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SYMPLECTIC SURFACES IN SYMPLECTIC 4-MANIFOLDS

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Abstract. Let a closed, minimal, symplectic 4-manifold X contain a symplectic surface F such that the genus g of F is greater than or equal to one and the value $c_1(TX)[F] > g$. Then we show that the space X is rational or ruled.

1. Introduction

Let X be a closed, connected, minimal symplectic 4-manifold with symplectic form ω. Let F be a symplectic surface in X satisfying $c_1(TX)[F] > 0$. In this case McDuff [10] proposed the following problem : Does it follow that the space X must be rational or ruled? This is true for minimal complex surfaces and when F is a rational curve with the intersection number $F \cdot F \geq 0$. Let us introduce a few theorems and use them in the process of proving our Theorem 1.4.

Theorem 1.1 [1]. *If a minimal complex surface X contains a curve C with $c_1(C) = -c_1(K_X) \cdot C > 0$, then X is rational or ruled.*

Theorem 1.2 [16]. *A minimal symplectic 4-manifold (X, ω) which contains a rational curve C with $C \cdot C \geq 0$ is symplectomorphic either to $\mathbb{CP}^2$ or to $S^2 \times S^2$ with the standard symplectic form.*

Remark. McDuff pointed out in [18] that Theorem 1.3 in [16] about the structure of symplectic S^2-bundles needs an extra hypothesis. The argument which proves uniqueness works only for a restricted range of cohomology

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classes, and the condition $a^2(V) > (a(F))^2$ need not hold when the base M of S^2-bundle V has genus > 0.

We say that a closed 2-manifold S is positively and symplectically immersed in (X, ω) if it is symplectically immersed (i.e. the restriction of ω to S does not vanish) and its only singularities are transverse double points of positive orientation.

In general, the singularities of a J-holomorphic curve need not all be transverse double points. However, it is proved in [15] that any J-holomorphic curve can be perturbed so that it is positively symplectically immersed in the above sense. McDuff [16] proved the problem for the case of immersed spheres.

Theorem 1.3 [16].

(1) *If a compact symplectic 4-manifold (X, ω) contains a positively symplectically immersed 2-sphere S with $c_1(S) \geq 2$, then (X, ω) is the blow up of a rational or ruled manifold.*

(2) *If S is not embedded, then X is rational.*

In this paper, we would like to give the solution, Theorem 1.4, of a restricted version of the problem proposed by McDuff. We introduce, in Section 2, some basic results about J-holomorphic curves in symplectic 4-manifolds, in Section 3, the Gromov invariant. In section 4, using McDuff's results and the Seiberg-Witten and Gromov invariants we will prove the following theorem:

Theorem 1.4. *Let X be a closed, connected, minimal symplectic 4-manifold containing a symplectic surface F with genus $g(F) = g \geq 1$ and $c_1(TX)[F] > g$. Then X is rational or ruled.*

2. J-Holomorphic Curves in 4-Manifold

In this section, we introduce the basic results about J-holomorphic curves on symplectic 4-manifolds which we will need later. For the convenience we will begin with a brief summary of the Gromov's theory. For more details see [12, 16, 9].

First recall that an almost complex structure J on a symplectic manifold (X, ω) is said to be ω-tame if ω is positive on all J-complex line in TX. We will denote the Sobolev space of all H^s-smooth ω-tame J by $\mathcal{J}(\omega)$, where s is suitably large. It is easy to check that $\mathcal{J}(\omega)$ is nonempty and contractible. Given a homology class $A \in H_2(X, \mathbb{Z})$ and an ω-tame J, a parameterized J-holomorphic A-curve is a map u from a Riemann surface (Σ_g, j) to (X, J) which represents the class A, and is J-holomorphic in the sense that $du \circ j = J \circ du$. We define the moduli space $\mathcal{M}_{A,g}$ by setting

$$\mathcal{M}_{A,g} = \{(u, j, J) \in \mathcal{F} \times \mathcal{J}(\Sigma_g) \times \mathcal{J}(\omega) : u \text{ is } (j, J) - \text{holomorphic}\},$$

