TAIWANESE JOURNAL OF MATHEMATICS
Vol. 7, No. 1, pp. 115-128, March 2003
This paper is available online at http://www.math.nthu.edu.tw/tjm/

EXISTENCE OF GENERALIZED NEAREST POINTS*

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Abstract. The relation between directional derivatives of generalized distance functions and the existence of generalized nearest points in Banach spaces is investigated. We show that if the generalized function generated by a closed set has a one-sided directional derivative equal to 1 or -1, then the existence of generalized nearest points follows. We also give a partial answer to an open problem proposed by S. Fitzpatrick.

1. Introduction

Let X be a real Banach space of dimension at least 2 and X^* be the dual of X. For a nonempty subset $A \subset X$, by int $A, \partial A$ we mean the interior of A, the boundary of A, respectively. We use $B(x, r)$ to denote the closed ball in X with center x and radius $r > 0$. In particular, we put $B = B(0, 1)$.

Throughout this paper, C will denote a closed bounded convex subset of X with $0 \in intC$. Clearly C is an absorbing subset of X but not neessarily symmetric. Recall that the Minkowski functional $P_C : X \to R$ generated by the set C is defined by

$$P_C(x) = \inf\{\alpha > 0 : x \in \alpha C\}.$$

For a closed nonempty subset G of X and $x \in X$,define the generalized distance function by

$$d_G(x) = \inf_{x \in G} P_C(x - z).$$

A point $z_0 \in G$ with $P_C(x - z_0) = d_G(x)$ is called a generalized nearest point (or generalized best approximation) of x from G. Moreover, if the one-side directional derivative

Received February 2, 2000; revised September 6, 2001.
Communicated by Sen-Yen Shaw.
2000 *Mathematics Subject Classification*: 41A65, 46B20.
Key words and phrases: directional derivatives of generalized distance functions, existence of generalized nearest points, (compact) locally uniformly convex, minimizing sequence.
*Project supported by the NSF of China. (Grant No. 19971013)

$$d'_G(x)(y) = \lim_{t\to 0^+} \frac{d_G(x+ty) - d_G(x)}{t}$$

exists,then $-P_C(-y) \le d'_G(x)(y) \le P_C(y)$.

Recently, De Blasi and Myjak [8] and Li [9] investigated the well posedness of the generalized best approximation problems. Their results improve and extend the corresponding results in [10, 11, 12, 13].

As shown in [1, 2, 3, 4, 5], in the case when $P_C(\cdot)$ is the norm $\|\cdot\|$, or equivalently, $C = B$, differentiability properties of $d_G(\cdot)$ are related to nonemptiness and continuity of the metric projection P_G, defined by

$$P_G(x) = \{z \in G : P_C(x - z) = d_G(x)\}.$$

In the present paper, we will investigate the relationship between directional derivatives of the generalized distance function and the existence of generalized nearest points in Banach spaces. It is proved that if the generalized distance function to a closed set in a Banach space has a one-side directional derivative equal to 1 or -1 then we have the existence of nearest points, which extends a result due to S. Fitzpatrick [6]. Moreover, we also give a partial answer to an open problem proposed by S. Fitzpatrick in [6].

2. Preliminaries and Lemmas

First, we recall some well known properties of the Minkowski functional which follow immediately from the definitions.

Proposition 2.1. *Let C be as above. Then for every $x, y \in X$ we have*

(i) $P_C(x) \ge 0$ *and* $P_C(x) = 0$ *if and only if* $x = 0$;

(ii) $P_C(x+y) \le P_C(x) + P_C(y)$;

(iii) $-P_C(y-x) \le P_C(x) - P_C(y) \le P_C(x-y)$;

(iv) $P_C(\lambda x) = \lambda P_C(x)$, *if* $\lambda \ge 0$;

(v) $P_C(-x) = P_{-C}(x)$;

(vi) $P_C(x) = 1$ *if and only if* $x \in \partial C$;

(vii) $P_C(x) < 1$ *if and only if* $x \in intC$;

