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SCHROEDER'S EQUATION IN SEVERAL VARIABLES*

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Abstract. In 1884, Koenigs showed that when φ is an analytic self-map of the unit disk fixing the origin, with $0 < |\varphi'(0)| < 1$, then Schroeder's functional equation, $f \circ \varphi = \varphi'(0)f$, can be solved for a unique analytic function f in the disk with $f'(0) = 1$. Here we consider a natural analogue of Schroeder's equation in the unit ball of $\mathbb{C}^N$ for $N > 1$, namely, $f \circ \varphi = \varphi'(0)f$ where φ is an analytic self-map of the unit ball fixing the origin and f is to be a $\mathbb{C}^N$-valued analytic map on the ball. Under some natural hypotheses on φ, we give necessary and sufficient conditions for the existence of a solution f satisfying $f'(0) = I$ and then describe all analytic solutions in the ball. We also discuss various phenomena which may occur in the several variable setting that do not occur when $N = 1$.

1. Introduction

When φ is an analytic map of the unit disk into itself, with $\varphi(0) = 0$ and $\lambda = \varphi'(0)$ satisfying $0 < |\lambda| < 1$, work of Koenigs [9] in 1884 gives an essentially unique solution f to Schroeder's functional equation $f \circ \varphi = \lambda f$ that is analytic in the disk. Koenigs realized f as the almost uniform limit of the sequence of normalized iterates φ_n/λ^n of φ. When the Koenigs function f lies in an appropriate space of functions analytic on the disk (e.g., the Hardy space or the Bergman space), then f serves as an eigenvector for the composition

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operator C_φ acting on this space. Thus, the solutions of Schroeder's equation play a basic role in the study of composition operators on spaces of functions analytic on the disk.

The goal of this paper is to consider a natural several-variable version of Schroeder's equation, with the unit ball in $\mathbb{C}^N$ replacing the unit disk. The idea of solving such a multidimensional analogue of Schroeder's equation for φ by considering the almost uniform convergence of a sequence of normalized iterates of φ (generalizing Koenig's original proof in the disk) has a long history, which includes work by Poincare [13, 14], Bieberbach [2] and Picard [12] (this latter is not completely correct in the case that certain special arithmetic relationships hold on the eigenvalues of $\varphi'(0)$; see [7] for a discussion of this issue and for further details on the history of work in this area). More recently, as part of an extensive study of holomorphic maps of $\mathbb{C}^N$ to $\mathbb{C}^N$, Rosay and Rudin [16] have shown that for a biholomorphic map φ of $\mathbb{C}^N$ onto $\mathbb{C}^N$ fixing 0, the iterates $\varphi'(0)^{-n}\varphi_n$ converge almost uniformly on the "region attracted to 0 by φ" under a hypothesis on the eigenvalues of $\varphi'(0)$ which excludes the possibility of these special eigenvalue relationships. They also give a proof which shows in general how to conjugate φ to a "normal form"; this becomes Schroeder's equation when no eigenvalue relationships hold. Earlier work of Reich [15] had claimed the same result, but there were problems with convergence under certain conditions in his argument. We thank the referee for pointing out the Rosay-Rudin reference to us.

Much of the previous work on Schroeder's equation in several variables, or related questions, such as the linearization of differential equations by a suitable change of variable ([11, p. 335], [8], [20] and [1, Chapter 5]) has focused on the existence of formal power series solutions or local analytic solutions. For example, references [10], [18], [19], and [11, pp. 332-336] discuss formal solutions or local analytic solutions and the obstruction that can arise when there are certain arithmetic relations (called resonances in the differential equations literature) among the eigenvalues of $\varphi'(0)$.

In contrast, our approach constructs global analytic solutions in the ball, without requiring any consideration of difficult issues of convergence of formal power series, by using the theory of compact composition operators on certain Hilbert spaces of functions analytic on the ball. This enables us to give necessary and sufficient conditions, under mild hypotheses on φ, for the solution of a natural form of Schroeder's functional equation in the ball. In particular, the global solutions are found to exist under no less general conditions than the formal power series solutions found by earlier researchers. With this approach, we find that the more subtle issue of the existence of an analytic solution in the ball that is locally univalent near the origin becomes a matter of the diagonalizability of certain matrices whose size depends on particular arithmetic relationships among the eigenvalues of $\varphi'(0)$; see Theorem 14.

We begin with a discussion of the desired form for a several-variable analogue of Schroeder's equation. If φ is an analytic map of the unit ball, B_N, into itself and f is a $\mathbb{C}^m$-valued, analytic function on the ball, then $f \circ \varphi$ is defined and is also a $\mathbb{C}^m$-valued, analytic function on the ball. If we were to proceed by analogy with the one-variable case where the Koenigs' function is obtained as a limit of normalized iterates of the given map, since φ and its iterates are $\mathbb{C}^N$-valued, we would expect the unknown function in Schroeder's equation to be $\mathbb{C}^N$-valued. The multiplier on the right-hand side of Schroeder's equation is a constant; in the several-variable context, this constant might be a scalar, but it is more general to consider a constant matrix. Therefore, we are motivated to seek a $\mathbb{C}^N$-valued analytic function f for which

$$f \circ \varphi = Af, \tag{1}$$

where A is some $N \times N$ matrix. As in the one-variable setting, we will also assume $\varphi(0) = 0$. If, for a mapping f satisfying Equation (1), $f'(0)$ is invertible, then $f'(\varphi(0))\varphi'(0) = Af'(0)$, and it follows that $A = f'(0)\varphi'(0)f'(0)^{-1}$. Therefore, the equation $g \circ \varphi = \varphi'(0)g$ has the function $g = f'(0)^{-1}f$ as a solution. Notice that this substitution also gives $g'(0) = I$. Thus we will define our analogue of Schroeder's functional equation to be

$$f \circ \varphi = \varphi'(0)f \tag{2}$$

and a solution f of Equation (2) will be called a *Schroeder map for* φ. We will be primarily concerned with seeking Schroeder maps that are locally univalent near 0. By the above computation, the inverse function theorem, and its converse (see, for example, [17]), this is equivalent to the existence of a Schroeder map whose derivative at 0 is the identity.

Recall that "φ unitary on a slice" means that there exist ζ and η in ∂B_N with $\varphi(\lambda\zeta) = \lambda\eta$ for all λ in the unit disk D. Since φ maps B_N into itself and $\varphi(0) = 0$, the Schwarz Lemma [17] implies $|\varphi'(0)| \leq 1$. Strict inequality occurs precisely when φ is not unitary on any slice. This can be seen by considering the self-maps of the disk defined by $g(\lambda) = \langle \varphi(\lambda\zeta), \eta \rangle$ for ζ and η in ∂B_N. Since $g'(0) = \langle \varphi'(0)\zeta, \eta \rangle$, the one-variable Schwarz Lemma gives the desired conclusion. To avoid certain pathologies, we will often assume that φ is not unitary on any slice. In particular, φ not unitary on any slice means $|\varphi'(0)| < 1$, which implies $\varphi'(0)$ has no eigenvalue of modulus 1. As a consequence, if φ is not unitary on any slice and f solves Equation (2), then $f(0) = 0$.

The organization of the rest of the paper is as follows. In the next section we give some preliminary observations on Schroeder maps in several variables and then show how to build solutions to Schroeder's equation from eigenfunctions for the composition operator C_φ when C_φ is compact on some weighted Hardy space, and conversely how to extract eigenfunctions for C_φ from Schroeder maps. In Section 3 we show that the hypotheses $\varphi(0) = 0$ and φ not unitary on any slice imply that C_φ *will* be compact on certain weighted Hardy spaces which are weighted Bergman spaces defined for weight functions that decay to 0 sufficiently rapidly. In Section 4 we give a number of examples of maps φ which either do or do not have Schroeder maps f satisfying the additional desired condition $f'(0) = I$. In these examples certain arithmetic relationships hold among the eigenvalues of $\varphi'(0)$ which can potentially make the existence of a locally univalent Schroeder map impossible. These examples are put into context in Section 5 where the main results (Theorems 13 and 14) give necessary and sufficient conditions for the existence of a Schroeder map satisfying $f'(0) = I$ under natural hypotheses on φ. Roughly speaking, the operative condition is the diagonalizability of a certain size upper-left corner of the matrix for C_φ with respect to the standard, nonnormalized orthogonal basis for a weighted Hardy space. From these results we are able to give a complete description of all Schroeder maps for φ.

2. Solutions to Schroeder's Equation

In one variable, the solutions to Schroeder's equation are unique up to a multiplicative constant, and, moreover, when φ is univalent in the disk, the Schroeder map will be univalent also. Both of these results can fail when $N > 1$. However, we do have the following restricted version of the latter result.

Propposition 1. *Let φ be an analytic map of B_N into itself such that φ is not unitary on any slice of B_N, $\varphi(0) = 0$. and $A = \varphi'(0)$ is invertible. If f is an analytic map of B_N into $\mathbb{C}^N$ that solves Schroeder's functional equation $f \circ \varphi = Af$ and $f'(0)$ is invertible, then f is univalent on B_N if and only if φ is univalent on B_N.*

Proof. Suppose first that φ is not univalent, that is, suppose that z and w are distinct points of the ball for which $\varphi(z) = \varphi(w)$. The functional equation gives

$$Af(z) = f(\varphi(z)) = f(\varphi(w)) = Af(w).$$

Since A is an invertible matrix, we see $f(z) = f(w)$ so that f is not univalent either.

