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ON THE IDENTITY $h(x) = af(x) + g(x)b$

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Abstract. A description of the generalized (α, β)-derivations f, g and h of a prime ring R which satisfying $h = af + gb$ is given.

Receently, Bresar [3] gave a description of derivations d, g and h of a prime ring R satisfying $d = ag+hb$, where a and b are some fixed noncentral elements of R. This results generalizes a theorem in Herstein's paper [9]. Latter on, the author [6] exteneded this result to (α, β)-derivations, the result we obtained generalizes several results simultaneously. In this note we will extend this result further to the so-called generalized (α, β)-derivations which are motivated by the same paper [3].

Throughout, R will be a prime ring with center Z, Q will denote the two sided Martindale quotient ring of R and C will be the extended centroid of R. Also, α and β will be the automorphisms of R. Recall that an additive mapping $\delta : R \to R$ is said to be an (α, β)-derivation if $\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y)$ for all $x, y \in R$. A typical (α, β)-derivation is so-called inner (α, β)-derivation defined by $\delta(x) = a\alpha(x) - \beta(x)a$ for all $x \in R$, where $a \in R$.

We begin with a definition.

Definition 1. Let R be a ring, α and β automorphisms of R and δ an (α, β)-derivation of R. An additive mapping $f : R \to R$ is said to be a right generalized (α, β)-derivation of R associated with δ if

$$f(xy) = f(x)\alpha(y) + \beta(x)\delta(y) \text{ for all } x, y \in R \tag{1}$$

and f is said to be a left generalized (α, β)-derivation of R associated with δ if

$$f(xy) = \delta(x)\alpha(y) + \beta(x)f(y) \text{ for all } x, y \in R \tag{2}$$

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f is said to be a generalized (α,β)-derivation of R associated with δ if it is both a left and right generalized (α,β)-derivation of R associated with δ.

Note that if R is a prime ring then any generalized (α,β)-derivation f of R is associated with a unique (α,β)-derivation δ. Also note that any (α,β)-derivation of R is clearly a generalized (α,β)-derivation. The following example gives us left(right) generalized (α,β)-derivations other than generalized (α,β)-derivations.

Example 1. Let $a,b \in R$ be such that one of them is not zero and let $\alpha,\beta \in Aut(R)$, the group of automorphisms of R. Define

$$f(x) = a\alpha(x) + \beta(x)b \text{ for all } x \in R$$

Then for $x,y \in R$, we have $f(x+y) = f(x) + f(y)$ and

$$\begin{aligned} f(xy) &= a\alpha(xy) + \beta(xy)b \\ &= a\alpha(x)\alpha(y) + \beta(x)\beta(y)b \\ &= (a\alpha(x) + \beta(x)b)\alpha(y) + \beta(x)(-b\alpha(y) - \beta(y)(-b)) \end{aligned}$$

That is, f is a right generalized (α,β)-derivation associated with δ_1, where $\delta_1(x) = -b\alpha(x) - \beta(x)(-b)$ for all $x \in R$. We also have

$$\begin{aligned} f(xy) &= a\alpha(xy) + \beta(xy)b \\ &= a\alpha(x)\alpha(y) + \beta(x)\beta(y)b \\ &= (a\alpha(x) - \beta(x)b)\alpha(y) + \beta(x)(a\alpha(y) + \beta(y)b) \end{aligned}$$

That is, f is also a left generalized (α,β)-derivation associated with δ_2, where $\delta_2(x) = a\alpha(x) - \beta(x)a$ for all $x \in R$. In general, it may not be true that $\delta_1 = \delta_2$, that is, f may not be a generalized (α,β)-derivation associated with δ_1 or δ_2. However, we have the following

Lemma 1. *Let R be a prime ring and let f be as in example 1. Then f is a generalized (α,β)-derivation of R associated with $\delta = \delta_1 = \delta_2$ if and only if either $a+b=0$, or $a+b$ is intertible in Q and $\alpha^{-1}\beta(x) = (\alpha^{-1}(a+b))^{-1}x\alpha^{-1}(a+b)$ for all $x \in R$.*

