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ON GENERALIZED EXPONENTS OF TOURNAMENTS

Bo Zhou* and Jian Shen

Abstract. We determine two types of generalized exponent sets for tournaments with given order. In the course of proving the main results we find the following result, which may be interesting in its own right: When n is large enough, almost all tournaments on n vertices have the property that there is a path of length 2 from each vertex u to each vertex $v \neq u$.

1. Introduction

Let $G = (V, E)$ be a digraph on n vertices. The notation $u \xrightarrow{k} v$ (resp. $u \overset{k}{\not\to} v$) is used to indicate that there is a walk (no walk) of length k from u to v. By $u \to v$ we denote $u \xrightarrow{1} v$. Similarly, for a set $X \subseteq V$, the notation $X \xrightarrow{k} v$ (resp. $X \overset{k}{\not\to} v$) means that $u \xrightarrow{k} v$ for some $u \in X$ (resp. $u \overset{k}{\not\to} v$ for each $u \in X$).

A digraph G is said to be primitive if there exists an integer $p \geq 1$ such that $u \xrightarrow{p} v$ for all $u, v \in V(G)$ (possibly $u = v$). The minimum such p is called the exponent of G, denoted by $\exp(G)$.

Suppose G is a primitive digraph of order n. The exponent of a set $X \subseteq V$, denoted by $\exp_G(X)$, is the smallest integer $p \geq 1$ such that $X \xrightarrow{p} v$ for each $v \in V$. For $x \in V$, we use $\exp_G(x)$ to denote $\exp_G(\{x\})$ for simplicity. Without loss of generality, let the vertices $v_1, v_2, \ldots, v_n$ of G be ordered such that $\exp_G(v_1) \leq \exp_G(v_2) \leq \cdots \leq \exp_G(v_n)$. Then $\exp_G(v_k)$ is called the first type k-th generalized exponent of G, denoted by $\exp(G, k)$. Let $F(G, k) = \max\{\exp_G(X) : |X| = k\}$ with $1 \leq k \leq n$. Then $F(G, k)$ is called the second type k-th generalized exponent of G. Clearly we have $\exp(G, n) = F(G, 1) = \exp(G)$.

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A tournament is a digraph in which every pair of distinct vertices is joined by exactly one arc. It is well known [3] that each vertex of a strong (strongly connected) tournament of order n is contained in at least one cycle of each length between 3 and n, inclusive, and hence [7] that a tournament of order n is primitive if and only if it is strong and $n \geq 4$.

Let PT_n be the set of all primitive tournament of order n, and let $E_1(n,k) = \{\exp(T,k) : T \in PT_n\}$ and $E_2(n,k) = \{F(T,k) : T \in PT_n\}$ be the two generalized exponent sets of PT_n. Let $e_1(n,k) = \max\{m : m \in E_1(n,k)\}$ and $e_2(n,k) = \max\{m : m \in E_2(n,k)\}$.

In [7], Moon and Pullman proved that $e_1(n,n) = n + 2$ for $n \geq 5$ and that $E_1(n,n) = \{3, \ldots, n + 2\}$ for $n \geq 6$. In [4, 5], Liu proved that for $n \geq 7$

$$e_1(n,k) = k + 2 \tag{1}$$

and

$$e_2(n,k) = \begin{cases} n - k + 3 & k = 1, 2, \\ n - k + 2 & 3 \leq k \leq n - 1, \\ 1 & k = n. \end{cases} \tag{2}$$

The main purpose of this note is to determine the sets $E_1(n,k)$ and $E_2(n,k)$.

2. The Exponent Set $E_1(n,k)$

In this section we will determine the exponent set $E_1(n,k)$ for $n \geq 7$. First we construct a family of tournaments $T_{n,r}$ which will give us a number of exponents in the sets $E_1(n,k)$ and $E_2(n,k)$.

Let n, r be integers with $1 \leq r \leq n - 2$, $n \geq 7$. The tournament $T_{n,r}$ with vertex set $V = \{1, 2, \ldots, n\}$ is defined as follows: The arcs $(n, n-1), \ldots, (2, 1)$ are in $T_{n,r}$ and so are the arcs (j, i) if $1 \leq i < j \leq r + 1$; arcs not yet specified are all oriented towards vertices with larger numbers. Clearly $T_{n,r}$ contains a cycle of length n and hence it is primitive.