where $\mathcal{F}$ is a suitable Sobolev space of somewhere injective maps $\Sigma_g \to X$, and $\mathcal{J}(\Sigma_g)$ is the genus g Teichmüller space. Let $P_A : \mathcal{M}_{A,g} \to \mathcal{J}(\omega)$ be the projection defined by $(u, j, J) \mapsto J$, and let $\mathcal{M}_p(A, J, \mathcal{J}_g)$ be the inverse image $P_A^{-1}(J)$. Then $\mathcal{M}_{A,g}$ is a Hilbert manifold and that the projection $P_A : \mathcal{M}_{A,g} \to \mathcal{J}(\omega)$ is Fredholm. For generic J in $\mathcal{J}(\omega)$, $\mathcal{M}_p(A, J, \mathcal{J}(\Sigma_g))$ is a manifold whose dimension is the index of the differential δP_A,

$$\mathrm{ind}(\delta P_A) = \dim \mathcal{J}(\Sigma_g) + 4(1-g) + 2c_1(A),$$

where $g = \mathrm{genus}(\Sigma_g)$, and $c_1(A)$ is the first Chern number of X on A.

The J-holomorphic image C in (X, ω) of a Riemann surface Σ_g of genus g must have $c_1(C) \geq 1-g$, for generic J. Indeed, given $K > 0$, we define $\mathcal{U}(K, g)$ to be the subset of $\mathcal{J}(\omega)$ containing of all J such that, for each holomogy class B such that $c_1(B) \leq -g$ and $\omega(B) \leq K$, there are no J-holomorphic images of Σ_g in the class B. Then $\mathcal{U}(K, g)$ is open, dense and path-connected.

Assume that J is C^∞. Then the virtual genus $g(C)$ of a closed curve C is de fined to be the number $g(C) = 1 + \frac{1}{2}(C \cdot C - c_1(TX)(C))$. If C is an embedded J-holomorphic curve of a Riemann surface Σ with genus g_Σ, the equalities $c_1(TX)(C) = [c_1(TC) + c_1(\nu_C)](C) = 2 - 2g_\Sigma + C \cdot C$ show that the virtual genus $g(C)$ equals the genus g_Σ of Σ. Let C be a J-holomorphic image of the closed Riemann surface of Σ with genus g_Σ. Then the virtual genus $g(C)$ is an integer which is greater than or equal to g_Σ, with equality if and only if C is embedded.

3. Pseudo-Holomorphic Curves.

In this section we prove that the moduli space of the pseudo-holomorphic curves is compact whenever suitable assumptions are made. We consider the moduli space $\mathcal{M}^G(X, e)$ of Gromov's pseudo-holomorphic curves which represent the Poincaré dual $\alpha = PD(e)$ of the class e.

Fix a Riemann surface Σ of genus g and consider the moduli space $\mathcal{M}^G(X, e, g)$ of all equivalence classes of pairs $[u, j]$ where $j \in \mathcal{J}(\Sigma)$ is a complex structure on Σ and $u : \Sigma \to X$ is a (j, J)-holomorphic map which represent the class α. The equivalence relation is given by the obvious action of the diffeomorphism group $\mathrm{Diff}(\Sigma)$ on $\mathrm{Map}(\Sigma, X) \times \mathcal{J}(\Sigma)$. If $2g-2 = c_1(K) \cdot e + e \cdot e$, the map $u : \Sigma \to X$ is an embedding for every pair $[u, j] \in \mathcal{M}^G(X, e, g)$. By [9], $\mathcal{M}^G(X, e, g)$ is the space of all embedded unparameterized pseudo-holomorphic curves representing α and dimension of $\mathcal{M}^G(X, e, g)$ is

$$\dim \mathcal{M}^G(X, e, g) = e \cdot e - c_1(K) \cdot e \equiv d.$$

The class $A \in H_2(X, \mathbb{Z})$ is called J-simple if it cannot be written as a sum $A_1 + \cdots + A_m$ for some $m \geq 2$ where

(1) each A_i has a J-holomorphic representative and
(2) if $A_i \neq A_j$, there is a sequence $A_i = A_{i_1}, A_{i_2}, \cdots, A_{i_n} = A_j$ such that positive intersection numbers $A_{i_p} \cdot A_{i_{p+1}} \geq 1$ for $p = 1, \cdots, n-1$.

By the same way as the proof of the completeness theorem in [24], we have the following lemma in the case of J-simple.