Proof. From example 1, it is easy to see that f is a generalized (α,β)-derivation if and only if $\delta_1 = \delta_2$. The latter says that $-b\alpha(x) + \beta(x)b = a\alpha(x) - \beta(x)a$ for all $x \in R$. Hence $(a+b)\alpha(x) = \beta(x)(a+b)$ and thus $\alpha^{-1}(a+b)x = \alpha^{-1}\beta(x)\alpha^{-1}(a+b)$ for all $x \in R$. If $a+b \neq 0$, then by [10;p.136], $\alpha^{-1}(a+b)$ and hence $a+b$ is intertible in Q and $\alpha^{-1}\beta(x) = \alpha^{-1}(a+$

$b)x(\alpha^{-1}(a+b))^{-1}$ for all $x \in R$. Conversely, if $a+b=0$, then $b=-a$ and $f(x) = a\alpha(x) + \beta(x)b = a\alpha(x) - \beta(x)a$ is an (α,β)-derivation of R. If $\alpha^{-1}\beta(x) = \alpha^{-1}(a+b)x(\alpha^{-1}(a+b))^{-1}$ for all $x \in R$, then $(a+b)\alpha(x) = \beta(x)(a+b)$ and $\delta_1(x) = -b\alpha(x) - \beta(x)(-b) = a\alpha(x) - \beta(x)\alpha = \delta_2(x)$ for all $x \in R$. Hence f is a generalized (α,β)-derivation associated with $\delta = \delta_1 = \delta_2$.

Let us examine the previous example $f(x) = a\alpha(x) + \beta(x)b$ more closely. We can rewrite f into the form $f(x) = (a+b)\alpha(x) + \delta_1(x) = \beta(x)(a+b) + \delta_2(x)$, where $\delta_1(x) = -b\alpha(x) - \beta(x)(-b)$ and $\delta_2(x) = a\alpha(x) - \beta(x)a$. On the other hand, since any automorphism of a prime ring R can be uniquely extended to both left and right Martindale quotient rings of R, we see that $f(x) = a\alpha(x) + \beta(x)b$ can also be uniquely extended to both left and right Martindale quotient rings of R. Moreover, we have $f(1) = a+b$ and $f(x) = f(1)\alpha(x) + \delta_1(x) = \beta(x)f(1) + \delta_2(x)$. In general, we have

Lemma 2. *Let R be a prime ring. If f is a left (right resp.) generalized (α,β)-derivation of R, then f can be uniquely extended to the left (right resp.) Martindale quotient ring ${}_RF$ (F_R resp.) of R and $f(x) = \beta(x)f(1) + \delta(x)$ ($f(x) = f(1)\alpha(x) + \delta(x)$ resp.) for all $x \in R$, where δ is an (α,β)-derivation of R.*

Proof. Assume that f is a left generalized (α,β)-derivation associated with δ. Let $T(x) = f(x) - \delta(x)$. Then $T(xy) = f(xy) - \delta(xy) = \delta(x)\alpha(y) + \beta(x)f(y) - (\delta(x)\alpha(y) + \beta(x)\delta(y)) = \beta(x)(f(y) - \delta(y)) = \beta(x)T(y)$ for all x, $y \in R$. For $s \in {}_RF$, there exists an ideal I_s of R such that $I_s s \in R$. Then T can be uniquely extended to ${}_RF$ by the rule $T(is) = \beta(i)T(s)$ for all $i \in I_s$. Since $f(x) = T(x) + \delta(x)$ for all $x \in R$ and δ can be uniquely extended to ${}_RF$, we conclude that f can be uniquely extended to ${}_RF$. Moreover, we have $f(x) = f(x \cdot 1) = \delta(x)\alpha(1) + \beta(x)f(1) = \beta(x)f(1) + \delta(x)$ for all $x \in R$. Similarly, every right generalized (α,β)-derivation associated with δ can be uniquely extended to F_R and $f(x) = f(1)\alpha(x) + \delta(x)$ for all $x \in R$.