Lemma 2.1. *Suppose that* $1 \leq r \leq n - 4$, $n \geq 7$. *Then*

$$\exp_{T_{n,r}}(k) = \begin{cases} 3 & 1 \leq k \leq r, \\ 3 + k - r & r + 1 \leq k \leq n. \end{cases}$$

Proof. There are two cases to consider.

Case 1. $1 \leq k \leq r$. Then $k \to r + 2 \to r + 1 \to j$ if $1 \leq j \leq r$, $k \to j + 2 \to j + 1 \to j$ if $r + 1 \leq j \leq n - 2$, $k \to n - 3 \to n \to n - 1$ if $j = n - 1$, and $k \to n - 1 \to n - 2 \to n$ if $j = n$. Hence $k \xrightarrow{3} j$ for any j with

$1 \le j \le n$. Note that the smallest positive length of walks from each vertex to itself is 3. We have $\exp_{T_{n,r}}(k) = 3$.

Case 2. $r + 1 \le k \le n$. Then $k \xrightarrow{k-r} r \xrightarrow{3} j$ and hence $k \xrightarrow{3+k-r} j$ for any j with $1 \le j \le n$. We have $\exp_{T_{n,r}}(k) \le 3 + k - r$. Since the smallest length from k to r is $k - r$, we have $k \overset{2+k-r}{\not\to} r$, which implies $\exp_{T_{n,r}}(k) \ge 3 + k - r$. Hence $\exp_{T_{n,r}}(k) = 3 + k - r$. ∎

Let $T'_{n,r}$ be the tournament obtained from $T_{n,r}$ by replacing the arc $(r, 1)$ with $(1, r)$. By using some similar arguments as in Lemma 2.1, one can prove the following lemma.

Lemma 2.2. *For* $n \ge 7$,

$$\exp_{T'_{n,n-4}}(k) = \begin{cases} 3 & 1 \le k \le n-3, \\ 4 & k = n-2, \\ 5 & k = n-1, \\ 6 & k = n; \end{cases}$$

$$\exp_{T'_{n,n-3}}(k) = \begin{cases} 3 & 1 \le k \le n-2, \\ 4 & k = n-1, \\ 5 & k = n; \end{cases}$$

$$\exp_{T'_{n,n-2}}(k) = \begin{cases} 3 & 1 \le k \le n-1, \\ 4 & k = n. \end{cases}$$

Theorem 2.3. $E_1(n, k) = \{3, \dots, k+2\}$ *for* $n \ge 7$.

Proof. For any $T \in PT_n$, clearly we have $\exp(T, k) \ge 3$. By (1) we have $E_1(n, k) \subseteq \{3, \dots, k+2\}$. In the following we are going to prove the converse conclusion.

Let r be an integer with $1 \le r \le n-4$. By Lemma 2.1,

$$\exp_{T_{n,r}}(k) = \begin{cases} 3 & 1 \le k \le r, \\ 3 + k - r & r + 1 \le k \le n. \end{cases}$$

Then (let $r = k, k-1, \dots, 1$)

$$\{3, \dots, k+2\} \subseteq E_1(n, k) \quad \text{if} \quad 1 \le k \le n-4 \tag{3}$$

and (let $r = n-4, n-5, \dots, 1$)

$$\{k - n + 7, \dots, k+2\} \subseteq E_1(n, k) \quad \text{if} \quad n-3 \le k \le n. \tag{4}$$

By Lemma 2.2, $k-n+6 = \exp(T'_{n,n-4},k) \in E_1(n,k)$ if $k = n-3, n-2, n-1, n$, $k-n+5 = \exp(T'_{n,n-3},k) \in E_1(n,k)$ if $k = n-2, n-1, n$, $k-n+4 = \exp(T'_{n,n-2},k) \in E_1(n,k)$ if $k = n-1, n$. Note that $\{3\} \subseteq E_1(n,n)$ (see [7]). Then

$$\{3,\ldots,k-n+6\} \subseteq E_1(n,k) \quad \text{if} \quad n-3 \le k \le n. \tag{5}$$