Lemma 3.1. *If A is J-simple, $\mathcal{M}^G(X, PD(A), g \geq 1)$ is compact.*

Let X be a closed, symplectic 4-manifold. Then the Gromov invariant is defined by the moduli space $\mathcal{M}^G(X, e, g)$ for each e in $H^2(X, \mathbb{Z})$. Let A is J-simple, $\mathcal{H} = \mathcal{H}(e, J, \Omega) = \{\{(C, 1)\}\}$. By the Proposition 4.3 in [23], we can choose a generic (J, Ω) such that C is non-degenerated. Hence $Gr(e) = \sum_{h \in \mathcal{H}} q(h) = q(\{(C,1)\}) = \prod_k r(C_k, m_k) = r(C, 1) = \pm 1$, where $r(\cdot)$ is an integer assignment to pairs (C, m) of positive integer m and pseudo-holomorphic submanifold $C \subset X$. Therefore we have the following Lemma.

Lemma 3.2. *If A is J-simple which is not represented by a torus with $A \cdot A = 0$, then $Gr(e) = \pm 1$, where $e = PD(A)$.*

4. Main Theorem

In this section we assume that X is a closed, minimal, symplectic 4-manifold containing a symplectic surface F of genus $g(F) = g \geq 1$ and $c_1(F) > g$. Since F is a symplectic 2-dimensional submanifold of X, there is a pseudo-holomorphic embedding $u : (\Sigma_g, j) \longrightarrow (X, J)$ such that $u(\Sigma_g) = F$, where Σ_g is a Riemann surface of genus g.

Lemma 4.1. *There is an element J of $\mathcal{U}_\infty$ such that the class $[F]$ can be represented by a J-holomorphic cusp-curve $S = S_1 \cup \cdots \cup S_m$, where, for each i, the class $A_i = [S_i]$ is J-simple and J is regular for A_i-curves.*

Proof. Let J_0 be an ω-tame almost complex structure for which F has a J_0-holomorphic parameterization. We may assume that J_0 is a regular value for the projection map P_F (defined in Section 2) and that it belongs to u_∞. As the corollary of the Gromov compactness theorem for pseudo-holomorphic curves, there is a neighborhood $\mathcal{N}(J_0)$ of J_0 in $\mathcal{J}(\omega)$ such that only finitely many classes A in $H_2(X, \mathbb{Z})$ with $\omega(A) \leq \omega(F)$ has J-holomorphic representative for some J in $\mathcal{N}(J_0)$. Hence we may further assume that J_0 is a regular value for all such P_A. Thus there is a finite-decomposition $A_1 + \cdots + A_m$ of F such that

(1) each A_i has a J-holomorphic representative S_i and

(2) if $A_i \neq A_j$, there is a sequence $A_i = A_{i_1}, A_{i_2}, \cdots, A_{i_n} = A_j$ such that intersection numbers $A_{i_p} \cdot A_{i_{p+1}} \geq 1$ for $p = 1, \cdots, n-1$.

Hence we may take $J = J_0$. Clearly, each such decomposition of $[F]$ gives rise either to a cusp-curve $S = S_1 \cup \cdots \cup S_m$ or a representation of $[F]$ as a multiply-covered curve. In the latter case, a multiply-covered curve can be represented by a cusp-curve. That is, by Corollary 2.3, $[F]$ has a decomposition with $m = 3$, $A_1 = A_2 = A_3$ and $A_1 \cdot A_2 = A_2 \cdot A_3 = A_3 \cdot A_1 = 1$, so that it also has a representation by a cusp-curve. If the constituent components of this cusp-curve are not simple, they may be decomposed further. And so, among the finite set of decompositions of $[F]$, there clearly is at least one such that each A_i is J_0-simple. ∎

Now we introduce McDuff's and Taubes's results which we will need later.