Conversely, suppose f is not univalent on B_N and z and w are distinct points of the ball for which $f(z) = f(w)$. Then, for every positive integer n,

$$f(\varphi_n(z)) = A^n f(z) = A^n f(w) = f(\varphi_n(w))$$

Since $f'(0)$ is invertible, there is a neighborhood of 0 on which f is univalent. Since φ is not unitary on a slice, the iterates of φ tend to 0 and there is an n large enough so that $\varphi_n(z)$ and $\varphi_n(w)$ are both in this neighborhood. This means that $\varphi_n(z) = \varphi_n(w)$. Since φ univalent on B_N would imply φ_n is also univalent, φ cannot be univalent on B_N. ■

The linear fractional maps discussed in Section 4 show that the "only if" direction can fail if A is not invertible. Example 1 in Section 4 shows that without the hypothesis that $f'(0)$ be invertible, the "if" direction of the above result need not hold.

In one variable the condition $\varphi'(0) = 0$ leads to a degenerate situation, since if φ is not identically 0, the equation $f \circ \varphi = 0$ has only the trivial solution $f = 0$. In several variables we note the following consequence of a zero eigenvalue for $\varphi'(0)$:

Proposition 2. *Let φ be a nonconstant map of B_N into itself with $\varphi(0) = 0$ and suppose 0 is an eigenvalue of $A = \varphi'(0)$. If f is a solution of Schroeder's functional equation $f \circ \varphi = Af$ such that f is an analytic map of B_N into $\mathbb{C}^N$ and $f'(0)$ is invertible, then there is a neighborhood Ω of 0 such that $\varphi(\Omega)$ is contained in an $(N-1)$-dimensional submanifold of B_N.*

Proof. Suppose that f is an analytic map of B_N into $\mathbb{C}^N$ with $f'(0)$ invertible such that $f \circ \varphi = Af$. Let v be an eigenvector of A^* for the eigenvalue 0. Then it follows that

$$\langle f(\varphi(z)), v\rangle = \langle Af(z), v\rangle = \langle f, A^* v\rangle = \langle f, 0\rangle = 0.$$

This says that f maps the range of φ into the $(N-1)$-dimensional subspace orthogonal to v. Since f' is continuous and $f'(0)$ is invertible, there is a neighborhood of 0 on which f' is invertible also. The inverse function theorem then guarantees that f is one-to-one in some neighborhood of 0 and has an analytic inverse g in a neighborhood U of $f(0) = 0$. Thus, if $\Omega = \varphi^{-1}(g(U))$, then $\varphi(\Omega)$ is contained in the submanifold $g(U \cap [v]^\perp)$. ■

We will see later a nontrivial example of a map φ for which 0 is an eigenvalue of $\varphi'(0)$, yet for which there is a univalent f solving $f \circ \varphi = \varphi'(0)f$.

Our approach to constructing solutions to Schroeder's Equation (2) will be through the theory of compact composition operators. A composition operator is defined from an analytic self-map φ of the ball by $C_\varphi(g) = g \circ \varphi$ for g analytic in B_N. In general we will be considering composition operators acting on weighted Hardy spaces $H^2_\beta(B_N)$ in the ball. By definition, these are Hilbert spaces of analytic functions in B_N for which the monomials z^α, $|\alpha| \geq 0$, form a complete orthogonal set of nonzero vectors satisfying

$$\beta(|\alpha|) \equiv \frac{\|z^\alpha\|}{\|z^\alpha\|_2} = \frac{\|z^{\tilde{\alpha}}\|}{\|z^{\tilde{\alpha}}\|_2}$$

whenever $|\alpha| = |\widetilde{\alpha}|$, where $\|\cdot\|$ denotes the norm in $H^2_\beta(B_N)$ and $\|\cdot\|_2$ denotes the norm in $L^2(\sigma_N)$, σ_N being the normalized Lebesgue measure on B_N. Here α is a multi-index $(\alpha_1, \alpha_2, \ldots, \alpha_N)$, $\alpha_j \geq 0$, and $|\alpha| = \sum \alpha_j$. If $f = \sum_0^\infty f_s$ is the homogeneous expansion of a function analytic in B_N, then f is in $H^2_\beta(B_N)$ if and only if $\|f\|^2 \equiv \sum_0^\infty \|f_s\|_2^2 \beta(s)^2 < \infty$.

We will ordinarily write matrices for operators on $H^2_\beta(B_N)$ with respect to the nonnormalized orthogonal basis $1, z_1, z_2, \ldots, z_N, z_1^2, \ldots$. This "standard basis" is ordered in the usual way: z^α precedes z^γ where $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $\gamma = (\gamma_1, \ldots, \gamma_N)$ are multi-indices, if either $|\alpha| < |\gamma|$ or, in the case $|\alpha| = |\gamma|$, if there is a j_0 so that $\alpha_j = \gamma_j$ for $j < j_0$ and $\alpha_{j_0} > \gamma_{j_0}$; we write $\alpha < \gamma$ in this case. If the jth monomial in this ordering is z^α, then the jth column of the matrix for C_φ has as its entries the coefficients of φ^α with respect to this standard basis. This matrix is related to the matrix of C_φ with respect to the corresponding normalized basis by a diagonal similarity, given by the diagonal matrix with entries $\|z^\alpha\|$.

It will sometimes be convenient to note that by a unitary change of variables we may assume that the matrix for C_φ is lower triangular.

Lemma 3. *Let $\varphi : B_N \to B_N$ be analytic with $\varphi(0) = 0$. There is a map $\psi : B_N \to B_N$ with C_ψ unitarily equivalent to C_φ on any weighted Hardy space $H^2_\beta(B_N)$, for which the matrix of C_ψ with respect to the standard basis is lower-triangular.*

Proof. Set $A = \varphi'(0)$. By the Schur theorem, there exists an $N \times N$ unitary matrix U so that UAU^{-1} is upper triangular. Set $\psi = U\varphi U^{-1}$, so that $\psi'(0)$ is the upper-triangular matrix UAU^{-1}. Since the jkth entry of $\psi'(0)$ is $D_k\psi_j(0)$, this says that

$$\psi_j(z) = a_{jj}z_j + \cdots + a_{jN}z_N + \text{ higher-order terms.}$$

The upper-left corner of the matrix of C_ψ is

$$\begin{pmatrix} 1 & 0 \\ 0 & \psi'(0)^t \end{pmatrix}.$$

In general, the description of the coordinate functions of ψ shows that if $\alpha > \beta$, for multi-indices α and β, then the coefficient of z^β in the power series expansion of ψ^α is 0. Thus the matrix of C_ψ is lower triangular. Moreover, the map $z \mapsto Uz$ induces a bounded composition operator C_U which is unitary on $H^2_\beta(B_N)$ (see, e.g., [5, Lemma 8.1]) and $C_\psi = C_{U^{-1}} C_\varphi C_U = C_U^{-1} C_\varphi C_U$. ■

An important tool for constructing solutions to Schroeder's equation will be a result describing the spectra of compact composition operators on general weighted Hardy spaces $H^2_\beta(B_N)$.

Theorem 4 [5]. *Suppose $\varphi : B_N \to B_N$ with $\varphi(0) = 0$. If C_φ is compact on some weighted Hardy space $H^2_\beta(B_N)$, then the spectrum of C_φ consists of $0, 1$ and all possible products of the eigenvalues of $\varphi'(0)$.*

We will see in Section 3 that the hypotheses $\varphi(0) = 0$ and φ not unitary on any slice are sufficient to guarantee that C_φ is compact on certain weighted Hardy spaces which are, in fact, weighted Bergman spaces. Thus, under these assumptions on φ we will have, for each eigenvalue λ_j of $\varphi'(0)$, an eigenfunction ψ_j, analytic in B_N, and satisfying $\psi_j \circ \varphi = \lambda_j \psi_j$. The next result shows how to use these to build a solution to Schroeder's equation.

Theorem 5. *Let $\varphi : B_N \to B_N$ be analytic with $\varphi(0) = 0$. Suppose $A = \varphi'(0)$ is diagonalizable, with eigenvalues $\lambda_1, \ldots, \lambda_N$ and corresponding eigenvectors $w_1, \ldots, w_N$. Suppose C_φ is compact on some weighted Hardy space H^2_β, so that C_φ has eigenfunctions $\psi_1, \ldots, \psi_N$ corresponding to $\lambda_1, \ldots, \lambda_N$. Then*

$$f(z) = \psi_1(z) w_1 + \cdots + \psi_N(z) w_N$$

is an analytic map of B_N into $\mathbb{C}^N$ satisfying $f \circ \varphi = Af$.