By combining (3), (4) and (5), we have $\{3,\ldots,k+2\} \subseteq E_1(n,k)$ for any k with $1 \le k \le n$. The proof is completed. ∎

3. The Exponent Set $E_2(n,k)$

In this section we will determine the exponent set $E_2(n,k)$ for $n \ge 7$. Note that $F(T,1) = \exp(T,n)$ and $F(T,n) = 1$ for any $T \in PT_n$. We have $E_2(n,1) = E_1(n,n)$ and $E_2(n,n) = \{1\}$. Thus we only need to consider the case $2 \le k \le n-1$.

Lemma 3.1. *For* $1 \le r \le n-4$,

$$F(T_{n,r},k) = \begin{cases} n-r+2 & k=2, \\ n-k-r+3 & 3 \le k \le n-r, \\ 3 & n-r+1 \le k \le n-1. \end{cases}$$

Proof. We write $G = T_{n,r}$. Let $X \subseteq \{1,\ldots,n\}$ with $|X| = k$ and let $1 \le i \le n$. Note that $j \xrightarrow{l} i$ for any $1 \le j \le r$ and $l \ge 3$ (see Lemma 2.1). If X contains some j with $1 \le j \le r$, then $j \xrightarrow{3} i$ and hence $X \xrightarrow{3} i$ for any $1 \le i \le n$, implying $\exp_G(X) \le 3$. Suppose X contains no j with $1 \le j \le r$.

Case 1. $k=2$. X contains some j with $r+1 \le j \le n-1$. Then $j \xrightarrow{j-r} r \xrightarrow{l} i$ for any $1 \le i \le n$ and $l \ge 3$. On setting $l = n-j+2$ we have $j \xrightarrow{n+r-2} i$ and hence $X \xrightarrow{n+r-2} i$ for any $1 \le i \le n$, which implies $F(G,2) \le n-r+2$. Take $X_0 = \{n,n-1\}$. Clearly $X_0 \not\xrightarrow{n-r+1} r$. Thus $F(G,2) \ge \exp_G(X_0) \ge n-r+2$. It follows that $F(G,2) = n-r+2$.

Case 2. $k \ge n-r$. Then $k = n-r$, $X = \{r+1,\ldots,n\}$, $r+3 \xrightarrow{3} i$ for any $1 \le i \le r$, $i \xrightarrow{3} i$ for any $r+1 \le i \le n$ and hence $X \xrightarrow{3} i$ for any $1 \le i \le n$. It follows that $F(G,k) \le 3$. Take $X_0 = \{1,\ldots,n\} \backslash \{r+2\}$. Then $X_0 \not\xrightarrow{2} r$ and we have $F(G,k) \ge \exp_G(X_0) \ge 3$. Thus $F(G,k) = 3$.

Case 3. $3 \le k \le n-r-1$. Let $j = \min\{u : u \in X\}$. Then $r+1 \le j \le n-k+1$. If $r+1 \le j \le n-k$, then $j \xrightarrow{j-r} r \xrightarrow{l} i$ for any $1 \le i \le n$ and

$l = n - k - j + 3 \geq 3$, which implies $j \xrightarrow{n-k-r+3} i$ and hence $X \xrightarrow{n-k-r+3} i$ for any $1 \leq i \leq n$. Suppose $j = n - k + 1$. Then $X = \{n-k+1, \dots, n\}$. Note that $n - k + 3 \xrightarrow{n-k-r+3} i$ for any $1 \leq i \leq r$, $n - k + 1 \xrightarrow{n-k-r+1} r \to i + 1 \to i$ for any $r + 1 \leq i \leq n - 1$, $n - k + 1 \xrightarrow{n-k-r+1} r \to n - 2 \to n$. We have $X \xrightarrow{n-k-r+3} i$. It follows that $F(G, k) \leq n - k - r + 3$. Take $X_0 = \{n-k, \dots, n\} \setminus \{n-k+2\}$. Then $X_0 \overset{n-k-r+2}{\not\to} r$ and so $F(G, k) \geq \exp_G(X_0) \geq n - k - r + 3$. Hence $F(G, k) = n - k - r + 3$. ■

By using some similar arguments as in Lemma 3.1, one can prove the following lemmas.