Remark. (1) A left (right resp.) generalized (α,β)-dervation f of a prime ring R is associated with a unique (α,β)-derivation δ.

(2) A left (right resp.) generalized (α,β)-derivation f of a prime ring R can be extended to Q if and only if $f(1) \in Q$.

We can sharpen the previous lemma little bit when f is a generalized (α,β)-derivation associated with δ.

Lemma 3. *Let R be a prime ring. Then f is a generalized (α,β)-derivation of R associated with δ if and only if one of the following holds:*

(i) *$f(x) = \delta(x)$ for all $x \in R$*

(ii) *$f(x) = f(1)\alpha(x) + \delta(x) = \beta(x)f(1) + \delta(x)$ for all $x \in R$, where $f(1)$ is invertible in Q and $\beta\alpha^{-1}(x) = f(1)xf(1)^{-1}$ for all $x \in R$.*

Proof. If f is a generalized (α, β)-derivation of R associated with δ, then as a right generalized (α, β)-derivationof R, we have $f(x) = s\alpha(x) + \delta(x)$ for all $x \in R$, where $s = f(1) \in F_R$. On the other hand, as a left generalized (α, β)-derivation of R, we have $f(xy) = \delta(x)\alpha(y) + \beta(x)f(y)$ for all $x, y \in R$. Subsititute $f(y) = s\alpha(y) + \delta(y)$ and $f(xy) = s\alpha(xy) + \delta(xy)$ into the last equation, we obtain $s\alpha(x)\alpha(y) = \beta(x)s\alpha(y)$for all $x, y \in R$. Therefore, $s\alpha(x) = \beta(x)s$ for all $x \in R$ and hence $s \in Q$. If $s = 0$, then $f(x) = \delta(x)$ for all $x \in R$. If $s \neq 0$, then $sx = \beta(\alpha^{-1}(x))s$ for all $x \in R$. Hence $\beta(\alpha^{-1}(x)) = sxs^{-1}$ for all $x \in R$ by [10; p136]. Since $s\alpha(x) = \beta(x)s$ for all $x \in R$, we also have $f(x) = \beta(x)s + \delta(x) = \beta(x)f(1) + \delta(x)$ for all $x \in R$.

The converse is obvious.

Definition 2. We say a generalized (α, β)-derivation of a prime ring R association with δ is proper if $f \neq \delta$.

Let $\delta \neq 0$ be an (α, β)-derivation of a prime R and let $a \in R$. It is shown in [5] that if $a\delta(x) = 0\,(\delta(x)a = 0)$ then $a = 0$. In the following lemma we show that this is still true for any nonzero generalized (α, β)-derivation associated with δ.

Lemma 4. *Let $f \neq 0$ be a generalized (α, β)-derivation of a prime ring R associated with δ and let $a \in R$.*

(i) *if $af(x) = 0$ for all $x \in R$, then $a = 0$*

(ii) *if $f(x)a = 0$ for all $x \in R$, then $a = 0$.*

Proof. (i) If $af(x) = 0$ for all $x \in R$, then $0 = af(xy) = a(f(x)\alpha(y) + \beta(x)\delta(y)) = \alpha\beta(x)\delta(y)$ for all $x, y \in R$. Assume on the contrary that $a \neq 0$. Then since R is prime and β is an automorphism, $\delta(y) = 0$ for all $y \in R$. On the other hand, we also have $0 = af(xy) = a(\delta(x)\alpha(y) + \beta(x)f(y)) = \alpha\beta(x)f(y)$ for all $x, y \in R$. Hence $f(y) = 0$ for all $y \in R$ which is contrary to the hypothesis. This completes the proof of (i).

The proof of (ii) is similar.

Note that Lemma 4 does not hold for neither left nor right generalized (α, β)-derivation. Indeed, we have the following examples.