Conversely, if f satisfies $f \circ \varphi = Af$ and if $v_1, \ldots, v_N$ are eigenvectors for A^* with eigenvalues $\overline{\lambda_1}, \ldots, \overline{\lambda_N}$, then $\psi_j(z) \equiv \langle f(z), v_j \rangle$ satisfies $\psi_j \circ \varphi = \lambda_j \psi_j(z)$.

Proof. Define $f : B_N \to \mathbb{C}^N$ by $f(z) = \psi_1(z) w_1 + \cdots + \psi_N(z) w_N$, where ψ_j is analytic in B_N with $\psi_j \circ \varphi = \lambda_j \psi_j$ and w_j is in $\mathbb{C}^N$ with $Aw_j = \lambda_j w_j$. Then

$$\begin{aligned} f \circ \varphi &= \psi_1(\varphi(z)) w_1 + \cdots + \psi_N(\varphi(z)) w_N \\ &= \lambda_1 \psi_1(z) w_1 + \cdots + \lambda_N \psi_N(z) w_N \end{aligned}$$

while

$$\begin{aligned} \varphi'(0) f(z) &= \varphi'(0)(\psi_1(z) w_1) + \cdots + \varphi'(0)(\psi_N(z) w_N) \\ &= \psi_1(z) \lambda_1 w_1 + \cdots + \psi_N(z) \lambda_N w_N \\ &= f \circ \varphi. \end{aligned}$$

This gives the first part of the theorem.

For the converse, suppose $f : B_N \to \mathbb{C}^N$ is analytic with $f \circ \varphi = \varphi'(0) f$ and let $v_1, \cdots, v_N$ be eigenvectors of $A^* = \varphi'(0)^*$ corresponding to the eigenvalues $\overline{\lambda_1}, \ldots, \overline{\lambda_N}$. Define $\psi_j(z) = \langle f(z), v_j \rangle$. Then ψ_j is analytic on B_N and

$$\begin{aligned}\psi_j(\varphi(z)) &= \langle f(\varphi(z)), v_j \rangle = \langle \varphi'(0) f(z), v_j \rangle \\ &= \langle f(z), A^* v_j \rangle = \langle f(z), \overline{\lambda_j} v_j \rangle = \lambda_j \psi_j. \end{aligned} \qquad \blacksquare$$

The hypothesis that $A = \varphi'(0)$ is diagonalizable in Theorem 5 is not necessary. Without this assumption an analytic f solving Schroeder's equation can still be constructed from the (nonindependent) eigenvectors of A and the eigenfunctions for C_φ as before. However since our ultimate interest is in solutions to Schroeder's equation which are locally univalent near 0, this leads us naturally to the additional requirement that $\varphi'(0)$ be diagonalizable.

3. Ccompactness of Composition Operators

We show in this section that whenever $\varphi : B_N \to B_N$ is analytic with $\varphi(0) = 0$ and φ not unitary on any slice, then C_φ is Hilbert–Schmidt, and thus compact, on the weighted Bergman space $A^2_G(B_N)$ defined from a weight function which decays to 0 sufficiently rapidly as $r \to 1$, e.g., $G(r) = \exp(-q/(1-r))$, $q > 0$. For $G(r)$ a positive, continuous, nonincreasing function on $[0, 1)$, the Bergman space $A^2_G(B_N)$ is the space of analytic functions f on B_N for which

$$\|f\|^2_G = \int_{B_N} |f|^2 G(|z|)\, d\nu_N(z) < \infty,$$

where $d\nu_N$ is the volume measure on B_N, normalized so that $\nu_N(B_N) = 1$. This is a weighted Hardy space $H^2_\beta(B_N)$, where

$$\beta(s) = \beta(|\alpha|) = \frac{\|z^\alpha\|_G}{\|z^\alpha\|_2}$$

for any multi-index α. It will be convenient to say that $G(r)$ is a *fast regular* weight if

$$\lim_{r \to 1} \frac{G(r)}{(1-r)^a} = 0$$

for every $a > 0$, and moreover this ratio is decreasing for r near 1 for all $a > 0$.

Though for our purposes it will be enough to find a specific positive, continuous, and nonincreasing weight function G so that C_φ is compact on the Bergman space $A^2_G(B_N)$, the next two lemmas are easily done in the context of a general weighted Bergman space $A^2_G(B_N)$. In the case $N = 1$, both of these lemmas appear, with similar proofs, in [5].

Lemma 6. *C_φ is Hilbert–Schmidt on $A^2_G(B_N)$ if and only if*

$$\int_{B_N} \|K_{\varphi(z)}\|^2 G(|z|)\, d\nu_N(z) < \infty,$$

where K_w denotes the kernel function for evaluation at w in $A^2_G(B_N)$.

Proof. Consider the orthonormal basis for $A^2_G(B_N)$:

$$\frac{z^\alpha}{\|z^\alpha\|_G} = \frac{z^\alpha}{\beta(|\alpha|)\|z^\alpha\|_2},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is an multi-index and $\|z^\alpha\|_2$ denotes the $L^2(\sigma_N)$ norm of z^α, given by

$$\|z^\alpha\|_2^2 = \frac{(N-1)!\alpha!}{(N-1+|\alpha|)!},$$

where $\alpha! = \alpha_1! \cdots \alpha_N!$. Thus C_φ is Hilbert–Schmidt if and only if

$$\sum_\alpha \|C_\varphi(z^\alpha)\|_G^2 \frac{(N-1+|\alpha|)!}{(\beta(|\alpha|))^2(N-1)!\alpha!} < \infty.$$

Now

$$\text{(3)}\quad \sum_\alpha \|C_\varphi(z^\alpha)\|_G^2 \frac{(N-1+|\alpha|)!}{(\beta(|\alpha|))^2(N-1)!\alpha!} = \sum_{s=0}^\infty \frac{(N-1+s)!}{\beta(s)^2(N-1)!s!} \sum_{|\alpha|=s} \frac{s!}{\alpha!}\|\varphi^\alpha\|_G^2,$$

where $\varphi^\alpha(z)$ denotes $\varphi_1(z)^{\alpha_1} \cdots \varphi_N(z)^{\alpha_N}$.

The inner sum has value

$$\begin{aligned}\sum_{|\alpha|=s} \frac{s!}{\alpha!} \int_{B_N} |\varphi^\alpha|^2 G(|z|)\, d\nu_N(z) &= \int_{B_N} \sum_{|\alpha|=s} \frac{s!}{\alpha!} |\varphi^\alpha|^2 G(|z|)\, d\nu_N(z) \\ &= \int_{B_N} \langle \varphi(z), \varphi(z)\rangle^s G(|z|)\, d\nu_N(z) \\ &= \int_{B_N} |\varphi(z)|^{2s} G(|z|)\, d\nu_N(z)\end{aligned}$$

by the multinomial theorem.

Thus (3) is equal to

$$\begin{aligned}
&\sum_{s=0}^{\infty} \frac{(N-1+s)!}{(N-1)!\beta(s)^2 s!} \int_{B_N} |\varphi(z)|^{2s} G(|z|)\, d\nu_N(z) \\
&\quad = \int_{B_N} \left(\sum_{s=0}^{\infty} \frac{(N-1+s)!|\varphi(z)|^{2s}}{(N-1)!\beta(s)^2 s!} \right) G(|z|)\, d\nu_N(z) \\
&\quad = \int_{B_N} \|K_{\varphi(z)}\|^2 G(|z|)\, d\nu_N(z)
\end{aligned}$$

since

$$\|K_w\|^2 = \sum_{s=0}^{\infty} \frac{(N-1+s)!}{(N-1)!s!} \frac{|w|^{2s}}{\beta(s)^2}$$

(see, for example, [4]). ∎

Lemma 7. *Let $G(r)$ be a positive, continuous, and nonincreasing function on $[0,1)$. Fix b in $(0,1)$. Then for any z in B_N,*

$$\|K_z\|_G^2 \le (1-b)^{-2N}(1-|z|)^{-2N}[G(1-b(1-|z|))]^{-1}.$$

Proof. Fix z in B_N and let $\delta = (1-b)(1-|z|)$. Consider the ball $B_\delta(z)$ centered at z with radius δ. Since $\nu_N(B_\delta) = \delta^{2N}$, we have, by the sub-mean value property,

$$\begin{aligned}
|K_z(z)| &\le \frac{1}{\delta^{2N}} \int_{B_\delta(z)} |K_z(w)|\, d\nu_N(w) \\
&= \frac{1}{\delta^{2N}} \frac{1}{\sqrt{G(|z|+\delta)}} \int_{B_\delta(z)} |K_z(w)| \sqrt{G(|z|+\delta)}\, d\nu_N(w) \\
&\le \frac{1}{\delta^{2N}} \frac{1}{\sqrt{G(|z|+\delta)}} \int_{B_\delta(z)} |K_z(w)| \sqrt{G(|w|)}\, d\nu_N(w)
\end{aligned}$$

since G is nonincreasing. Now $d\nu_N(w)/\delta^{2N}$ is a probability measure on $B_\delta(z)$, so

$$\begin{aligned}
\left(\int_{B_\delta(z)} |K_z(w)| \sqrt{G(|w|)}\, \frac{d\nu_N(w)}{\delta^{2N}} \right)^2 &\le \int_{B_\delta(z)} |K_z(w)|^2 G(|w|)\, \frac{d\nu_N(w)}{\delta^{2N}} \\
&\le \frac{1}{\delta^{2N}} \|K_z\|_G^2,
\end{aligned}$$

which gives

$$|K_z(z)|^2 = \|K_z\|_G^4 \le \frac{1}{G(|z|+\delta)} \frac{1}{\delta^{2N}} \|K_z\|_G^2$$

or

$$\|K_z\|_G^2 \le (1-b)^{-2N}(1-|z|)^{-2N}[G(1-b(1-|z|))]^{-1}$$

by the definition of δ. ■

Theorem 8. *If $\varphi : B_N \to B_N$ is analytic with $\varphi(0) = 0$ and φ is not unitary on any slice, then C_φ is Hilbert–Schmidt on any Bergman space $A_G^2(B_N)$ for which G is continuous, positive, and nonincreasing on $[0,1)$ satisfying*

$$\frac{G(r)}{(1-r)^{2N}G(1-\rho(1-r))}$$

is bounded near 1 for any $\rho > 1$. In particular, C_φ is Hilbert–Schmidt on the Bergman space $A_G^2(B_N)$ for $G(r) = \exp(-q/(1-r))$, $q > 0$.