Lemma 4.2 [16]. *Let J be a regular value for the projection P_A defined in section 2. If A is a J-simple class which can be represented by an embedded J-holomorphic 2-sphere. Then $p = A \cdot A \leq 1$.*

Lemma 4.3 [13]. *If X contains a J-simple class which can be represented by an embedded 2-sphere with self-intersection 1, then $X = \mathbb{CP}^2$.*

Lemma 4.4 [14]. *Let C be a symplectically embedded rational A-curve in a symplectic 4-manifold (X, ω), where A is a simple homology class of self-intersection zero. Then there is a fibration $\pi : X \to M$ which is compatible with ω and has one fiber equal to C, where M is a compact 2-manifold.*

Theorem 4.5 [21]. *Let (X, ω) be a closed symplectic 4-manifold with $b_2^+(X) \geq 2$ and $C \subset X$ be a 2-dimensional symplectic submanifold with $C \cdot C \geq 0$. Then $c_1(K) \cdot C = -c_1(TX)(C) \geq 0$ and every line bundle $E \to X$ with $SW(X, L_E) \neq 0$ satisfies*

$$0 \leq c_1(E) \cdot C \leq c_1(K) \cdot C.$$

Corollary 4.6. *Let X be a compact symplectic 4-manifold containing a symplectically embedded 2-dimensional manifold F with $c_1(TX)(F) > 0$ and $g(F) \geq 1$. Then $b_2^+(X) = 1$.*

Proof. Suppose X is a closed symplectic 4-manifold with $b_2^+(X) \geq 2$. Let F be a symplectically embedded surface with $c_1(TX)(F) > 0$ and $g(F) \geq 1$. Then by the adjunction formula, $F \cdot F = c_1(TX)(F) + 2g - 2 > 0$. This contradicts Theorem 4.5. ∎

Since X is a closed symplectic 4-manifold containing a symplectically embedded surface F with $g(F) = g \geq 1$ and $c_1(TX)(F) > g > 0$, by Corollary 4.6, $b_2^+(X) = 1$. For simplicity, we denote $\mathcal{M}^G(X, PD(A), g)$ by $\mathcal{M}(X, J)$ when A is J-simple.

Theorem 4.7. *Let an almost complex structure J on X be generic. Suppose that A is a J-simple class which can be represented by an embedded J-holomorphic surface with genus $g \geq 1$. Then the intersection number $p = A \cdot A \leq g$.*

Proof. Since A is J-simple, by Lemma 3.1, $\mathcal{M}(A, J) \equiv \mathcal{M}^G(X, PD(A), g)$ is compact and the dimension of the moduli space $\mathcal{M}(A, J)$ is

$$\begin{aligned} \dim \mathcal{M}(A, J) &= 2c_1(TX)(A) + 2g - 2 \\ &= 2(A \cdot A + 2 - 2g) + 2g - 2 \\ &= 2p + 2 - 2g. \end{aligned}$$

Suppose that $p > g$. Let $\mathcal{M}(A, \mathcal{J})$ be the set of pairs (f, J) where $J \in \mathcal{J}(\omega)$ and $f \in \mathcal{M}(A, J)$. Consider the evaluation map coupled by $(p+1-g)$-tuple of Σ_g

$$e_A : \mathcal{M}(A, \mathcal{J}) \times \Sigma_g \times \cdots \times \Sigma_g \longrightarrow X \times \cdots \times X$$

given by $(f, J, z_1, \cdots, z_{p+1-g}) \mapsto (f(z_1), \cdots, f(z_{p+1-g}))$, where Σ_g is the Riemann surface of genus g and there are $p + 1 - g$ factors in each product. Let $j : X \to X \times \cdots \times X$ be an inclusion given by $z \mapsto (z, x_2, \cdots, x_{p+1-g})$, where $x_2, \cdots x_{p+1-g}$ are distinct fixed points in X. Then e_A is transverse to the inclusion j. let $R = e_A^{-1}(\mathrm{im} j) \cap P_A^{-1}(J)$ for a generic J. Then R is a compact 4-manifold. Let $e = pr \circ e_A$ be a map from R into X, where pr is the projection onto the first factor. Since $\mathcal{M}(A, J) \times \Sigma_g \times \cdots \times \Sigma_g$ is compact and $X \times \cdots \times X$ is connected, $e : R \to X$ is surjective, by the following Lemma 4.10. Let

$$\pi : \mathcal{M}(A, J) \times \Sigma_g \times \cdots \times \Sigma_g \longrightarrow \mathcal{M}(A, J)$$

be the obvious projection. Since R is a submanifold of $\mathcal{M}(A, J) \times \Sigma_g \times \cdots \times \Sigma_g$, $\pi(R)$ is a compact 2-dimensional submanifold of $\mathcal{M}(A, J)$. Then $R \simeq \pi(R) \times \Sigma_g$. We now claim that $\pi(R) \simeq S^2$. To see this, we identify the tangent space to X at x_2 with $\mathbb{C}^2$ and consider the map $R \to \mathbb{CP}^1 = S^2$ given by

$$(f, J, z_1, \cdots, z_{p+1-g}) \mapsto T_{x_2} f \equiv \text{ the tangent space to } \mathrm{im} f \text{ at } f(z_2) = x_2.$$