Example 3. Let $R = M_2(F)$. Let $a = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $b = s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s^{-1}$ and $c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $c(a+b) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$. Now, define $f(x) = ax + s^{-1}xsb$ for all $x \in R$. Then $f(x) = ax + bx = (a+b)x$ for all $x \in R$. Clearly, $cf(x) = c(a+b)x = 0$ for all $x \in R$. But, $c \neq 0$. Similarly, if let $g(x) = bs^{-1}xs + xa$ for all $x \in R$, then $g(x) = x(a+b)$ for all $x \in R$. Now let $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $e \neq 0$ but $g(x)e = 0$ for all $x \in R$.

Now we come to our main theorem

Theorem 1. *Let R be a prime ring, and let f, g and h be the generalized (α, β)-derivations of R associated with γ, δ and τ respectively. Assume that there exists $a, b \in Q \setminus C$ such that*

$$h(x) = af(x) + g(x)b \text{ for all } x \in R. \tag{6}$$

If either $f \neq 0$ or $g \neq 0$, then there exists $s \in Q$ such that $s\alpha(x)s^{-1} = \beta(x)$ for all $x \in R$ and one of the following holds:

(i) *$f(x) = s\alpha(x)$, $g(x) = s\alpha(x)$ and $h(x) = as\alpha(x) + \beta(x)sb$ for all $x \in R$, where $as + sb = s$ or 0.*

(ii) *$f(x) = s[b, \alpha(x)], g(x) = [a, \beta(x)]s$ and $h(x) = s[s^{-1}asb, \alpha(x)]$ for all $x \in R$.*

(iii) *$f(x) = s\alpha(x) + \eta s[b, \alpha(x)]$, $g(x) = s\alpha(x) + \eta[a, \beta(x)]s$ and $h(x) = s[\eta s^{-1}asb - b,\ \alpha(x)]$ for all $x \in R$, where $as + sb = 0$, $\eta \in C$.*

(iv) *$f(x) = s\alpha(x) + \eta s[b, \alpha(x)]$, $g(x) = s\alpha(x) + \eta[a, \beta(x)]s$ and $h(x) = s\alpha(x) + s[\eta s^{-1}asb - b, \alpha(x)]$ for all $x \in R$, where $as + sb = s$, $\eta \in C$.*

Proof. If f and g are all (α, β)-derivation, then (ii) holds by Theorem 1 in [6]. So we may assume that either f or g is a proper generalized (α, β)-derivation.

Substituting xy for x into (6), we have $af(x)\alpha(y) + a\beta(x)\gamma(y) + g(x)\alpha(y)b + \beta(x)\delta(y)b = h(x)\alpha(y) + \beta(x)\tau(y) = af(x)\alpha(x) + g(x)b\alpha(y) + \beta(x)\tau(y)$. Hence

$$[a, \beta(x)]\gamma(y) + g(x)[\alpha(y), b] = \beta(x)(\tau(y) - a\gamma(y) - \delta(y)b) \text{ for all } x, y \in R. \tag{7}$$

Using Lemma 3 and the hypothesis $h = af + gb$, we can rewrite (7) into the following form

$$[a, \beta(x)]\gamma(y) + \delta(x)[\alpha(y), b] = \beta(x)A\alpha(y) \text{ for all } x, y \in R. \tag{8}$$

where A is one of the values in $\{as+sb-s, as+sb, as-s, sb-s, as, sb, -s\}$. Substituting yz for y into (8), the left hand side of (8) becomes

$$\begin{aligned}
&[a,\beta(x)]\gamma(yz)+\delta(x)[\alpha(yz),b]\\
=&[a,\beta(x)](\gamma(y)\alpha(z)+\beta(y)\gamma(z))+\delta(x)[\alpha(y)\alpha(z),b]\\
=&[a,\beta(x)]\gamma(y)\alpha(z)+[a,\beta(x)]\beta(y)\gamma(z)+\delta(x)[\alpha(y),b]\alpha(z)+\delta(x)\alpha(y)[\alpha(z),b]\\
=&([a,\beta(x)]\gamma(y)+\delta(x)[\alpha(y),b])\alpha(z)+[a,\beta(x)]\beta(y)\gamma(z)+\delta(x)\alpha(y)[\alpha(z),b]\\
=&\beta(x)A\alpha(y)\alpha(z)+[a,\beta(x)]\beta(y)\gamma(z)-\delta(x)\alpha(y)[b,\alpha(z)].
\end{aligned}$$