(viii) $\mu\|x\| \le P_C(x) \le \nu\|x\|$, *where*

$$\mu = \inf_{x\in\partial B} P_C(x) \quad and \quad \nu = \sup_{x\in\partial B} P_C(x).$$

Proposition 2.2. *Define*

$$q_c(x^*) = \sup_{x \in C}(x^*, x), \forall x^* \in X^*.$$

Then we have

(i) $q_c(x^* + y^*) \leq q_c(x^*) + q_c(y^*) \quad \forall x^*, y^* \in X^*$;

(ii) $q_c(\lambda x^*) = \lambda q_c(x^*), \forall \lambda \geq 0$ *and* $x^* \in X^*$;

(iii) $P_C(x) = \sup\{\langle x^*, x\rangle : x^* \in X^*, q_c(x^*) \leq 1\}$.

Proposition 2.3. *Let G be a closed subset of X. Then for every $x, y \in X$, we have*

$$-P_C(y - x) \leq d_G(x) - d_G(y) \leq P_C(x - y),$$

and

$$| \, d_G(x) - d_G(y) \, | \leq \nu \|x - y\|.$$

Definition 2.1. (i) C is said to be compact locally uniformly convex at $y \in \partial C$, if every sequence $\{y_n\} \subset \partial C$ with $\lim_{n\to\infty} P_C(y_n + y) = 2$ implies that $\{y_n\}$ has a converging subsequence.

(ii) C is said to be locally uniformly convex at $y \in \partial C$, if for every sequence $\{y_n\} \subset \partial C, \lim_{n\to\infty} P_C(y_n + y) = 2$ implies that $\lim_{n\to\infty} P_C(y_n - y) = 0$.

(iii) C is said to be (compact) locally uniformly convex, if C is (compact) locally uniformly convex at y for every $y \in \partial C$.

Definition 2.2. C is called strictly convex, if for every $x, y \in \partial C, P_C(x + y) = P_C(x) + P_C(y)$ implies $x = y$.

Definition 2.3. C is said to be (sequentially) Kadec if every sequence $\{x_n\} \subset \partial C$, with $x_n \to x_0 \in \partial C$ weakly, converges strongly to x_0.

The following calculation is useful for constructing examples.

Lemma 2.1. *Let G be a closed nonempty subset of $X, x \in X \backslash G$, and $y \in \partial C$. Suppose that*

$$\lim_{t\to 0^+} \sup \frac{d_G(x + ty) - d_G(x)}{t} = 1.$$

If $\{z_n\}$ is a minimizing sequence for x(i.e.$\{z_n\} \subset G$ satisfies $\lim_{n\to\infty} P_C(x - z_n) = d_G(x)$), then $\lim_{n\to\infty} P_C(y_n + y) = 2$, where $y_n = \frac{x - x_n}{P_C(x - z_n)}$.

Proof. Let $t_n \to 0^+$ such that

$$\lim_{n\to\infty} \frac{d_G(x + t_n y) - d_G(x)}{t_n} = 1.$$

We may assume that $0 < t_n < d_G(x) \le P_C(x - z_n)$ and $t_n^2 > P_C(x-z_n) - d_G(x)$. Note that the function $h(t)$ defined by

$$h(t) = \frac{P_C(x - z_n + ty) - P_C(x - z_n)}{t} \tag{1}$$

is nondecreasing with respect to t. It follows that

$$\begin{aligned}\frac{d_G(x + t_n y) - d_G(x)}{t_n} &\le \frac{P_C(x + t_n y - z_n) - P_C(x - z_n) + t_n^2}{t_n} \\ &\le \frac{P_C[(x - z_n) + P_C(x - z_n)y] - P_C(x - z_n)}{P_C(x - z_n)} + t_n \\ &= P_C(y_n + y) - 1 + t_n,\end{aligned}$$

and

$$\begin{aligned}2 &\le \liminf_{n\to\infty} P_C(y_n + y) \\ &\le \limsup_{n\to\infty} P_C(y_n + y) \\ &\le \limsup_{n\to\infty} P_C(y_n) + P_C(y) = 2.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} P_C(y_n + y) = 2.$$

Thus we complete the proof.