Lemma 3.2. *For* $n \geq 7$,

$$F(T'_{n,n-3}, k) = \begin{cases} 4 & k = 2; \\ 3 & 3 \leq k \leq n-3; \\ 2 & k = n-2, n-1; \end{cases}$$

$$F(T'_{n,n-2}, k) = \begin{cases} 3 & 2 \leq k \leq n-3; \\ 2 & k = n-2, n-1. \end{cases}$$

Lemma 3.3. *Let* $T''_{n,n-3}$ *be the tournament obtained from* $T_{n,n-3}$ *by replacing the arc* $(n-3, 2)$ *with* $(2, n-3)$. *Then*

$$F(T''_{n,n-3}, 2) = 5.$$

Let T_n *be the tournament containing the arc* (i, j) *if and only if* $0 < j - i \leq \lfloor \frac{1}{2} n \rfloor$, *where subtraction is taken modulo* n *or* $n+1$ *according to* n *is odd or even. Then*

$$F(T_n, k) = \begin{cases} 3 & k = 2, 3; \\ 2 & 4 \leq k \leq \lfloor \frac{n}{2} \rfloor + 1; \\ 1 & k \geq \lfloor \frac{n}{2} \rfloor + 2; \end{cases}$$

Let T'_n *be the tournament obtained from* T_n *by replacing the arcs* $(i, 1)$ *with* $(1, i)$ *for all* $\lfloor \frac{n}{2} \rfloor \leq i \leq n-3$. *Then*

$$F(T'_n, k) = \begin{cases} 3 & k = 2; \\ 2 & 4 \leq k \leq n-3; \\ 1 & k = n-2, n-1; \end{cases} \quad \text{if } n \text{ is odd};$$

$$F(T'_n, k) = \begin{cases} 3 & k = 2, 3, 4 \\ 2 & 5 \leq k \leq n-3; \\ 1 & k = n-2, n-1; \end{cases} \quad \text{if } n \text{ is even}.$$

Lemma 3.4. *Suppose* $T \in PT_n$ *and* $k \le \lfloor n/2 \rfloor + 1$. *Then* $F(T,k) \ge 2$.

Proof. Suppose $F(T,k) = 1$. Then the in-degree of every vertex is at least $n-k+1$ and hence T contains at least $n(n-k+1)$ arcs. This yields $n(n-1)/2 \ge n(n-k+1)$; that is, $k \ge \lfloor n/2 \rfloor + 2$, a contradiction. ■

In order to show $2 \in E_2(n,2)$, one needs to find a tournament T on n vertices with $F(T,2) = 2$. For some special n, we know that an explicit construction of such a tournament is possible. For example, when n is prime, $n \ge 7$ and $n \equiv 3 \pmod 4$, it can be proved that every Paley tournament on n vertices has the following property: there is a path of length 2 from each vertex u to each vertex $v \ne u$. This property implies $F(T,2) = 2$. (See [2, page 193] for the definition of a Paley tournament.) Our next result shows that a desired tournament on n vertices always exists for every $n \ge 24$, although it is not constructible from our proof.

Theorem 3.5. *For any integer* $n \ge 24$, *there exists a tournament* T *on* n *vertices with the following property*:

there is a path of length 2 *from each vertex* u *to each vertex* $v \ne u$.

Furthermore, *for sufficiently large* n, *almost all tournaments on* n *vertices have this property.*

Proof. Let $\mathcal{A}$ be the set of all tournaments with vertex set $[n] = \{1, 2, \ldots, n\}$. Then $|\mathcal{A}| = 2^{\binom{n}{2}}$. For any $i, j \in [n]$ with $i \ne j$, let $\text{Prob}\{i \xrightarrow{2} j\}$ (resp. $\text{Prob}\{i \overset{2}{\nrightarrow} j\}$) denote the probability of $i \xrightarrow{2} j$ (resp. $i \overset{2}{\nrightarrow} j$) among all tournaments in $\mathcal{A}$. Then

$$\text{Prob}\{i \overset{2}{\nrightarrow} j\} = \text{Prob}\{i \to k \to j \text{ holds for no } k \text{ with } k \ne i, j\} = \left(\frac{3}{4}\right)^{n-2},$$

and

$$\begin{aligned}\text{Prob}\{i \xrightarrow{2} j \text{ for all distinct pairs of vertices } i, j\} &\ge 1 - \sum_{i \ne j \in [n]} \text{Prob}\{i \overset{2}{\nrightarrow} j\} \\ &= 1 - n(n-1)\left(\frac{3}{4}\right)^{n-2} \\ &> 0 \text{ when } n \ge 24.\end{aligned}$$