Also the right hand side of (8) becomes $\beta(x)A\alpha(y)\alpha(z)$. Therefore, we have

$$[a,\beta(x)]\beta(y)\gamma(z)=\delta(x)\alpha(y)[b,\alpha(z)] \text{ for all } x,y,z\in R. \tag{9}$$

Form (9), it is easy to see that $\gamma=0$ if and only if $\delta=0$. So, if $g=0$ then $\delta=0$ and hence $\gamma=0$. But, by the hypothesis $f\neq 0$, hence $f=s\alpha$ by Lemma 3. By (7), we have $\beta(x)\tau(y)=0$ for all $x,y\in R$. Therefore, $\tau(x)=0$ and $h(x)=af(x)=as\alpha(x)$ for all $x\in R$. By Lemma 3, we have $as=s$, which is not the case. So $g\neq 0$. Similiarly, using an analogue of (7), we can show that $f\neq 0$. Therefore, $f\neq 0$ and $g\neq 0$.

The last paragraph tells us that either both f and g have zero (α,β)-derivations or both f and g have nonzero (α,β)-derivations. Suppose both f and g having zero (α,β)-derivations, then $f=s\alpha=g$ and $s\alpha=\beta s$ by Lemma 3. Therefore $h=af+gb=as\alpha+s\alpha b=as\alpha+\beta sb$. Since h is a generalized (α,β)-derivations, we must have $as+sb=s$ or 0 by Lemma 1 and Lemma 3. Hence (i) holds. Now suppose that f and g have nonzero (α,β)-derivations. Applying α^{-1} on each term of (9), we have

$$\begin{aligned}
&\alpha^{-1}([a,\beta(x)])\alpha^{-1}\beta(y)\alpha^{-1}\gamma(z)\\
&=\alpha^{-1}\delta(x)y\alpha^{-1}([b,\alpha(z)]) \text{ for all } x,y,z\in R.
\end{aligned}\tag{10}$$

Since either f or g is a proper generalized (α,β)-derivations, $\alpha^{-1}\beta$ is Q-inner by Lemma 3. In fact, $\alpha^{-1}\beta(x)=txt^{-1}$ for all $x\in R$, where $t=\alpha^{-1}(s)$ and $s\alpha=\beta s$. Substituting this into (10), we have

$$\begin{aligned}
&\alpha^{-1}([a,\beta(x)])tyt^{-1}\alpha^{-1}\gamma(z)\\
&=\alpha^{-1}\delta(x)y\alpha^{-1}([a,\alpha(z)]) \text{ for all } x,y,z\in R.
\end{aligned}\tag{11}$$

By a similar argument as we did before (e.g. [6]), there exists $\lambda\in C$ such that $\alpha^{-1}\delta(x)=\lambda\alpha^{-1}([a,\beta(x)])t$ and $t^{-1}\alpha^{-1}\gamma(z)=\lambda\alpha^{-1}([b,\alpha(z)])$ for all $x,z\in R$. Therefore $\delta(x)=[a,\beta(x)]\eta s$, $\gamma(x)=\eta s[b,\alpha(x)]$ for all $x\in R$, where $\eta=\alpha(\lambda)$.