Lemma 2.2. *Let $\{y_n\} \subset \partial C, y \in \partial C$, be such that $\lim_{n\to\infty} P_C(y_n + y) = 2$. Let*

$$G_0 = \Big\{ z_n = -\Big(1 + \frac{1}{n}\Big)\frac{y_n + y}{P_C(y_n + y)} : n = 1, 2, \cdots \Big\}.$$

Then $d'_{G_0}(0)(y) = 1$ and $d'_{G_0}(0)(-y) = -1$.

Proof. For every $t > 0$, we have

$$\begin{aligned}d_{G_0}(ty) - d_{G_0}(0) &= \inf_n P_C(ty - z_n) - 1 \\ &= \inf_n \Big\{ P_C(ty - z_n) - P_C(-z_n) + \frac{1}{n} \Big\}.\end{aligned}$$

Let $n_t \in \{1, 2, \cdots\}$ be such that

$$\inf_n \Big[P_C(ty - z_n) - P_C(-z_n) + \frac{1}{n} \Big] \ge P_C(ty - z_{n_t}) - P_C(-z_{n_t}) + \frac{1}{n_t} - t^2.$$

From the last inequality, we have $\lim_{t\to 0^+} n_t = +\infty$, (otherwise the last formula gives $0 \geq 0 + \lim\limits_{t\to 0^+} \frac{1}{n_t}$, a contradiction). By virtue of the convexity of $P_C(\cdot)$, it follows that

$$\frac{P_C[(-t)(-y) - z_n] - P_C(-z_n)}{-t} \leq \frac{P_C[\alpha_n(-y) - z_n] - P_C(-z_n)}{\alpha_n},$$

whenever $-t < \alpha_n = \frac{1+\frac{1}{n}}{P_C(y_n+y)}$ and $t > 0$. Thus

$$\frac{P_C(ty - z_n) - P_C(-z_n)}{t} \geq \frac{P_C(-\alpha_n y - z_n) - P_C(-z_n)}{-\alpha_n}.$$

It follows that

$$\begin{aligned}
1 = P_C(y) &\geq \limsup_{t\to 0^+} \tfrac{d_{G_0}(ty) - d_{G_0}(0)}{t} \\
&\geq \liminf_{t\to 0^+} \tfrac{d_{G_0}(ty) - d_{G_0}(0)}{t} \\
&\geq \liminf_{t\to 0^+} \big(\tfrac{P_C(ty - z_{n_t}) - P_C(-z_{n_t}) + \frac{1}{n_t}}{t} - t\big) \\
&\geq \liminf_{t\to 0^+} \big(\tfrac{P_C(ty - z_{n_t}) - P_C(-z_{n_t})}{t} - t\big) \\
&\geq \liminf_{t\to 0^+} \big(\tfrac{P_C(-\alpha_{n_t} y - z_{n_t}) - P_C(-z_{n_t})}{-\alpha_{n_t}} - t\big) \\
&\geq \liminf_{t\to 0^+} \big(\tfrac{P_C(-\alpha_{n_t} y - z_{n_t}) - P_C(-z_{n_t})}{-\alpha_{n_t}} - t\big) + \liminf_{t\to 0^+}(-t) \\
&\geq \lim_{n_t\to\infty} \big(\tfrac{P_C[-\alpha_{n_t} y + \alpha_{n_t}(y_{n_t} + y)] - P_C[\alpha_{n_t}(y_{n_t} + y)]}{-\alpha_{n_t}}\big) \\
&= \liminf_{n_t\to\infty}(-P_C(y_{n_t}) + P_C(y_{n_t} + y)) = 2 - 1 = 1,
\end{aligned}$$

so that $d'_{G_0}(0)(y) = 1$.

