounded, that is precompact in X. Now we shall show that $F(E_0) = S = \{Fu : u \in E_0\}$ is an equicontinuous family of functions. For $0 < s < t$, we have

$$\begin{aligned}\|(Fu)(t) - (Fu)(s)\| \leq& \|B^{-1}(T(t) - T(s))Bu_0\| \\ &+\|B^{-1}(T(t) - T(s))Bg(t_1, ..., t_p, u(t_1), ..., u(t_p))\| \\ &+ \int_0^t \|B^{-1}\| \; \|T(t-\theta) - T(s-\theta)\| \\ &\quad \|f(\theta, u(\theta), \int_0^\theta a(\theta,\tau)k(\tau, u(\tau))d\tau)\|d\theta \\ &+ \int_s^t \|B^{-1}\| \; \|T(s-\theta)\| \\ &\quad \|f(\theta, u(\theta) \int_0^\theta a(\theta\tau)k(\tau, u(\tau))d\tau)\|d\theta \\ \leq& (R\|Bu_0\| + RG)\|T(t) - T(s)\| \\ &+RL \int_0^t \|T(t-\theta) - T(s-\theta)\|d\theta + RML|t-s|.\end{aligned}$$

The right hand side of the above inequality is independent of $u \in E_0$ and tends to zero as $s \to t$ as a consequence of the continuity of $T(t)$ in the uniform operator topology for $t > 0$. It is also clear that S is bounded in E. Thus by Arzela-Ascoli's theorem, S is precompact. Hence by the Schauder fixed point theorem, F has a fixed point in E_0 and any fixed point of F is a mild solution of (3)–(4) on I such that $u(t) \in X$ for $t \in I$.

Next we prove that the problem (3)–(4) has a strong solution.

Theorem 3.2. *Assume that*

(i) *Conditions $H_1 \sim H_7$ hold,*

(ii) *Y is a reflexive Banach space with norm $\|.\|$,*

(iii) *There exists a constant $K > 0$ such that*

$$\|k(t,u)\| \leq K \text{ for } \ t \in I \ \text{ and } \ u \in B_r,$$

(iv) *There exists a constant $L_1 > 0$ such that*

$$\|f(t,u_1,u_2) - f(s,v_1,v_2)\| \leq L_1[|t-s| + \|u_1 - v_1\| + \|u_2 - v_2\|], \\ s,t \in I, u_i, v_i \in B_r$$

(v) *There exist constants $L_2 > 0, a_0 > 0$ such that*

$$|a(t,\tau) - a(s,\tau)| \leq L_2|t-s|, \quad \text{for} \ \ t,\tau,s \in I,$$
$$|a(t,s)| \leq a_0 \ \text{ for } \ s,t \in I,$$

(vi) *u is the unique mild solution of the problem* (3)-(4),

(vii) *$Bu_0 \in D(-AB^{-1})$ and $Bg(t_1,....,t_p,u(t_1),...,u(t_p)) \in D(-AB^{-1})$.*

Then u is a unique strong solution of the problem (3)–(4) on I.

Proof. Since all the assumptions of Theorem 3.1 are satisfied, then the problem (3)–(4) has a mild solution belonging to $C(I,B_r)$. By the assumption (vi), u is the unique mild solution of the problem (3)-(4). Now, we shall show that u is a strong solution of problem (3)-(4) on I.

For any $t \in I$, we have

$$
\begin{aligned}
u(t+h)-u(t) &= B^{-1}[T(t+h)-T(t)]Bu_0 \\
&\quad -B^{-1}[T(t+h)-T(t)]Bg(t_1,....,t_p,u(t_1),...,u(t_p)) \\
&\quad +\int_0^h B^{-1}T(t+h-s)f(s,u(s),\int_0^s a(s,\tau)k(\tau,u(\tau))d\tau)ds \\
&\quad -\int_h^{t+h} B^{-1}T(t+h-s)f(s,u(s),\int_0^s a(s,\tau)k(\tau,u(\tau))d\tau)ds \\
&\quad -\int_0^t B^{-1}T(t-s)f(s,u(s),\int_0^s a(s,\tau)k(\tau,u(\tau))d\tau)ds \\
&= B^{-1}T(t)[T(h)-I]Bu_0 - B^{-1}T(t)[T(h)-I] \\
&\quad Bg(t_1,....,t_p,u(t_1),...,u(t_p)) \\
&\quad +\int_0^h B^{-1}T(t+h-s)f(s,u(s),\int_0^s a(s,\tau)k(\tau,u(\tau))d\tau)ds \\
&\quad +\int_0^t B^{-1}T(t-s)[f(s+h,u(s+h), \\
&\qquad \int_0^{s+h} a(s+h,\tau)k(\tau,u(\tau))d\tau)) \\
&\quad -f(s,u(s),\textstyle\int_0^s a(s,\tau)k(\tau,u(\tau))d\tau)]ds.
\end{aligned}
$$