Subsituting $\gamma(y) = \eta s[b, \alpha(y)]$ and $\delta(x) = [a, \beta(x)]\eta s$ into (8), we have $\beta(x)A\alpha(y) = 0$ for all $x, y \in R$ and hence $A = 0$. Since A must be one of the values in $\{as + sb - s, as + sb, as - s, sb - s, as, sb, -s\}$, it follows that either $as+sb-s = 0$ or $as+sb = 0$. If $as+sb-s = 0$, then $f(x) = s\alpha(x)+\eta s[b, \alpha(x)]$, $g(x) = s\alpha(x) + [a, \beta(x)]\eta s$ and $h(x) = as\alpha(x) + s\alpha(x)b + a\eta s[b, \alpha(x)] + [a, \beta(x)]\eta sb = s\alpha(x) - sb\alpha(x) + s\alpha(x)b + \eta as[b, \alpha(x)] + \eta[s^{-1}as, \alpha(x)]b = s\alpha(x) - s[b, \alpha(x)] + \eta s[s^{-1}asb, \alpha(x)] = s\alpha(x) + s[\eta s^{-1}asb - b, \alpha(x)]$ for all $x \in R$. If $as + sb = 0$, then $f(x) = s\alpha(x) + \eta s[b, \alpha(x)]$, $g(x) = s\alpha(x) + \eta[a, \beta(x)]s$ and $h(x) = as\alpha(x)+s\alpha(x)b+\eta as[b, \alpha(x)]+\eta[a, \beta(x)]sb = -sb\alpha(x)+s\alpha(x)b+\eta ss^{-1}as[b, \alpha(x)]+\eta s[s^{-1}as, \alpha(x)]b = -s[b, \alpha(x)]+s[\eta s^{-1}asb, \alpha(x)] = s[\eta s^{-1}asb - b, \alpha(x)]$ for all $x \in R$. Therefore, either (iii) or (iv) holds. This completes the proof of Theorem 1.

One should note that if there exists $s \in Q$ such that $s\alpha(x)s^{-1} = \beta(x)$ for all $x \in R$ and one of (i), (ii), (iii), (iv) holds, then $h(x) = af(x) + g(x)b$ for all $x \in R$.

As a corollary, we have

Corollary 1. *Let R be a prime ring, f and g generalized (α, β)-derivations of R associated with γ and δ respectively. Assume that there exists $a, b \in Q \setminus C$ such that $af(x) + g(x)b = 0$ for all $x \in R$. If either $f \neq 0$ or $g \neq 0$, then there exists $s \in Q$ such that $s\alpha(x)s^{-1} = \beta(x)$ for all $x \in R$ and one of the following holds*:

(i) *$f(x) = s[b, \alpha(x)]$, $g(x) = [a, \beta(x)]s$ for all $x \in R$ and $s^{-1}asb \in C$ $(\beta^{-1}(a)\alpha^{-1}(b) \in C)$.*

(ii) *$f(x) = s\alpha(x) + \eta s[b, \alpha(x)]$, $g(x) = s\alpha(x) + \eta[a, \beta(x)]s$ for all $x \in R$,* $as + sb = 0$ and $\eta sas^{-1}b - b \in C(\lambda\beta^{-1}(a)\alpha^{-1}(b) - b \in C)$.

Proof. This is a consequence of Theorem 1.

Corollary 2. *Let R be a prime ring, f a nonzero generalized (α, β)-derivations of R associated with δ. Let $a \in Q \setminus C$ be such that $[a, f(x)] = 0$ for all $x \in R$. Then there exists $s \in Q$ such that $s\alpha(x)s^{-1} = \beta(x)$ for all $x \in R$ and one of the following holds*:

(i) *$f(x) = [a, \beta(x)]s$ for all $x \in R$, $a + sas^{-1} \in C$ and $s^{-1}asa \in C$.*

(ii) *Char $R = 2$, $f(x) = s\alpha(x) + \eta s[a, \alpha(x)]$ for all $x \in R$, $[a, s] = 0$ and $\eta a^2 + a \in C$.*

Proof. We can appeal to corollary 1 with $b = -a$ and $g = f$. If Corollary 1 (i) holds, then $f(x) = [a, \beta(x)]s = s[-a, \alpha(x)] = [-sas^{-1}, \beta(x)]s$ for all $x \in R$

since $s\alpha(x)s^{-1} = \beta(x)$ for all $x \in R$. Therefore, $[a + sas^{-1}, \beta(x)] = 0$ for all $x \in R$. Hence $a + sas^{-1} \in C$. Also, $s^{-1}asa \in C$.