Proof. Since φ is not unitary on any slice, there exists $r_0 < 1$ and $R > 1$ so that

$$\frac{1-|\varphi(z)|}{1-|z|} \ge R$$

if $r_0 \le |z| < 1$ [4]. We will show that $\|K_{\varphi(z)}\|^2 G(|z|)$ is bounded in $r_0 \le |z| < 1$, and hence in B_N; from this and Lemma 6 the result follows.

Choose $b < 1$ close enough to 1 so that $bR > 1$. For $|z| \ge r_0$ we have $1 - |\varphi(z)| \ge R - R|z|$ and thus

$$\|K_{\varphi(z)}\|_G^2 = \|K_{|\varphi(z)|}\|_G^2 \le \|K_{1-R(1-|z|)}\|_G^2$$

since $\|K_w\|$ increases with $|w|$. By Lemma 7, this is bounded above by

$$\frac{(1-b)^{-2N}(1-(1-R(1-|z|)))^{-2N}}{G(1-b(1-(1-R(1-|z|))))},$$

which is equal to

$$\frac{(1-b)^{-2N}R^{-2N}(1-|z|)^{-2N}}{G(1-bR(1-|z|))}$$

so that

$$\|K_{\varphi(z)}\|^2 G(|z|) \le \frac{G(|z|)}{(1-b)^{2N}R^{2N}(1-|z|)^{2N}G(1-bR(1-|z|))}.$$

The hypothesis on $G(r)/((1-r)^{2N}G(1-\rho(1-r)))$ guarantees that this is bounded for $r_0 \le |z| < 1$. Using $G(|z|) = \exp(-q/(1-|z|))$, $q > 0$, a computation shows that this hypothesis on $G(r)$ is satisfied. ■

4. Examples

We consider in this section some examples which show some of the complications that can arise in several variables, but are never present in one variable.

Example 1. Let $\varphi(z_1, z_2) = (\frac{1}{2}z_1, \frac{1}{4}z_2 + \frac{1}{2}z_1^2)$. Clearly, φ maps B_2 into B_2 and is univalent. In the notation of Theorem 5, we set

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}, \ \lambda_1 = \frac{1}{2}, \ \lambda_2 = \frac{1}{4}, \ w_1 = (1,0)^t, \ w_2 = (0,1)^t.$$

We seek eigenfunctions $\psi_1(z)$ and $\psi_2(z)$ for C_φ corresponding to λ_1 and λ_2. Setting $\psi_1(z) = \sum_\gamma c_\gamma z^\gamma$, we seek to solve

$$\psi_1 \circ \varphi = \frac{1}{2}\psi_1 \tag{4}$$

in a neighborhood of 0. Note that since $\varphi(0) = 0$, we must have $\psi_1(0) = 0$. Order the multi-indices γ as previously described. By comparing the power series expansions of the left- and right-hand sides of Equation (4) we will show that the only solutions to this equation are $\psi_1(z) = cz_1$. To this end, suppose that ψ_1 is a different solution and let α be the least multi-index greater than $(1,0)$ for which c_α is nonzero in the power series for ψ_1. Now

$$\psi_1 \circ \varphi = \sum_\gamma c_\gamma \varphi^\gamma = \sum_{\gamma=(\gamma_1,\gamma_2)} c_\gamma \Big(\frac{1}{2}z_1\Big)^{\gamma_1} \Big(\frac{1}{4}z_2 + \frac{1}{2}z_1^2\Big)^{\gamma_2}.$$

We examine the multi-indices γ for which the expansion of $c_\gamma(\frac{1}{2}z_1)^{\gamma_1}(\frac{1}{4}z_2 + \frac{1}{2}z_1^2)^{\gamma_2}$ will contribute a nonzero term of the form kz^α. When $|\gamma| > |\alpha|$, a comparison of the total order in the terms of $c_\gamma(\frac{1}{2}z_1)^{\gamma_1}(\frac{1}{4}z_2 + \frac{1}{2}z_1^2)^{\gamma_2}$ shows that none are of the desired form. When $|\gamma| < |\alpha|$, the hypothesis that $c_\gamma = 0$ for $(1,0) < \gamma < \alpha$ yields no contribution of the form kz^α, $k \neq 0$.

When $|\gamma| = |\alpha|$ we note that

$$c_\gamma\Big(\frac{1}{2}z_1\Big)^{\gamma_1}\Big(\frac{1}{4}z_2 + \frac{1}{2}z_1^2\Big)^{\gamma_2} = c_\gamma \frac{1}{2}^{\gamma_1} z_1^{\gamma_1} \sum_{\beta_1+\beta_2=\gamma_2} c(\beta_1,\beta_2)\Big(\frac{1}{4}z_2\Big)^{\beta_1}\Big(\frac{1}{2}z_1^2\Big)^{\beta_2}.$$

In order that any term in this is of the form kz^α we must have

- $\gamma_1 + 2\beta_2 = \alpha_1$,
- $\beta_1 = \alpha_2$,
- $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2$,

- $\beta_1 + \beta_2 = \gamma_2$,

which together imply $\alpha_1 = \gamma_1$ and $\alpha_2 = \gamma_2$. In other words, the only term kz^α arising from $c_\gamma \varphi^\gamma$ for $|\gamma| = |\alpha|$ occurs when $\gamma = \alpha$ and will be $c_\alpha \frac{1}{2}^{\alpha_1} \frac{1}{4}^{\alpha_2} z^\alpha$. Then Equation (4) implies that

$$\left(\frac{1}{2}\right)^{\alpha_1} \left(\frac{1}{4}\right)^{\alpha_2} = \frac{1}{2}$$

since our assumption is that $c_\alpha \neq 0$. This is impossible, since $\alpha > (1, 0)$. This contradiction shows that the only solutions to Equation (4) are $\psi_1(z) = cz_1$.

A similar analysis is possible in finding solutions to

$$\psi_2 \circ \varphi = \frac{1}{4} \psi_2. \tag{5}$$

Set $\psi_2 = \sum_\gamma b_\gamma z^\gamma$ and note first that $\psi_2(0) = 0$. Comparing the coefficients of z_1 and z_1^2 on the left- and right-hand sides of Equation (5) yields $b_\gamma = 0$ for $\gamma = (1, 0)$ and $(0, 1)$. Assume, for a contradiction, that there is a solution ψ_2 with a nonzero coefficient b_γ for some multi-index $\gamma > (2, 0)$; let α denote the least such multi-index.

Again we determine for which γ, $b_\gamma z^\gamma$ will contribute a nonzero term of the form kz^α. As before, a comparison of total orders shows that there is no such contribution when $|\gamma| > |\alpha|$. If $|\gamma| < |\alpha|$, recall that $b_\gamma = 0$ for $\gamma = (1, 0), (0, 1)$ or $(2, 0) < \gamma < \alpha$. Thus we can only have $|\gamma| < |\alpha|$ and $b_\gamma \neq 0$ if $\gamma = (2, 0)$. In this case, $b_\gamma \varphi^\gamma = b_\gamma (\frac{1}{2} z_1)^2$ and by our choice of α this is not of the form kz^α.