Now let us prove that $d'_{G_0}(0)(-y) = -1$. Take $\alpha_n = \frac{1+\frac{1}{n}}{P_C(y_n+y)}$. Note that the function $h(t)$ is nondecreasing with respect to t. We have

$$h(-z_n, t) \leq h(-z_n, \alpha_n), \forall 0 < t \leq \alpha_n,$$

i.e.,

$$\frac{P_C(-ty - z_n) - P_C(-z_n)}{t} \leq \frac{P_C(-\alpha_n y - z_n) - P_C(-z_n)}{\alpha_n}.$$

It follows that

$$\begin{aligned}
-1 &= -P_C(y) \\
&\le \liminf_{t\to 0^+} \tfrac{d_{G_0}(-ty)-d_{G_0}(0)}{t} \\
&\le \limsup_{t\to 0^+} \tfrac{d_{G_0}(-ty)-d_{G_0}(0)}{t} \\
&= \limsup_{t\to 0^+} \tfrac{\inf_n P_C(-ty-z_n)-1}{t} \\
&\le \lim_{t\to 0^+} \inf_n \tfrac{P_C(-ty-z_n)-1}{t} \\
&= \limsup_{t\to 0^+} \inf_n \big[\tfrac{P_C(-ty-z_n)-P_C(-z_n)+\frac{1}{n}}{t}\big] \\
&\le \limsup_{t\to 0^+} \tfrac{P_C(-\alpha_n y-z_n)-P_C(-z_n)}{\alpha_n} + \lim_{n\to\infty} \tfrac{\frac{1}{n}}{\alpha_n} \\
&= \lim_{n\to\infty} [P_C(y_n) - P_C(y_n+y)] + 0 = -1.
\end{aligned}$$

This implies that $d'_{G_0}(0)(-y) = -1$ and completes the proof.

Remark 2.1. Set

$$G_0 = \Big\{ z_n = -\Big(1+\frac{1}{n}\Big)\frac{y_n - y}{P_C(y_n - y)} : n = 1, 2, \cdots \Big\},$$

where $\{y_n\} \subset \partial C$ and $-y \in \partial C$. If $\lim_{n\to\infty} P_C(y_n - y) = 2$ then $d'_{G_0}(0)(y) = -1$ by Lemma 2.2.

Remark 2.2. In the case when $P_C(\cdot)$ is a norm, Lemma 2.2 was given by S. Fitzpatrick [6], but the author did not show that $d'_{G_0}(0)(-y) = -1$. In Lemma 2.2 of [7], the authors gave a proof of this fact using the homogeneity of $d'_{G_0}(0)(y)$ with respect to y. However, $d'_{G_0}(0)(y)$ is not homogenous, in general. Hence we could not deduce $d'_{G_0}(0)(-y) = -1$ directly from $d'_{G_0}(0)(y) = 1$.

3. Main Results

Theorem 3.1. *Let X be a Banach space, $y \in \partial C$. Then the following statements are equivalent*:

(i) *for any nonempty closed subset G of X and $x \in X\backslash G$, if*

$$\limsup_{t\to 0^+} \frac{d_G(x+ty) - d_G(x)}{t} = 1,$$

then G is approximatively compact for x, in the sense that any sequence $\{x_n\} \subset G$ satisfying $lim_{n\to\infty} P_C(x - z_n) = d_G(x)$ has a converging subsequence;

(ii) *for any nonempty closed subset G of X and $x \in X\backslash G$, if $d'_G(x)(y) = 1$, then G is approximatively compact for x;*

(iii) *C is compact locally uniformly convex at y.*

Proof. (i)$\Rightarrow$(ii). It is obvious.

(ii)$\Rightarrow$(iii). Suppose (iii) does not hold. Then there exists a sequence $\{y_n\} \subset \partial C$ such that $lim_{n\to\infty} P_C(y_n+y) = 2$, but $\{y_n\}$ has no converging subsequence. Let

$$G = \Big\{x - \Big(1 + \frac{1}{n}\Big)\frac{y_n + y}{P_C(y_n + y)} : n = 1, 2, \cdots\Big\}.$$

Then G is closed and

$$d_G(x) = \inf_{z\in G} P_C(x - z) = 1 < 1 + \frac{1}{n} = P_C(x - z)$$

for every n and $z \in G$ so that x has no nearest point in G. From Lemma 2.2 we have that $d'_G(x)(y) = d'_{G_0}(0)(y) = 1$, which contradicts (ii).