This map is well-defined since all the elements of R are embeddings by (2.3) of section 2. Further, it clearly factors through π, so that we get a map

$$\theta : \pi(R) \longrightarrow S^2.$$

Then θ is injective. If not, that is, $T_{x_2}f = T_{x_2}g$ and $f \neq g$ in $\pi(R)$, $(f \cdot g)_{x_2} \geq 2$. Where for simplicity, we denote $f \equiv \mathrm{im} f$, $g \equiv \mathrm{im} g$. Since f and g meet at $x_2, \cdots, x_{p+1-g}$ and $(f \cdot g) = p$ the surface representing $f \cup g$ has at most genus $2g + p - 2$. By the adjunction formula

$$2g(f+g) - 2 \geq (f+g)^2 - c_1(f+g) = 4g - 4 + 2p.$$

This is impossible. Hence θ must be a homeomorphism. That is, R is a Σ_g-bundle over S^2. Since an intersection point of two distinct J-holomorphic curves C and C' always occurs with positive orientation, e preserves the orientation. Hence by Lemma 3.4 $\deg(e) = Gr(e) > 0$. Then $e^* : H^*(X, \mathbb{R}) \to H^*(R, \mathbb{R})$ is injective. Since $b_2^+(X) = 1$, $b_2^-(X) = 0$, or 1.

If $b_2^- = 0$, then by the following Theorem 4.8, the intersection form Q_X of X is $Q_X = (+1)$. Let $H^*(X)$ be the integral cohomology of X modulo torsion. Then $H^2(X) = \mathbb{Z}[\alpha]$ and $\alpha \cdot \alpha = 1$. Since $A \in H_2(X, \mathbb{Z})$, $a = PD(A) \in H^2(X, \mathbb{Z})$. Let $\pi : H^2(X, \mathbb{Z}) \to H^2(X)$ be the projection map and $\tilde{a} = \pi(a)$. Then $\tilde{a} \in H^2(X) = \mathbb{Z}[\alpha]$. Since X is a symplectic 4-manifold with $b_2^+(X) = 1$ and $b_2^-(X) = 0$,

$$c_1(TX) = k\alpha \quad \text{and} \quad c_1(TX)^2 = k^2 = 2\chi(X) + 3\sigma(X) = 9 - 4b_1(X).$$

Hence $c_1(TX) = 3\alpha$ or $c_1(TX) = \alpha$. If $c_1(TX) = 3\alpha$, then $c_1(TX) \cdot \alpha = 3$ and

$$\dim \mathcal{M}^G(X, \alpha, g) = 2g - 2 + 2c_1(TX)(\alpha) = 4 > 0.$$

Hence $PD(\alpha)$ is represented by an embedded J-holomorphic curve of genus 0. If $c_1(TX) = \alpha$, then $c_1(TX) \cdot \alpha = 1$ and

$$\dim \mathcal{M}^G(X, \alpha, g) = 2g - 2 + 2c_1(TX)(\alpha) = 2 > 0.$$

Hence $PD(\alpha)$ is represented by an embedded J-holomorphic curve of genus 1. Since $PD(\alpha)$ can be represented by an embedded J-holomorphic curve of genus $g = 0$ or 1 and A is J-simple, $\tilde{a} = \pm\alpha$. $a^2 = \tilde{a}^2 = (\pm\alpha)^2 = 1$. This contradicts the fact $p > g = 1$. Therefore $b_2^-(X) = 1$. Then by the following Theorem 4.9, the intersection from Q_X of X is either

$$Q_X = (100-1) \quad \text{or} \quad Q_X = (0110)$$

Therefore $H^2(X) = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta]$ and $\alpha \cdot \alpha = 0 = \beta \cdot \beta$ or $\alpha \cdot \alpha = 1$ and $\beta \cdot \beta = -1$. Similarly, since X is a symplectic 4-manifold with $b_2^+(X) = 1$ and $b_2^-(X) = 1$,

$$c_1(TX) = k_1\alpha + k_2\beta \quad \text{and} \quad c_1(TX)^2 = 2\chi(X) + 3\sigma(X) = 8 - 4b_1.$$