Finally, if $|\gamma| = |\alpha|$, consider

$$b_\gamma \left(\frac{1}{2} z_1\right)^{\gamma_1} \left(\frac{1}{4} z_2 + \frac{1}{2} z_1^2\right)^{\gamma_2},$$

where

$$\left(\frac{1}{4} z_2 + \frac{1}{2} z_1^2\right)^{\gamma_2} = \sum_{\beta_1 + \beta_2 = \gamma_2} c(\beta_1, \beta_2) \left(\frac{1}{4} z_2\right)^{\beta_1} \left(\frac{1}{2} z_1^2\right)^{\beta_2}.$$

As before, the expansion of this contributes a term of the form kz^α only if

- $\gamma_1 + 2\beta_2 = \alpha_1$,
- $\beta_1 = \alpha_2$,
- $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2$,
- $\beta_1 + \beta_2 = \gamma_2$,

which together imply $\alpha_1 = \gamma_1$ and $\alpha_2 = \gamma_2$. Thus the only term in $\sum b_\gamma \varphi^\gamma$ of the form kz^α comes from $\gamma = \alpha$ and is

$$b_\alpha \left(\frac{1}{2}\right)^{\alpha_1} \left(\frac{1}{4}\right)^{\alpha_2} z^\alpha.$$

For Equation (5) to hold we must have

$$\frac{1}{2}^{\alpha_1}\frac{1}{4}^{\alpha_2} = \frac{1}{4}.$$

But $\alpha > (2,0)$ so either $\alpha_1 > 2$, or $\alpha_2 \geq 2$, or α_1 and α_2 are both positive. Thus we have a contradiction.

Thus the only solutions to $\psi_2 \circ \varphi = \frac{1}{4}\psi_2$ are $\psi_2(z) = bz_1^2$.

By Theorem 5, we have solutions to Schroeder's equation of the form $f(z_1, z_2) = (az_1, bz_1^2)$. The second part of Theorem 5 guarantees that this is a complete list of all solutions; in particular, there are no solutions f of Schroeder's equation with $f'(0) = I$.

Example 2. Let $\varphi(z_1, z_2) = (c_1 z_1, c_1^3 z_2 + c_2 z_1^2)$ for nonzero values of c_1, c_2 that are small enough that φ maps B_2 into itself. Note that φ is univalent in B_2 and $\varphi'(0) = \mathrm{diag}(c_1, c_1^3)$. Furthermore, $\psi_1(z) = z_1$ is an eigenfunction of C_φ with eigenvalue c_1 and both $\psi_2(z) = z_1^3$ and $\psi_3(z) = z_2 + c_2 z_1^2/(c_1^3 - c_1^2)$ are eigenfunctions of C_φ corresponding to c_1^3. Thus Schroeder's equation has both a nonunivalent solution

$$f(z_1, z_2) = (z_1, z_1^3)$$

and a univalent solution

$$f(z_1, z_2) = \left(z_1, z_2 + \frac{c_2 z_1^2}{c_1^3 - c_1^2}\right).$$

One can show by direct computation that this example also provides an instance in which the "normalized iterates" $\varphi'(0)^{-n}\varphi_n$ fail to converge in any neighborhood of 0; also see [16, p. 74], where a similar example with this observation is given.

Example 3: Linear Fractional Maps. Let φ be a linear fractional map of B_N into itself. This means φ can be written as

$$\varphi(z) = \frac{Az}{\langle z, C\rangle + 1},$$

where A is an $N \times N$ matrix, C and z are (column) vectors in $\mathbb{C}^N$, and $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product. Note that $\varphi'(0) = A$. The requirement that $\varphi(B_N) \subset B_N$ implies that $|C| < 1$. The case $C = 0$ is uninteresting, so we assume $C \neq 0$. In [6], we showed that when A has no eigenvalue of modulus 1, there is a $\mathbb{C}^N$-valued f defined and univalent on B_N with $f \circ \varphi = \varphi'(0)f$. Specifically,

$$f(z) = \frac{z}{\langle z, P\rangle + 1},$$

where $P = (I - A^*)^{-1}C$.

Note that for linear fractional maps φ, we always get a univalent solution to Schroeder's equation, regardless of the invertibility of $\varphi'(0)$ or the univalence of φ.

5. Codifying the Examples

In this section we give some general results which explain the examples in the previous sections. We begin with several lemmas.

Lemma 9. *Let T be an operator with block matrix*

$$T = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

with respect to an orthogonal decomposition of a Hilbert space. If λ is an eigenvalue of T and $|\lambda| > \|Z\|$, then λ is an eigenvalue of X, and the multiplicity of λ as an eigenvalue of X is the same as its multiplicity as an eigenvalue of T.

Proof. If

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix},$$

we must have $Xu + Yv = \lambda u$ and $Zv = \lambda v$. Now if $|\lambda| > \|Z\|$, then $v = 0$ and thus $Xu = \lambda u$ as desired. Moreover, if

$$w_j = \begin{pmatrix} u_j \\ v_j \end{pmatrix}$$

are linearly independent eigenvectors for T corresponding to λ, then by the above calculation $v_j = 0$ for each j and the u_j's are linearly independent eigenvectors for X. Conversely, given linearly independent eigenvectors u_j for X, we have

$$\begin{pmatrix} u_j \\ 0 \end{pmatrix}$$

are linearly independent eigenvectors for T. This gives the statement on multiplicities. ∎

Lemma 10. *Suppose A is an $n \times n$ lower-triangular matrix and suppose $i = j_1, j_2, \cdots, j_k$ are the indices such that $a_{ii} = 0$. If the rank of A is $n - k$, then there are vectors $v_1, v_2, \cdots, v_k$ that are orthogonal to the columns of A such that $v_i(j_i) = 1$ and $v_i(\ell) = 0$ for $\ell > j_i$.*

Proof. We will construct, inductively, a basis w_1, w_2, $\cdots$, w_n for $\mathbb{C}^n$ consisting of columns of A and vectors orthogonal to the columns of A so that the vectors $v_i = w_{j_i}$ satisfy the conclusion of the theorem.

We assume, without loss of generality, that $1 \le j_1 < j_2 < \cdots < j_k \le n$. Let A_i denote the ith column of A. Since the rank of A is $n-k$, the columns $\{A_\ell : \ell \ne j_q, 1 \le q \le k\}$ are a basis for the range of A. If v is a vector in $\mathbb{C}^n$, let $[v]_q$ be the vector in $\mathbb{C}^q$ such that $[v]_q(i) = v(i)$ for $1 \le i \le q$.

If $j_1 = 1$, that is, if $a_{11} = 0$, let $w_1 = (1, 0, \cdots, 0)^t$, but if $j_1 > 1$, that is, if $a_{11} \ne 0$, let $w_1 = A_1$. Then $[w_1]_1$ is a basis for $\mathbb{C}^1$.

Now suppose $w_1, w_2, \cdots, w_{p-1}$ have been chosen so that $[w_1]_{p-1}, [w_2]_{p-1}, \cdots,$ $[w_{p-1}]_{p-1}$ form a basis for $\mathbb{C}^{p-1}$, and for $i \le p-1$, $w_i = A_i$ if $a_{ii} \ne 0$, but if $a_{ii} = 0$, then w_i is orthogonal to the columns $\{A_\ell : \ell \ne j_q, 1 \le q \le k\}$, hence all the columns of A, $w_i(i) = 1$, and $w_i(\ell) = 0$ for $\ell > i$.

If $p = n+1$, then the vectors $v_i = w_{j_i}$ for $i \le k$ satisfy the conclusion of the theorem. On the other hand, suppose $p \le n$. If $a_{pp} \ne 0$, let $w_p = A_p$. By the induction hypothesis, $[w_1]_{p-1}$, $\cdots$, $[w_{p-1}]_{p-1}$ is a basis for $\mathbb{C}^{p-1}$. Since A is lower triangular, $[A_p]_{p-1} = 0$ and it follows that $[w_1]_p$, $\cdots$, $[w_{p-1}]_p$, $[w_p]_p$ are linearly independent. That is, they form a basis for $\mathbb{C}^p$ and the conclusion holds for p.

Now suppose $a_{pp} = 0$ so that $[A_p]_p = 0$. By the induction hypothesis, $[w_1]_{p-1}, \cdots, [w_{p-1}]_{p-1}$ is a basis for $\mathbb{C}^{p-1}$. This means that $[w_1]_p$, $\cdots$, $[w_{p-1}]_p$ span a $(p-1)$-dimensional subspace of $\mathbb{C}^p$. Now let u be a nonzero vector in $\mathbb{C}^p$ that is orthogonal to each of $[w_1]_p$, $\cdots$, $[w_{p-1}]_p$. In particular, $[w_1]_p$, $\cdots$, $[w_{p-1}]_p$, u form a basis for $\mathbb{C}^p$. If $u(p) = 0$, then the vector $[u]_{p-1}$ is orthogonal to each of $[w_1]_{p-1}$, $\cdots$, $[w_{p-1}]_{p-1}$. This would mean that $[u]_{p-1} = 0$ and that $u = 0$. This contradiction implies $u(p) \ne 0$ and we may suppose that $u(p) = 1$. Then, we take w_p to be the vector in $\mathbb{C}^n$ such that $[w_p]_p = u$ and $w_p(i) = 0$ for $i > p$. By construction, w_p is orthogonal to $\{A_\ell : \ell < p$ and $\ell \ne j_q$, $1 \le q \le k\}$. Moreover, because the first p components of A_i are 0 for $i \ge p$, w_p is also orthogonal to $A_p, \cdots, A_n$. Thus, the conclusion holds for p in this case as well.

By induction, then, the conclusion of the lemma holds. ∎

Although we do not need the extra information, it is clear from the construction that the vectors v_i are orthogonal to each other. We will use the lemma to prove the existence of a nice basis for the nullspace of certain upper-triangular matrices.