(iii)$\Rightarrow$(i). Assume that (iii) holds and $x \in X\backslash G$ satisfies

$$\lim_{t\to 0^+} \sup \frac{d_G(x + ty) - d_G(x)}{t} = 1.$$

Then by virtue of Lemma 2.1, it follow that $\lim_{n\to\infty} P_C(y_n + y) = 2$ exists for any minimizing sequence $\{z_n\}$ for x, where $y_n = \frac{x-z_n}{P_C(x-z_n)}$. Observe that C is compact locally uniformly convex at y. Then $\{y_n\}$ has a converging subsequence, again denoted by $\{y_n\}$. Hence $\{z_n\}$ has a converging subsequence too because $\lim_{n\to\infty} P_C(x - z_n) = d_G(x) > 0$. Thus we complete the proof.

Corollary 3.1. *The following statements are equivalent*:

(i) *For each closed nonempty subset G of X and $x \in X\backslash G$, if there is $y \in \partial C$ with $d'_G(x)(y) = 1$, then G is approximatively compact for x;*

(ii) *C is compact locally uniformly convex.*

Theorem 3.2. *Let $y \in \partial C$. The following statements are equivalent*:

(i) *For each nonempty closed subset G of X and $x \in X\backslash G$, if*

$$\lim_{t\to 0^+} \sup \frac{d_G(x + ty) - d_G(x)}{t} = 1,$$

then G is approximatively compact for x and $P_G(x) = \{x - d_G(x)y\}$;

(ii) *For each nonempty closed subset G of X and $x \in X\backslash G$, if $d'_G(x)(y) = 1$, then G is approximatively compact for x and $P_G(x) = \{x - d_G(x)y\}$;*

(iii) *C is locally uniformly convex at y.*

Proof. (i)$\Rightarrow$(ii). It is obvious.

(ii)$\Rightarrow$(iii). Suppose C is not locally uniformly convex at y. Then there is $\{y_n\} \subset \partial C$ such that

$$\lim_{n\to\infty} P_C(y_n + y) = 2 \quad \text{and} \quad P_C(y_n - y) \geq \delta > 0$$

for all n. By virtue of Theorem 3.1, $\{y_n\}$ has a converging subsequence, again denoted by $\{y_n\}$. Let $y_0 \in \partial C$ such that $P_C(y_n - y_0) \to 0$. Clearly, $P_C(y_0 - y) \geq \delta$ and $P_C(y_0 + y) = 2$. Let

$$G = \{x - y, x - y_0\}.$$

Thus, for each $t > 0$, we have $P_C(ty + y_0) = 1 + t$. Indeed, by Hahn-Banach Theorem, we may choose $x^* \in X^*$ with $q_c(x^*) \leq 1$ such that

$$\left\langle x^*, \frac{y_0 + y}{2} \right\rangle = P_C\left(\frac{y_0 + y}{2}\right) = 1,$$

and so $\langle x^*, y_0\rangle = \langle x^*, y\rangle = 1$. It follows that

$$\begin{aligned} 1 + t &= tP_C(y) + P_C(y_0) \\ &\geq P_C(ty + y_0) \\ &\geq \langle x^*, ty + y_0\rangle \\ &= \langle x^*, y_0\rangle + t\langle x^*, y\rangle \\ &= 1 + t. \end{aligned}$$

This implies that

$$P_C[(x + ty) - (x - y_0)] = P_C(ty + y_0) = 1 + t,$$

and

$$P_C[(x + ty) - (x - y)] = P_C((t + 1)y) = 1 + t,$$

so that $d_G(x + ty) = 1 + t, d_G(x) = 1$. From this, we have

$$d'_G(x)(y) = \lim_{t\to 0^+} \frac{d_G(x + ty) - d_G(x)}{t} = 1.$$

But $P_G(x) = \{x - y, x - y_0\} = G$, contradicting (ii).