If Corollary 1 (ii) holds, then $f(x) = s\alpha(x) + \eta s[-a, \alpha(x)] = s\alpha(x) + \eta s[s^{-1}as, \alpha(x)]$ for all $x \in R$ since $s\alpha(x)s^{-1} = \beta(x)$ for all $x \in R$. Therefore $[a + s^{-1}as, \alpha(x)] = 0$ for all $x \in R$, and hence $a + s^{-1}as \in C$. On the other hand, we also have $as - sa = 0$ by Corollary 1 (ii). Hence $2a \in C$. If Char $R \neq 2$, then $a \in C$ which is not the case. So Char $R = 2$. Since $\eta sas^{-1}a + a \in C$ and since $as = sa$, we have, $\eta a^2 + a \in C$. This completes the proof.

Sharpen Corollary 2, we have

Corollary 3. *Let R be a prime ring of characteristic not two and let f be a nonzero proper generalized (α, β)-derivations of R. If $a \in R$ is such that $[a, f(x)] = 0$ for all $x \in R$, then $a \in Z$.*

Note that Corollary 3 is no longer true if f is just merely a left (right) generalized (α, β)-derivation of R. In fact, we have the following example.

Example 4. *Let $R = M_2(F)$, where F is a field with Char $F \neq 2$. Let*

$$a = b = c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Define $f(x) = ax + xb$ for all $x \in R$. Then $f(x) = (a + b)x + [x, b]$ is not a generalized $(1, 1)$-derivation since $(a + b)$ is a nonzero divisor in R. However, f is a right generalized $(1, 1)$-derivation associated with d_{-b}, where $d_{-b}(x) = [-b, x]$. Clearly, $c \notin Z$ and $[c, f(x)] = 0$ for all $x \in R$.

For the next result, we need a lemma.

Lemma 5. *Let f be a nonzero proper generalized (α, β)-derivation of a prime ring R with ceter Z. If $f(x) \in Z$ for all $x \in R$, then R is commutative.*

Proof. For $x \in R$ and $z \in Z$, we have $f(xz) = f(x)\alpha(z) + \beta(x)\delta(z)$. Since $f(x) \in Z$ for all $x \in R$, we have $\beta(x)\delta(z) \in Z$ for all $x \in R$ and for all $z \in Z$. Therefore, $y\delta(z) \in Z$. If $\delta(z) \neq 0$ for some $z \in Z$, then R is commutative and we are done. So, we may assume that $\delta(Z) = 0$. By Lemma 3, $f(x) = s\alpha(x) + \delta(x)$ for all $x \in R$. Therefore, $f(z) = s\alpha(z) \in Z$ for all $z \in Z$. In particular, $s \in C$ and $\alpha = \beta$.

If $\delta = 0$, then $f(x) = s\alpha(x) \in Z$ for all $x \in R$ and hence R is commutative. So we have assume that $\delta \neq 0$. Commute $f(x)$ with $\alpha(x)$, we get $[\delta(x), \alpha(x)] =$

0 for all $x \in R$ and hence $[\alpha^{-1}(\delta(x)), x] = 0$ for $x \in R$. Since $\alpha = \beta$, $\alpha^{-1}\delta$ is a derivation. By [13], R is commutative.

Note that Lemma 5 still holds for left (right) generalized (α, β)-derivations, but its proof is little bit harder than that of Lemma 5. We will prove it latter.

Combine Corollary 3 and Lemma 5, we have

Corollary 4. *Let R be a prime ring of characteristic not two and let f be a nonzero proper generalized (α, β)-derivation of R. If $[f(x), f(y)] = 0$ for all $x, y \in R$, then R is commutative.*

As usual, we can extend Corollary 3 and Corollary 4 in the following way.