Corollary 11. *Suppose B is an $n \times n$ upper-triangular matrix and suppose $i = j_1, j_2, \cdots, j_k$ are the indices such that $b_{ii} = 0$. If the nullity of B is k, then there is a basis, v_1, v_2, $\cdots$, v_k, for the nullspace of B such that $v_i(j_i) = 1$*

and $v_i(\ell) = 0$ *for* $\ell > j_i$.

Proof. Apply Lemma 10 to the matrix B^* which is lower-triangular. The lemma asserts the existence of the vectors $v_1, v_2, \cdots, v_k$, such that $v_i(j_i) = 1$ and $v_i(\ell) = 0$ for $\ell > j_i$ and such that $v_1, v_2, \cdots, v_k$ are orthogonal to the range of B^*. Since the orthogonal complement of the range of B^* is the nullspace of B, these vectors are in the nullspace of B. Since the vectors are clearly linearly independent and the nullity of B is k, they form a basis for the nullspace of B. ∎

The proof of Lemma 10, which deals with lower-triangular matrices, starts at the upper-left corner of the matrix and works down to the lower-right corner. It is clear that an analogous result is true for upper-triangular matrices with an analogous proof starting at the lower-right corner of the matrix and working up to the upper-left corner. Using this result and duality as in the above corollary, we get the following corollary on constructing bases for the nullspaces of certain lower-triangular matrices.

Corollary 12. *Suppose* B *is an* $n \times n$ *lower triangular matrix and suppose* $i = j_1, j_2, \cdots, j_k$ *are the indices such that* $b_{ii} = 0$. *If the nullity of* B *is* k, *then there is a basis,* $v_1, v_2, \cdots, v_k$, *for the nullspace of* B *such that* $v_i(j_i) = 1$ *and* $v_i(\ell) = 0$ *for* $\ell < j_i$.

In the proofs of Theorems 13 and 14 we will let P_k denote the orthogonal projection of $H^2_\beta(B_N)$ onto the subspace span$\{z^\alpha : |\alpha| \le k\}$ for k any positive integer. Let $\mathcal{H}_k$ denote the abstract vector space spanned by the set $\{z^\alpha : |\alpha| \le k\}$. If we equip $\mathcal{H}_k$ with the inner product arising from $H^2_\beta(B_N)$, then $\mathcal{H}_k$ is isometrically isomorphic to $P_k H^2_\beta(B_N)$ and we denote by P the map from $P_k H^2_\beta(B_N)$ onto $\mathcal{H}_k$ that identifies polynomials; P is unitary if $\mathcal{H}_k$ is regarded as a Hilbert space as above, and an isomorphism of vector spaces if it is regarded as an abstract vector space.

Now suppose $\varphi : B_N \mapsto B_N$, $\varphi(0) = 0$, and $A = \varphi'(0)$ is upper-triangular. By Lemma 3, when A is upper-triangular, the matrix for C_φ on $H^2_\beta(B_N)$ is lower-triangular with respect to the standard orthogonal basis $\{z^\alpha\}$ for $H^2_\beta(B_N)$ for any β. Since C^*_φ has span$\{z^\alpha : |\alpha| \le k\}$ as an invariant subspace, we have

$$P_k C^*_\varphi P_k = C^*_\varphi P_k.$$

Taking adjoints gives

$$P_k C_\varphi P_k = P_k C_\varphi.$$

From this we get the useful observation that if f is in the nullspace of $C_\varphi - \lambda I$, then $P_k f$ is in the nullspace of $P_k C_\varphi P_k - \lambda I$, and if X is the matrix in the upper-left-hand corner of the matrix for C_φ with dimensions $K \times K$ where $K = \dim \mathcal{H}_k$, then PP_kf is in the nullspace of $X - \lambda I$. Similar results apply if we fix a multi-index τ and let P_τ denote the orthogonal projection of $H^2_\beta(B_N)$ onto span$\{z^\alpha : \alpha \le \tau\}$, and choose the size of X to correspond to the position of τ in the standard ordering.

For the next result, recall that there is no loss of generality in assuming that $A = \varphi'(0)$ is upper-triangular, since Schroeder's equation for φ has a locally univalent solution if and only if Schroeder's equation for $U\varphi U^{-1}$ for U unitary has a locally univalent solution.

Theorem 13. *Suppose φ is an analytic map of B_N into B_N with $\varphi(0) = 0$ and $A = \varphi'(0)$ is upper-triangular and diagonalizable, with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that $0 < |\lambda_j| < 1$. Assume further that φ is not unitary on any slice. Let X be any size square upper-left corner of the matrix for C_φ with respect to the standard (nonnormalized) basis for any weighted Hardy space $H^2_\beta(B_N)$, ordered in the usual way.*

If Schroeder's equation has a solution f on B_N, where $f \circ \varphi = Af$ and $f'(0) = I$, then X is diagonalizable.

Proof. Suppose X is of size $Q \times Q$. Find $m \ge 1$ so that $M \equiv \dim \mathcal{H}_m \ge Q$. If $f \circ \varphi = Af$ where $f'(0) = I$, write $f = (f_1, f_2, \cdots, f_N)$. We may find a fast regular weight $\widehat{G}(r)$ with the property that $\widehat{G}(|z|)|f_j(z)| \le 1$ for all z in B_N and $j = 1, 2, \cdots, N$. Set $G(r) = (\widehat{G}(r) \exp(-1/(1-r)))^{2m}$. A calculation shows that $G(r)$ satisfies the hypothesis of Theorem 8 (Exercise 5.2.2 of [5] is relevant here). Thus C_φ is Hilbert-Schmidt, and hence compact, on $A^2_G(B_N)$.

Using Corollary 11, we may choose a basis of eigenvectors v_j, with eigenvalues $\overline{\lambda}_j$, for the diagonalizable upper-triangular matrix A^* with $v_j(j) = 1$ and $v_j(k) = 0$ for $k > j$. By Theorem 5, $\psi_j(z) \equiv \langle f(z), v_j\rangle$ is an eigenfunction for C_φ with eigenvalue λ_j. Since by the choice of $\widehat{G}$, each f_k is in $A^2_G(B_N)$, and the ψ_j's are linear combinations of the f_k's, we have ψ_j in $A^2_G(B_N)$. Moreover, any product of at most m functions from the set $\{\psi_1, \psi_2, \cdots, \psi_N\}$ will also be in $A^2_G(B_N)$.

By assumption, $f'(0) = I$, so the homogeneous expansion of f_j is

$$f_j(z) = z_j + \text{higher-order terms},$$

where "higher-order terms" means terms of order greater than or equal to 2. Using this and the special form of the eigenvectors v_j we see that

$$\psi_j(z) = a^j_1 z_1 + a^j_2 z_2 + \cdots + a^j_{j-1} z_{j-1} + z_j + \text{terms of order at least 2.}$$

Notice that the ψ_j's are linearly independent, since if

$$\sum_{j=1}^{N} c_j \psi_j = 0, \tag{6}$$

we must have $c_N = 0$, since only ψ_N contains a nonzero z_N term. From this it follows that $c_{N-1} = 0$ since only ψ_{N-1} and ψ_N could contain nonzero z_{N-1} terms. Continuing in this matter we see that each coefficient c_j is 0.

Now let τ be the Qth multi-index for $\mathbb{C}^N$, with respect to the usual ordering. Consider the collection of all products $\psi^\gamma = \psi_1^{\gamma_1} \cdots \psi_N^{\gamma_n}$, where $\gamma \leq \tau$. Notice that each of these is a product of not more than m of the ψ_j's, counting repetitions, so that each of the products ψ^γ, $\gamma \leq \tau$, is in the collection

$$\psi_1, \psi_2, \cdots, \psi_N, \psi_1^2, \psi_1\psi_2, \cdots, \psi_N^2, \psi_1^3, \cdots, \psi_N^m \tag{7}$$

of at most m-fold products of the ψ_j's. These functions lie in $A^2_G(B_N)$, C_φ is compact on $A^2_G(B_N)$, and any such product

$$\Pi_1^N \psi_j^{k_j} \text{ with } k_j \geq 0 \text{ and } \sum k_j \leq m$$

is an eigenfunction for C_φ with eigenvalue

$$\Pi_1^N \lambda_j^{k_j}.$$

We claim that this collection (7) of at most m-fold products is linearly independent. Suppose

$$\sum c(\alpha)\psi^\alpha = 0, \tag{8}$$

where $\psi^\alpha = \psi_1^{\alpha_1}\psi_2^{\alpha_2}\cdots\psi_n^{\alpha_N}$, $\sum \alpha_j \leq m$, $\alpha_j \geq 0$. If $|\alpha| \geq 2$ then from the form of the ψ_j's we see that ψ^α contains no first-order terms, so by comparing the coefficients of the first-order terms in Equation (8) we see that

$$\sum_{|\alpha|=1} c(\alpha)\psi^\alpha = 0.$$

Since $\psi_1, \psi_2, \cdots, \psi_N$ are linearly independent, $c(\alpha) = 0$ for all $|\alpha| = 1$. Since this implies that only the terms with $|\alpha| = 2$ can contribute any second-order terms, we must have

$$\sum_{|\alpha|=2} c(\alpha)\psi^\alpha = 0.$$

Note that among all multi-indices of total order 2, only for $\alpha = (0, 0, \cdots, 0, 2)$ does ψ^α contain a nonzero z_N^2 term, so the corresponding coefficient $c(\alpha)$ is

0. By considering next the $z_{N-1}z_N$ terms we see that the coefficient $c(\alpha)$ for $\alpha = (0, \cdots, 0, 1, 1)$ must also be 0. Continuing in this manner we conclude that $c(\alpha) = 0$ for all α of total order 2, and hence Equation (8) becomes

$$\sum_{|\alpha|\geq 3} c(\alpha)\psi^\alpha = 0.$$

Proceeding in the same manner we see that all coefficients $c(\alpha)$ are 0 as desired.