(iii)⇒(i). Suppose that $x \in X\backslash G$ and

$$\lim_{t\to 0^+}\sup\frac{d_G(x+ty)-d_G(x)}{t}=1.$$

By Lemma 2.1, it follows that any minimizing sequence $\{z_n\}$ for x satisfies $\lim_{n\to\infty} P_C(y_n+y)=2$, where $y_n=\frac{x-z_n}{P_C(x-z_n)}$. Observe that C is locally uniformly convex at y. We have $\lim_{n\to\infty} P_C(y_n-y)=0$ so that $z_n\to x-d_G(x)y$ because $\lim_{n\to\infty} P_C(x-z_n)=d_G(x)$. Therefore (i) holds and we completes the proof.

Corollary 3.2. *The following statements are equivalent*:

(i) *For each closed nonempty subset G of X and $x\in X\backslash G$, if there is $y\in\partial C$*
such that $d'_G(x)(y)=1$, then G is approximatively compact for x and $P_G(x)=\{x-d_G(x)y\}$;

(ii) *C is locally uniformly convex.*

Theorem 3.3. *Let $-y\in\partial C$. The following statements are equivalent*:

(i) *If G is nonempty closed subset of X and $x\in X\backslash G$ with $\lim_{t\to 0^+}\inf\frac{d_G(x+ty)-d_G(x)}{t}$ = 1, then $P_G(x)\neq\emptyset$;*

(ii) *If G is nonempty closed subset of X and $x\in X\backslash G$ with $d'_G(x)(y)=-1$, then $P_G(x)=\emptyset$;*

(iii) *C is compact locally uniformly convex at $-y$*

Proof. (i)⇒(ii). It is obvious.

(ii)⇒(iii). Suppose C is not compact locally uniformly convex at $-y$, then there is $\{y_n\}\subset\partial C$ such that $\lim_{n\to\infty} P_C(y_n-y)=2$, but $\{y_n\}$ has no converging subsequence. Let

$$G=\left\{x-\left(1+\frac{1}{n}\right)\frac{y_n-y}{P_C(y_n-y)}:n=1,2,\cdots\right\}.$$

Then G is closed and

$$d_G(x)=\inf_{z\in G}P_C(x-z)=1<1+\frac{1}{n}=P_C(x-z),$$

for every n and $z\in G$ so that $P_G(x)=\emptyset$. However Remark 2.1 yields that $d'_G(x)(y)=d'_{G_0}(0)(y)=-1$, contradicting (ii).

(iii)⇒(i). Let $t_n \to 0^+$ with $\lim_{n\to\infty} \frac{d_G(x+t_ny)-d_G(x)}{t_n} = -1$. Choose $\{z_n\} \subset G$ such that

$$P_C(x+t_ny-z_n) < d_G(x+t_ny)+t_n^2.$$

Note that $h(t)$ given by (1) is nondecreasing with respect to t and $t_n > -P_C(x-z_n)$. We have

$$\frac{P_C(x+t_ny-z_n)-P_C(x-z_n)}{t_n} \geq \frac{P_C[x-P_C(x-z_n)y-z_n]-P_C(x-z_n)}{-P_C(x-z_n)},$$

and so

$$\begin{aligned}
&\frac{d_G(x+t_ny)-d_G(x)}{t_n}\\
\geq&\frac{P_C(x+t_ny-z_n)-P_C(x-z_n)}{t_n}-t_n\\
\geq&-t_n+\frac{P_C(x-z_n)-P_C[(x-z_n)-P_C(x-z_n)y]}{P_C(x-z_n)}\\
=&-t_n+1-P_C(y_n-y),
\end{aligned}$$

where $y_n = \frac{x-z_n}{P_C(x-z_n)}$. Thus ,$\{y_n\} \subset \partial C$ and $lim_{n\to\infty}P_C(y_n-y) = 2$, which implies that $\{y_n\}$ has a converging subsequence, say, $\{y_n\}$. Observe that

$$\begin{aligned}
d_G(x) &\leq P_C(x-z_n)\\
&\leq P_C(x+t_ny-z_n)+P_C(-t_ny)\\
&\leq (d_G(x+t_ny)+t_n^2)+t_n\\
&\leq d_G(x)+\nu t_n\|y\|+t_n^2+t_n.
\end{aligned}$$

We have

$$\lim_{n\to\infty} P_C(x-z_n) = d_G(x).$$

Thus $lim_{n\to\infty}z_n = z_0 \in G$ and $P_C(x-z_0) = d_G(x)$. (i) holds and we complete the proof.