First, consider the case $\alpha \cdot \alpha = 0 = \beta \cdot \beta$. Then $c_1(X)^2 = 2k_1k_2 = 8 - 4b_1$. Hence $b_1 = 0$ and $k_1 = 1$, $k_2 = 4$ or $k_1 = 2$, $k_2 = 2$. Therefore $c_1(TX) = \alpha + 4\beta$ or

$2\alpha+2\beta$. By the adjunction formula, $c_1(TX)\cdot\alpha = 2$ or $c_1(TX)\cdot\beta = 2$. Hence $\dim\mathcal{M}^G(X,\alpha,g) = 2g-2+2c_1(TX)(PD\alpha) = 2 > 0$ and $\dim\mathcal{M}^G(X,\beta) = 2g-2+2c_1(TX)(PD(\beta)) = 2 > 0$. Therefore $PD(\alpha)$ is represented by an embedded J-holomorphic curve of genus 0 or $PD(\beta)$ is represented by an embedded J-holomorphic curve of genus 0. Similarly we consider the case $\alpha\cdot\alpha = 1$ and $\beta\cdot\beta = -1$. Then $c_1(TX) = 3\alpha+\beta$ or $\alpha+\beta$. Since $c_1(TX)\cdot\alpha = 1$ or 3 and $c_1(TX)\cdot\beta = 1$, $\dim\mathcal{M}^G(X,\alpha,g) > 0$ and $\dim\mathcal{M}^G(X,\beta,g) > 0$. Hence $PD(\beta)$ is represented by an embedded J-holomorphic curve of genus 0. That is $PD(\beta)$ is an exceptional curve. Since X is minimal, this is impossible. Since Poincaré duals of α and β can be represented by an embedded J-holomorphic curves of genus $g = 0, 1$ or 2 and A is J-simple, $\tilde{a} = \pm\alpha$ or $\pm\beta$. Therefore $a^2 = \tilde{a}^2 \leq 1$. This is impossible. ∎

Theorem 4.8 (Donaldson). *If X is a compact oriented smooth 4-manifold with definite intersection form then Q_X is diagonalizable over the integers.*

Theorem 4.9 (Hasse-Minkowski). *Let Q be a unimodular quadratic form over the integers. If Q is odd and indefinite then it can be diagonalized over $\mathbb{Z}$ and thus*

$$Q \sim l(1) \oplus m(-1)$$

for some positive integers l and m. If Q is even and indefinite then it is equivalent to the form

$$Q \sim lE_8 \oplus mH$$

for some integers l and $m \geq 1$.

Let the evaluation map

$$e_0 : \mathcal{M}(A,\mathcal{J}) \longrightarrow X$$

for a fixed $z_0 \in \Sigma_g$ be defined by $e_0(f,J) = f(z_0)$.

Lemma 4.10. *For every point $z_0 \in \Sigma_g$ the map $e_0 : \mathcal{M}(A,\mathcal{J}) \to X$ is a submersion.*

Proof. It can be proved by the same way as the proof of Theorem 6.1.1 in [13].

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Now we assume that F is a symplectically embedded surface with $g(F) = g \geq 1$ and $c_1(TX)(F) > g$. Then by Lemma 4.1, there is an almost complex structure J of $\mathcal{U}_\infty$ such that the class $[F]$ can be represented by a J-holomorphic cusp-curve $S = S_1 \cup \cdots \cup S_m$, where for each i, the class $A_i = [S_i]$

is J-simple and J is regular for A_i-curves. By the adjunction formula,

$$\begin{aligned}2g(F)-2 &= F\cdot F - c_1(TX)(F)\\ &= (A_1+\cdots+A_m)\cdot(A_1+\cdots+A_m)\\ &\quad - (c_1(TX)(A_1)+\cdots+c_1(TX)(A_m))\\ &= 2g(S_1)+\cdots+2g(S_m)-2m+2(A_1\cdot A_2+\cdots A_{m-1}\cdot A_m).\end{aligned}$$

Since S is a cusp-curve, $0\le g(S_1)+\cdots+g(S_m)\le g$.