Theorem 2. *Let R be a prime ring of characteristic not two and let f be a nonzero proper generalized (α, β)-derivation of R. If $a \in R$ is such that $[a, f(x)] \in Z$ for all $x \in R$, then $a \in Z$.*

Proof. Assume that f is a nonzero proper generalized (α, β)-derivation associated with δ. By Lemma 3, $f(x) = s\alpha(x) + \delta(x)$ for all $x \in R$. If $\delta = 0$, then $f(x) = s\alpha(x)$ for all $x \in R$. By the hypothesis, $[a, s\alpha(x)] \in Z$ for all $x \in R$. But since $s\alpha(R)$ is an ideal of R, so $a \in Z$. So we may assume that $\delta \neq 0$. Since $\alpha^{-1}\beta(x) = txt^{-1}$ for all $x \in R$, it follows easily that $\alpha^{-1}\delta(x) = td(x)$ for all $x \in R$, where d is a derivation of Q and $d(I) \subset R$ for some nonzero ideal I of R. Hence $\alpha^{-1}f(x) = tx + td(x)$ for all $x \in R$. By hypothesis, we have

$$[[b, tx + td(x)], y] = 0 \text{ for all } x, y \in R \tag{12}$$

where $b = \alpha^{-1}(a)$. If d is Q-inner, that is, there exists $c \in Q$ such that $d(x) = cx - xc$ for all $x \in Q$, then by Theorem 2 in [1], we have from (12) that

$$[[b, tx + t(cx - xc)], y] = 0$$

for all $x, y \in Q$. So without loss of generality, we may assume that $t, c \in R$, t is invertible in R and

$$[[b, tx + t(cx - xc)], y] = 0 \text{ for all } x, y \in R. \tag{13}$$

Substitute $0 \neq x = z \in Z$ into (13), we have $[b, t] \in Z$. Substitute $x = cz$ into (13), where $0 \neq z \in Z$, then we have $[b, tc] \in Z$. Since $[t, [b, tc]] = 0$ and $[tc, [t, b]] = 0$, we also have $[b, t[c, t]] = [b, [tc, t]] = 0$. Now substitute $x = t$ into (13), then we have $[b, t^2] = t[b, t] + [b, t]t = 2t[b, t] \in Z$. Hence $[b, t] = 0$ since Char $R \neq 2$. Therefore, $t[b, x + (cx - xc)] \in Z$ for all $x \in R$. Substitute $x = c^2$ into (13), then we have $[b, tc^2] = [b, tc]c + tc[b, c] = c[b, tc] + tct^{-1}[b, tc] =$

$(c+tct^{-1})[b,tc] \in Z$ and hence we have either $c+tct^{-1} \in Z$ or $[b,tc]=0$ since $[b,tc] \in Z$. If $c+tct^{-1} \in Z$, then $0=[b,c+tct^{-1}]=t^{-1}[b,tc]+[b,tc]t^{-1}=2t^{-1}[b,tc]$. Since Char $R \neq 2$, we have $[b,tc]=0$. So we can conclude that $[b,tc]=0$. Now replace x by cx in (13), we get $[b,tcx+tc(cx-xc)]=tc[b,x+(cx-xc)] \in Z$ for all $x \in R$. Since $tct^{-1}[b,x+(cx-xc)]=tc[b,x+(cx-xc)] \in Z$ for all $x \in R$ and since $t[b,x+(cx-xc)] \in Z$ for all $x \in R$, we have either $tct^{-1} \in Z$ or $t[b,x+(cx-xc)]=[b,tx+t(cx-xc)]=0$. If $tct^{-1} \in Z$, then $c \in Z$ and hence $d(x)=0$ for all $x \in R$. Therefore, $\delta(x)=0$ for all $x \in R$ which is not the case. If $[b,tx+t(cx-xc)]=0$ for all $x \in R$, then $[a,s\alpha(x)+\delta(x)]=0$ for all $x \in R$ and hence $a \in Z$ by Corollary 3. So we may assume that d is Q-outer. In this case, we have $[[b,tx+td(x)],y]=0$ for all $x,y \in Q$ by [12, Remark; p.14] and by [7, Theorem 1]. By a result of Kharchenko [11], $[[b,tx+tw],y]=0$ for all $x,w,y \in R$. In particular, $[b,x] \in C$ for all $x \in Q$. Hence again $a=\alpha(b) \in Z$.

As a Corollary, we have

Corollary 5. *Let R be a prime ring of characteristic not two and let f be a nonzero proper generalized (α,β)-derivation of R. If $[f(x),f(y)] \in Z$ for all $x,y \in R$, then R is commutative.*

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