Corollary 3.3. *The following statements are equivalent:*

(i) *for each closed nonempty subset G of X and $x \in X\backslash G$, there is $-y \in \partial C$ with $d'_G(x)(y) = -1$, if and only if $P_G(x) \neq \emptyset$;*

(ii) *C is compact locally uniformly convex.*

Proof. By virtue of Theorem 3.3, it suffices to show that there is $-y \in \partial C$ with $d'_G(x)(y) = -1$, if $P_G(x) \neq \emptyset$. For this purpose, choose $g_0 \in P_G(x), i.e.,\ \ 0 < d_G(x) = P_C(x-g_0)$. Then $g_0 \in P_G(g_0+t(x-g_0))$ for every $t \in (0,1]$. Put $y = \frac{g_0-x}{P_C(x-g_0)}$. We have $d_G(x+ty) = P_C(x-g_0)-t$ and $-y \in \partial C$.

Thus

$$\begin{aligned} d'_G(x)(y) &= \lim_{t\to 0^+} \frac{d_G(x+ty)-d_G(x)}{t} \\ &= \lim_{t\to 0^+} \frac{(P_C(x-g_0)-t)-P_C(x-g_0)}{t} \\ &= -1, \end{aligned}$$

and we complete the proof.

Corollary 3.4. *Let G be a closed nonempty subset of a Banach space X and C be compact locally uniformly convex. Then G is proximinal set (i.e., $P_G(x) \neq \emptyset$ for every $x \in X$) if and only if for every $x \in X\backslash G$, there is $-y \in \partial C$ with $d'_G(x)(y) = -1$.*

Remark 3.1. In the case when $P_C(\cdot) = \|\cdot\|$, it is easy to show that C is compact locally uniformly convex at $-y \in \partial C$ if and only if C is compact locally uniformly convex at $y \in \partial C$.

Remark 3.2. In Theorem 3.1 and Corollaries 3.3 and 3.4, the assumption that G is proximinal cannot be replaced by the condition that G is approximatively compact Furthermore, even in the case that C is locally uniformly convex, we cannot obtain that G is a Chebyshev set. For example, let X be an arbitrary locally uniformly convex infinite dimensional Banach space, and let C be the closed unit ball in X. Define $G = \{x \in X; \|x\| \geq 1\}$. Obviously, G is a proximinal set, but G is neither a Chebyshev set nor an approximatively compact set. But, from Corollary 3.4, there is $-y \in \partial C$ with $d'_G(x)(y) = -1$ for any $x \in X\backslash G$.

Theorem 3.4. *Let G be a nonempty closed subset of Banach space X. If G is approximatively compact for $x \in X\backslash G$ and $P_G(x) = \{g_0\}$, then there exists $y \in \partial C$ with $d'_G(x)(y) = 1$.*

Proof. Let $x_t = x + t(x - g_0), t \in (0, 1)$. By virtue of definition of $d_G(x_t)$, there is $g_t \in G$ with $P_C(x_t - g_t) < d_G(x_t) + t^2$. Clearly, $x_t \to x$ as $t \to 0^+$ and $P_C(x - g_t) - P_C(x - g_0) \geq 0$. Choose $x_t^* \in X^*$ with $q_c(x_t^*) = 1$ satisfying