Suppose λ appears exactly ℓ times on the diagonal of X. Then λ can be written in ℓ ways as a product $\Pi_1^N \lambda_j^{k_j}$, where $k_j \geq 0$ and the multi-index $(k_1, k_2, \ldots, k_N) \leq \tau$. We have produced in the above argument ℓ linearly independent eigenfunctions $\Pi_1^N \psi_j^{k_j}$ for C_φ with eigenvalue λ. Consider the projections $PP_\tau(\Pi_1^N \psi_j^{k_j})$, where P_τ denotes the projection of $H^2_\beta(B_N)$ onto $\text{span}\{z^\gamma : \gamma \leq \tau\}$ and P is the map from $P_\tau H^2_\beta(B_N)$ to this span (equipped with the inner product from $H^2_\beta(B_N)$) which identifies polynomials. The projections $PP_\tau(\Pi_1^N \psi_j^{k_j})$ are still linearly independent polynomials of degree Q, since the above independence argument only makes use of the terms of the functions $\Pi_1^N \psi_j^{k_j}$ corresponding to multi-indices $\leq \tau$, and these terms are still present in $PP_\tau(\Pi_1^N \psi_j^{k_j})$. Thus when λ appears ℓ times on the diagonal of X we have exhibited ℓ linearly independent eigenvectors in $\mathbb{C}^Q$ for X corresponding to the eigenvalue λ. This shows that X is diagonalizable. ■

Compare this result with Example 1 in the previous section, where $\varphi(z_1, z_2) = (\frac{1}{2}z_1, \frac{1}{4}z_2 + \frac{1}{2}z_1^2)$. The eigenvalues there are $\lambda_1 = 1/2$ and $\lambda_2 = 1/4$. If X is the upper-left 4×4 corner of the matrix for C_φ, X has diagonal entries 1, 1/2, 1/4, 1/4 and one nonzero off-diagonal entry; namely a 1/2 in the (4, 3) position. This matrix is not diagonalizable, and by Theorem 13 there cannot exist an analytic $f : B_2 \to C^2$ with $f \circ \varphi = \varphi'(0)f$ and $f'(0) = I$. This agrees with our previous calculations in Example 1.

The hypothesis in Theorem 13 that $A = \varphi'(0)$ is diagonalizable cannot be omitted. For example, consider the linear fractional map

$$\varphi(z) = \frac{Az}{\langle z, C\rangle + 1},$$

where

$$A = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix},$$

where $0 < |\lambda| < 1$, $a \neq 0$, and λ and $|C| \neq 0$ are chosen sufficiently small so that φ maps B_2 into itself. When $a \neq 0$, A is not diagonalizable. We know that there is a univalent solution to Schroeder's equation of the form

$$f(z) = \frac{z}{\langle z, P\rangle + 1},$$

where $P = (I - A^*)^{-1}C$. However the upper-left 3×3 corner of the matrix for C_φ is clearly not diagonalizable.

The main result of this section is a converse to Theorem 13 with a particular choice for X.

Theorem 14. *Suppose that φ is an analytic map of B_N into B_N with $\varphi(0) = 0$ and $A = \varphi'(0)$ an upper-triangular diagonalizable matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that $0 < |\lambda_j| < 1$. Assume further that φ is not unitary on any slice. Suppose that $\lambda_j = \lambda_1^{k_1} \cdots \lambda_N^{k_N}$ is the longest expression (maximal $\sum k_i$) for one eigenvalue of A as a product of any number of the eigenvalues of A. Set $m = k_1 + \cdots + k_N$ and $M =$ the number of multi-indices for $\mathbb{C}^N$ of total order less than or equal to m, equivalently, the dimension of $\mathcal{H}_m$. Let $\mathcal{M}$ be the upper-left $M \times M$ corner of the matrix for C_φ with respect to the standard (non-normalized) basis for any weighted Hardy space $H^2_\beta(B_N)$, ordered in the usual way. If $\mathcal{M}$ is diagonalizable, then Schroeder's equation has a solution F with $F'(0)$ invertible.*

Note that we always have $m \geq 1$, since the relation $\lambda_1 = \lambda_1$ (for example) always holds, and that when $m = 1$ either all of the eigenvalues of $\varphi'(0)$ are distinct and none is a product of any number of the other eigenvalues, or some eigenvalues are repeated ($\lambda_i = \lambda_j$ for some $i \neq j$), but none is a product of more than 1 of the other eigenvalues. When $m = 1$, $M = N + 1$ and the requirement that $\mathcal{M}$ be diagonalizable is automatically satisfied by virtue of the hypothesis that $\varphi'(0)$ is diagonalizable. So the existence of a Schroeder map which is univalent in a neighborhood of 0 is guaranteed in this case. Notice also that in the case that $m = 1$ and all eigenvalues of $\varphi'(0)$ are distinct, the hypothesis in Theorem 14 on the diagonalizability of $\varphi'(0)$ is automatically satisfied. On the other hand, one can construct linear fractional maps φ with $\varphi'(0)$ not diagonalizable and yet φ has a Schroeder map with invertible derivative at the origin.

Proof. Since φ is not unitary on any slice, C_φ is compact on $H^2_\beta(B_N) = A^2_G(B_N)$ for $G(r) = \exp(-1/(1-r))$ by Theorem 8. Let λ be any eigenvalue of $\varphi'(0)$. Suppose λ appears j times on the diagonal of the lower-triangular matrix for C_φ. By our choice of $\mathcal{M}$, all j of these diagonal entries lie on the diagonal of $\mathcal{M}$. Choose $M' \geq M$ so that

$$C_\varphi^* s \sim \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix},$$

where X is of size $M' \times M'$ and $\|Z\| < |\lambda_k|$ for $k = 1, 2, \ldots, N$.

Now the multiplicity of $\overline{\lambda}$ as an eigenvalue of C_φ^* is the same as its multiplicity as an eigenvalue of X by Lemma 9. Since $\mathcal{M}^*$ is diagonalizable and $\overline{\lambda}$ appears j times on the diagonal of $\mathcal{M}^*$, there must be j linearly independent eigenvectors for $\mathcal{M}^*$ corresponding to $\overline{\lambda}$. Since X is upper-triangular with $\mathcal{M}^*$ in the upper-left corner, these give rise to j linearly independent eigenvectors for X. Thus the multiplicity of $\overline{\lambda}$ as an eigenvalue of C_φ^* is at least j. Since $\overline{\lambda}$ appears j times on the diagonal of X, the multiplicity of $\overline{\lambda}$ as an eigenvalue of X is no more than j and we have $\dim\ker(C_\varphi^* - \overline{\lambda}) = j$; by compactness of C_φ on $A_G^2(B_N)$ we have $\dim \ker(C_\varphi - \lambda) = j$.

Now suppose λ occurs in positions $p_1, p_2, \cdots, p_j$ along the diagonal. Let $f_1, f_2, \cdots, f_j$ be j linearly independent eigenfunctions for C_φ acting on $A_G^2(B_N)$, with eigenvalue λ. There is a $q \geq m$ sufficiently large so that $P_q f_1, P_q f_2, \cdots, P_q f_j$ are linearly independent vectors. Let Q be the number of multi-indices for $\mathbb{C}^N$ of total order less than or equal to q and let X' be the upper-left $Q \times Q$ corner of the matrix for C_φ. Then $PP_q f_1, PP_q f_2, \cdots, PP_q f_j$ are a basis for $\ker(X' - \lambda)$. By Corollary 12, we have a basis $v_1, v_2, \cdots, v_j$ for $\ker(X' - \lambda)$ with $v_l(p_l) = 1$ and $v_l(r) = 0$ for $r < p_l$. Write $v_l = b_1^l PP_q f_1 + \cdots + b_j^l PP_q f_j$ and set

$$g_l^\lambda = b_1^l f_1 + \cdots + b_j^l f_j,$$

so that g_l^λ is an eigenfunction for C_φ in $A_G^2(B_N)$ with eigenvalue λ, and $PP_q(g_l^\lambda) = v_l$ has its first nonzero entry a 1 in the p_lth position (since the eigenfunctions are all 0 at 0, we count so that the "1st" position corresponds to the coefficient of z_1).