Using our assumptions we observe that

$$
\begin{aligned}
\|u(t+h)-u(t)\| &\leq R\|Bu_0\|Mh\|AB^{-1}\| + MRGh\|AB^{-1}\| \\
&\quad +hRML + RM\int_0^t L_1[h+\|u(s+h)-u(s)\| \\
&\quad [\int_0^s |a(s+h,\tau)-a(s,\tau)|\|k(\tau,u(\tau)\|d\tau \\
&\quad +\int_s^{s+h} |a(s+h,\tau)|d\tau\|k(\tau,u(\tau)\|]ds \\
&\leq R\|Bu_0\|Mh\|AB^{-1}\| + MRGh\|AB^{-1}\| \\
&\quad +hRML + RML_1\int_0^t [h+\|u(s+h)-u(s)\|]ds \\
&\quad +RML_1bh(a_0K+L_2Kb) \\
&\leq Ph + Q\int_0^t \|u(s+h)-u(s)\|ds,
\end{aligned}
$$

where $P = R\|Bu_0\|M\|AB^{-1}\| + MRG\|AB^{-1}\| + RML + Qb(1 + Ka_0 + L_2Kb)$ and $Q = RML_1$.

By Gronwall's inequality

$$\|u(t+h) - u(t)\| \leq Phe^{Qb}, \text{ for } t \in I.$$

Therefore, u is Lipschitz continuous on I. The Lipschitz continuity of u on I combined with (iv) and (v) imply that

$$t \to f(t, u(t), \int_0^t a(t,s)k(s, u(s))ds)$$

is Lipschitz continuous on I. Using the Corollary 2.11 in Section 4.2 of [16] and the definition of strong solution we observe that the linear Cauchy problem:

$$\begin{aligned}
&(Bv(t))' + Av(t) = f(t, u(t), \int_0^t a(t,s)k(s, u(s))ds), \quad t \in (0, b], \\
&v(0) = u_0 - g(t_1, ..., t_p, u(t_1), ..., u(t_p)),
\end{aligned}$$

has a unique strong solution v satisfying the equation

$$\begin{aligned}
v(t) &= B^{-1}T(t)Bu_0 - B^{-1}T(t)Bg(t_1, ..., t_p, u(t_1), ..., u(t_p)) \\
&\quad + \int_0^t B^{-1}T(t-s)f(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau)ds, \ t \in I, \\
&= u(t).
\end{aligned}$$

Consequently, u is a strong solution of problem (3)–(4) on I.

4. Example

Consider the following partial integrodifferential equation

$$\frac{\partial}{\partial t}(z(t,x) - z_{xx}(t,x)) - z_{xx}(t,x) = \mu(t, z(t,x), \int_0^t \eta(t,s)z(s,x)ds) \quad x \in [0,\pi], t \in I, \tag{5}$$

$$\begin{aligned}
&z(t,0) = z(t,\pi) = 0, \quad t \in I, \\
&z(0,x) + c\int_0^a z(s,x)ds = z_0(x), \quad 0 < a \leq b, \quad c > 0, \quad x \in [0,\pi].
\end{aligned} \tag{6}$$

Let us take $X = Y = L^2[0,\pi]$. Define the operators $A : D(A) \subset X \to Y, B : D(B) \subset X \to Y$ by

$$\begin{aligned}
Az &= -z_{xx}, \\
Bz &= z - z_{xx},
\end{aligned}$$

where $D(A) = D(B) = \{z \in X : z, z_x$ are absolutely continuous, $z_{xx} \in X$, $z(0) = z(\pi) = 0\}$. Define the operators $f : I \times X \times X \to Y$, $k : I \times I \times X \to X$ by

$$f(t, z, (kz)(t))(x) = \mu(t, z(t, x),\ (kz)(t, x)),\quad (kz)(t, x) = \int_0^t \eta(t, s) z(s, x) ds$$

and satisfy the conditions (H_5) and (H_6) on a bounded closed set $B_r \subset X$ for some $r > 0$. Then the above problem (5) can be formulated abstractly as

$$(Bz(t))' + Az(t) = f(t, z(t), \int_0^t a(t, s) k(s, z(s)) ds),\quad t \in I.$$

Note that $-AB^{-1}$ generates a uniformly continuous semigroup $T(t)$ on Y and it is given by [13]

$$T(t)z = \sum_{n=1}^{\infty} e^{\frac{-n^2}{(1+n^2)}t} < z, z_n > z_n$$

where $z_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$, $n = 1, 2, ...$, is an orthogonal set of eigenfunctions. Further, $T(t)$ is compact and $\|T(t)\| \le e^{-t}$ for each $t > 0$. Hence by Theorem 3.1 the equation (5) with nonlocal condition (6) has a mild solution on I.

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