$\langle x_t^*, x - g_t\rangle = P_C(x - g_t)$. We have

$$
\begin{aligned}
&\frac{d_G(x_t) - d_G(x)}{t} - P_C(x - g_0)\\
\geq &\frac{P_C(x_t - g_t) - P_C(x - g_0)}{t} - P_C(x - g_0) - t\\
= &\frac{\langle x_t^*, x_t - g_t\rangle - P_C(x - g_0)}{t} - P_C(x - g_0) - t\\
= &\frac{\langle x_t^*, x_t - x\rangle}{t} + \frac{P_C(x - g_t) - P_C(x - g_0)}{t} - P_C(x - g_0) - t\\
\geq &\langle x_t^*, x_t - g_0\rangle - P_C(x - g_0) - t\\
= &(\langle x_t^* - g_t\rangle + \langle x_t^*, g_t - g_0\rangle) - P_C(x - g_0) - t\\
\geq &(P_C(x - g_t) - P_C(x - g_0)) - P_C(g_t - g_0) - t\\
\geq &-P_C(g_t - g_0) - t.
\end{aligned}
$$

Thus

$$\frac{d_G(x_t) - d_G(x)}{t} \geq P_C(x - g_0) - P_C(g_t - g_0) - t. \tag{2}$$

Since

$$
\begin{aligned}
d_G(x) &\leq P_C(x - g_t)\\
&= P_C[x_t - t(x - g_0) - g_t]\\
&\leq P_C(x_t - g_t) + tP_C[-(x - g_0)]\\
&\leq d_G(x_t) + t^2 + t\nu\|x - g_0\|\\
&\leq d_G(x) + \nu t\|x - g_0\| + t^2 + t\nu\|x - g_0\|\\
&= d_G(x) + 2\nu t\|x - g_0\| + t^2,
\end{aligned}
$$

it follows that

$$\lim_{t\to 0^+} P_C(x - g_t) = d_G(x) = P_C(x - g_0) > 0.$$

Note that G is approximatively compact for x and $P_G(x) = \{g_0\}$. Then $\lim_{t\to 0^+} P_C(x_t - g_0) = 0$. From this and (2), we have

$$\liminf_{t\to 0^+} \frac{d_G(x_t) - d_G(x)}{t} \geq P_C(x - g_0).$$

Obviously,

$$\limsup_{t\to 0^+} \frac{d_G(x_t) - d_G(x)}{t} \leq P_C(x - g_0).$$

Thus $d'_G(x)(x-g_0) = P_C(x-g_0)$. By virtue of the positive homogeneity of $d'_G(x)(u)$ with respect to u, we have $d'_G(x)(y) = 1$ for $y = \frac{x-g_0}{P_C(x-g_0)}$. This completes the proof.

Corollary 3.5. *Let C be locally uniformly convex and G nonempty closed subset of X. Then for every $x \in X \backslash G$ there exists a $y \in \partial C$ such that $d'_G(x)(y) = 1$ if and only if G is approximatively compact and Chebyshev subset of X.*

Proof. This follows from Corollary 3.2 and Theorem 3.4.

Theorem 3.5. *Let G be a nonempty closed subset of X. If X is reflexive and C is both strictly convex and Kadec, then the set*

$$D = \{x \in X \backslash G; \exists\, y \in \partial C \text{ with } d'_G(x)(y) = 1\}$$

is residual in $X \backslash G$.

Proof. Let $X_0(G)$ be the set of all point $x \in X \backslash G$ such that problem $min_C(x, G)$ is well posed, by which we mean that there exists a unique point $z \in G$ satisfying $P_C(x-z) = d_G(x)$ and every minimizing sequence for x converges strongly to z. From Theorem 3.3 in [9], $X_0(G)$ is residual in $X \backslash G$. Furthermore, for every $x \in X_0(G), G$ is approximatively compact for x and $P_G(x)$ has exactly one element. Thus, by Theorem 3.4, there exists $y \in \partial C$ with $d'_G(x)(y) = 1$. It follows that $X_0(G) \subset D$. This completes the proof.

Remark 3.3. In the case when $P_C(\cdot) = \|\cdot\|$, S. Fitzpatrick [6] put forth the following open problem:

Problem F. *If G is a closed subset of reflexive Banach space X, is the set D residual in $X \backslash G$? Our Theorem 3.5 gives a partial answer to the above problem when C is both strictly convex and Kadec. We do not know whether it remains true without this assumption.*

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