(1) If $g(S_1)+\cdots+g(S_m)=0$, then $g(S_i)=0$ for all $i=1,\cdots,m$. Since $c_1(TX)(A_i)\ge 1$, $A_i\cdot A_i = c_1(TX)(A_i)-2\ge -1$ for all $i=1,\cdots,m$. By Lemma 4.2, $A_i\cdot A_i=-1,0$ or 1 for all $i=1,\cdots,m$. But since X is minimal, X does not contain an exceptional sphere. Hence $A_i\cdot A_i=0$ or 1 for all $i=2,\cdots,m$. Then by Lemma 4.3 and Lemma 4.4, X is rational or ruled.

(2) If $g(S_1)+\cdots+g(S_m)=1$, then $g(S_1)=1$ and $g(S_i)=0$ for all $i=2,\cdots,m$. By (1), $A_i\cdot A_i=0,1$ for all $i=2,\cdots,m$. If there is a class A_i such that $A_i\cdot A_i=1$, then $m>1$ and by Lemma 4.3, $X=\mathbb{CP}^2$. Since $m>1$, X contains a J-simple class A_1 which can be represented by a symplectically embedded surface of genus $g=1$. Every homology class in $H_2(\mathbb{CP}^2,\mathbb{Z})$ is of the form dH and dH is a J-simple iff $d=1$. Hence $d=1$. Since
$$g(dH)=\frac{d^2-3d+2}{2}=0,$$
this contradicts our assumption. Hence $A_i\cdot A_i=0$ for all $i=2,\cdots,m$. Also, by the adjunction formula, $A_1\cdot A_1 = 2g-2+c_1(TX)(A_1)\ge 2g-2+1-g=g-1=0$ and by Lemma 4.7, $A_1\cdot A_1=0$ or 1. Hence $c_1(TX)(A_1)=0$ or 1. If $c_1(TX)(F)=c_1(TX)(A_1)+\cdots c_1(TX)(A_m)>g=1$, then $m>1$. Therefore X contains a J-simple which can be represented by a symplectically embedded 2-sphere with self-intersection 0. Then by Lemma 4.4, X is ruled.

(3) Suppose that $g(S_1)+\cdots+g(S_m)=t\le g$ (i.e. $1<t\le g$). Let k be the number of S_i such that $g(S_i)=1$. Then $0\le k\le t$.

  (i) If $k=0$, then $g(S_i)\ge 2$ or $g(S_i)=0$. If $g(S_i)=0$, then by (2), $S_i\cdot S_i=0$. We have $c_1(TX)(S_i)=2$. If $g(S_i)\ge 2$, then by Theorem 4.8, $S_i\cdot S_i=g-1$, or g. We have $c_1(TX)(S_i)\le 0$. Since $c_1(TX)(F)=c_1(TX)(A_1)+\cdots+c_1(TX)(A_m)>g$, there is a J-simple class A which can be represented by an embedded 2-sphere with self-intersection 0. Then X is ruled.

  (ii) If $k>0$, then $g(S_1)=\cdots=g(S_k)=1$ and $g(S_i)\ge 2$ or $g(S_i)=0$ for all $i=k+1,\cdots,m$ and $k+g(S_{k+1})+\cdots+g(S_m)=t$. By (i), if $g(S_i) = 0$, then $c_1(TX)(A_i) = 2$, and if $g(S_i) \geq 2$, then $c_1(TX)(A_i) < 0$. If $g(S_i) = 0$, then $c_1(TX)(A_i) = 0$ or 1. Since $c_1(TX)(F) = c_1(TX)(A_1) + \cdots + c_1(TX)(A_m) > g$, there is a J-simple class A which can be represented by an embedded 2-sphere with self-intersection 0. Then X is ruled.

Therefore we have the following theorem:

Theorem 4.11. *Let X be a closed, minimal, symplectic 4-manifold containing a symplectic surface F satisfying $g(F) = g \geq 1$ and $c_1(TX)[F] > g$. Then X is rational or ruled.*

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