By hypothesis, $\varphi'(0)$ is diagonalizable. Let $W_1, W_2, \ldots, W_N$ be a basis of eigenvectors for $\varphi'(0)$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ and write $W_i = (W_i(1), W_i(2), \cdots, W_i(N))^t$. Let W be the $N \times N$ matrix whose ith column is W_i.

We construct a $\mathbb{C}^N$-valued analytic function F. First choose N functions $G_1, G_2, \ldots, G_N$ as follows. For $1 \leq k \leq N$, if the kth diagonal entry of $\varphi'(0)$ is the lth occurrence of λ on the diagonal of $\varphi'(0)$, let G_k be g_l^λ. By construction, G_k has leading term 1 in the kth position, i.e.,

$$G_k(z) = z_k + a_{k,k+1} z_{k+1} + \cdots + a_{k,N} z_N + \text{ higher-order terms.}$$

For notational convenience, define $a_{m,n} = 0$ if $1 \leq n < m \leq N$ and $a_{m,m} = 1$ for $1 \leq m \leq N$. Our desired analytic function F is defined as

$$F(z) = G_1(z)W_1 + G_2(z)W_2 + \cdots + G_N(z)W_N.$$

By Theorem 5, F satisfies $F \circ \varphi = \varphi'(0)F$.

We need only verify that $F'(0)$ is invertible. A computation shows that the first column of $F'(0)$ is W_1, the second column is $a_{12}W_1 + W_2$ and in general the ith column is $a_{1i}W_1 + a_{2i}W_2 + \cdots + a_{Ni}W_N = a_{1i}W_1 + a_{2i}W_2 + \cdots + W_i$ since $a_{ki} = 0$ if $k > i$ and $a_{ii} = 1$.

Denote the rows of $F'(0)$ generically by $R_1, R_2, \ldots, R_N$ and suppose some linear combination of the rows is 0, say,

$$c_1R_1 + c_2R_2 + \cdots + c_NR_N = 0.$$

Looking at the first entry of each row, this yields

$$c_1W_1(1) + c_2W_1(2) + \cdots + c_NW_1(N) = 0. \tag{9}$$

Using Equation (9) and considering next the second entry of each row gives

$$a_{12}(c_1W_1(1)+c_2W_1(2)+\cdots+c_NW_1(N))+c_1W_2(1)+c_2W_2(2)+\cdots+c_NW_2(N),$$

so that

$$c_1W_2(1) + c_2W_2(2) + \cdots + c_NW_2(N) = 0. \tag{10}$$

Similarly, Equations (9) and (10) together with the description of $F'(0)$ yield, by consideration of the third entries in each row,

$$c_1W_3(1) + c_2W_3(2) + \cdots + c_NW_3(N) = 0.$$

Continuing, we see that if some linear combination of the rows of $F'(0)$ is zero, the same linear combination of the rows of the matrix W is zero. By hypothesis, W is invertible, so we must have $c_1 = c_2 = \cdots = c_N = 0$, which says that $F'(0)$ is invertible as desired. ∎

An examination of the proof of the theorem above also gives the following corollary.

Corollary 15. *Suppose φ is an analytic map of B_N into B_N with $\varphi(0) = 0$ and $A = \varphi'(0)$ an upper-triangular diagonalizable matrix. Assume further that φ is not unitary on any slice. If $f \circ \varphi = \varphi'(0)f$ has a locally univalent solution, then for each number λ, the dimension of* $\ker(C_\varphi - \lambda I)$ *is the number of times λ occurs on the diagonal of the matrix for C_φ. Conversely, if for each number λ the dimension of* $\ker(\mathcal{M} - \lambda I)$ *is the number of times λ occurs on the diagonal of $\mathcal{M}$, then $f \circ \varphi = \varphi'(0)f$ has a locally univalent solution.*

As we noted in Section 1, the existence of a solution to Schroeder's equation with invertible derivative at 0 is equivalent to the existence of a solution with derivative equal to the identity. Thus when the hypotheses of Theorem 14 hold, Theorem 13 implies that *every* upper-left corner of the matrix for C_φ must be diagonalizable.

Theorem 14 explains the second example of Section 4. The eigenvalues are c_1 and c_1^3, so $m = 3$ and $M = 10$. The upper-left 10×10 corner of the matrix for C_φ can be seen to be diagonalizable, consistent with our observation in Section 4 that a Schroeder map with invertible derivative at 0 exists.

Corollary 16. *Suppose the hypotheses of Theorem* 14 *hold and that in addition $A = \varphi'(0)$ is diagonal. Then all solutions to Schroeder's equation can be described as $f = g \circ F$, where F is the Schroeder map, univalent in a neighborhood of* 0*, which was constructed in Theorem* 14 *and $g = (g_1, g_2, \cdots, g_N)$ is a mapping on $\mathbb{C}^N$ with polynomial coordinate functions. Moreover, if $g_k = \sum c(\gamma) z^\gamma$, then the coefficients $c(\gamma)$ are* 0 *unless $\lambda_k = \lambda_1^{\gamma_1} \lambda_2^{\gamma_2} \cdots \lambda_N^{\gamma_N}$, in which case $c(\gamma)$ can be chosen arbitrarily. If λ_k appears K times on the diagonal of the matrix of C_φ, then in g_k all but K of the coefficients of g_k must be* 0*.*

If $A = \varphi'(0)$ is merely diagonalizable, let S be any matrix which diagonalizes A, i.e., $SAS^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Then an arbitrary Schroeder map has the form $S^{-1} \circ g \circ S \circ F$ with F and g as just described.

Proof. First suppose that $\varphi'(0)$ is diagonal. Let $F : B_N \to \mathbb{C}^N$ be the Schroeder map constructed in Theorem 14 with $F'(0)$ invertible. Suppose $f : B_N \to \mathbb{C}^N$ is an arbitrary solution to $f \circ \varphi = \varphi'(0) f$. Since $F'(0)$ is invertible, F is a univalent map of some neighborhood V of 0 onto some neighborhood W of 0, with analytic inverse in W. Hence, for each z near 0, $\varphi'(0) z = F(\varphi(F^{-1}(z)))$ and we have

$$\begin{aligned}(f \circ F^{-1})(\varphi'(0)z) &= f(F^{-1}(\varphi'(0)z)) = f(F^{-1}(F(\varphi(F^{-1}(z))))) \\ &= f(\varphi(F^{-1}(z))) = \varphi'(0)(f \circ F^{-1})(z),\end{aligned}$$

so that $f \circ F^{-1}$ commutes with multiplication by the diagonal matrix $A = \varphi'(0) = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$. Let $g = f \circ F^{-1}$, a $\mathbb{C}^N$-valued analytic map in some neighborhood of 0. From the equation $gA = Ag$ we see that

$$g(\lambda_1 z_1, \cdots, \lambda_N z_N) = (\lambda_1 g_1(z), \cdots, \lambda_N g_n(z)),$$

so that for $k = 1, 2 \cdots, N$,

$$g_k(\lambda_1 z_1, \cdots, \lambda_N z_N) = \lambda_k g_k(z).$$

Writing g_k in terms of its homogeneous expansion we see that the coefficient of z^γ in g_k is 0 if $\lambda_k \neq \lambda_1^{\gamma_1} \cdots \lambda_N^{\gamma_N}$ and the coefficient is arbitrary if $\lambda_k = \lambda_1^{\gamma_1} \cdots \lambda_N^{\gamma_N}$. Since the diagonal entries of the matrix for C_φ are $1, \lambda_1, \lambda_2, \cdots, \lambda_N, \lambda_1^2, \lambda_1\lambda_2, \cdots, \lambda_N^2, \lambda_1^3, \cdots$, the coefficients of g_k must be zero except for the multi-indices corresponding to the diagonal entries which are equal to λ_k. This gives the desired form for an arbitrary Schroeder map when $\varphi'(0)$ is diagonal, and it is easy to see that any mapping in this form will be a Schroeder map for φ.

Now suppose that $A = \varphi'(0)$ is not diagonal, but is diagonalizable, with $SAS^{-1} = \Delta \equiv \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$, and that the other hypotheses of Theorem 14 hold. We observe that $SfF^{-1}S^{-1}$ must commute with Δ on a neighborhood of 0 whenever $f \circ \varphi = Af$:

$$\begin{aligned}(SfF^{-1}S^{-1})\Delta &= SfF^{-1}S^{-1}(SAS^{-1}) \\ &= SfF^{-1}AS^{-1} = SfF^{-1}(F\varphi F^{-1})S^{-1} \\ &= Sf\varphi F^{-1}S^{-1} = SAfF^{-1}S^{-1} \\ &= \Delta(SfF^{-1}S^{-1}).\end{aligned}$$

The calculations above show that $SfF^{-1}S^{-1} = g$, where g_k is a polynomial with the coefficient of $z^\gamma = 0$ if $\lambda_k \neq \lambda_1^{\gamma_1} \cdots \lambda_N^{\gamma_N}$ and arbitrary otherwise. So any Schroeder map f for φ has the form $f = S^{-1}gSF$ for a polynomial mapping g as described. Conversely, a calculation shows that any such mapping $S^{-1}gSF$ is a Schroeder map for φ. ■

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