Thus, for every $n \ge 24$, there exists some tournament $T \in \mathcal{A}$ such that $i \xrightarrow{2} j$ for all distinct vertices i, j. Furthermore, since $n(n-1)\left(\frac{3}{4}\right)^{n-2}$ approaches to 0 when n approaches to infinity, the second part of the theorem follows immediately. ■

Remark 1: The condition $n \geq 24$ in Theorem 3.5 is not tight and the bound 24 can almost certainly be decreased. On the other hand, the condition $n \geq 24$ cannot be removed entirely since Theorem 3.5 fails for $n = 4$ at least.

Remark 2: Theorem 3.5 may be interesting in its own right. It echoes the following result of Moon and Moser [6]: Almost all n by n (0,1)-matrices are primitive with exponent 2. (Recall that the adjacency digraph of a (0,1)-matrix may have digons, while a tournament contains none of them.)

Theorem 3.6. *For* $n \geq 7$ *and* $4 \leq k \leq n-1$ *or* $n \geq 24$ *and* $k = 2, 3$,

$$E_2(n,k) = \begin{cases} \{2, \ldots, n-k+3\} & k = 2, \\ \{2, \ldots, n-k+2\} & 3 \leq k \leq \lfloor n/2 \rfloor + 1, \\ \{1, \ldots, n-k+2\} & \lfloor n/2 \rfloor + 2 \leq k \leq n-1. \end{cases}$$

Proof. Let r be an integer with $1 \leq r \leq n-4$. By Lemma 3.1,

$$F(T_{n,r}, k) = \begin{cases} n-r+2 & k = 2, \\ n-k-r+3 & 3 \leq k \leq n-r, \\ 3 & n-r+1 \leq k \leq n-1. \end{cases}$$

Then (let $r = n-4, n-5, \ldots, 1$)

$$\left. \begin{array}{ll} \{6, \ldots, n+1\} & \text{for } k = 2 \\ \{4, \ldots, n-1\} & \text{for } k = 3 \\ \{3, \ldots, n-k+2\} & \text{for } 4 \leq k \leq n-1 \end{array} \right\} \subseteq E_2(n,k). \tag{6}$$

By Lemmas 3.2 and 3.3,

$$3 = F(T'_{n,n-2}, 2),\ 4 = F(T'_{n,n-3}, 2),\ 5 = F(T''_{n,n-3}, 2) \in E_2(n,2);$$

$$3 = F(T'_{n,n-2}, 3) \in E_2(n,3);$$

$$2 \in E_2(n,k) \text{ for } 4 \leq k \leq n-1;$$

$$1 \in E_2(n,k) \text{ for } \lfloor n/2 \rfloor + 2 \leq k \leq n-1.$$

By Theorem 3.5, $2 \in E_2(n,2)$ and hence $2 \in E_2(n,3)$ for $n \geq 24$. Hence we have

$$\{2, \ldots, n-k+3\} \subseteq E_2(n,k) \text{ for } k = 2;$$

$$\{2, \ldots, n-k+2\} \subseteq E_2(n,k) \text{ for } k = 3;$$

$$\{2, \ldots, n-k+2\} \subseteq E_2(n,k) \text{ for } 4 \leq k \leq \lfloor \tfrac{n}{2} \rfloor + 1;$$

$$\{1, \ldots, n-k+2\} \subseteq E_2(n,k) \text{ for } \lfloor n/2 \rfloor + 2 \leq k \leq n-1.$$

The proof is completed. ∎

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Bo Zhou
Research Center for Fundamental Science, Xi'an Jiaotong University
Xi'an 710049, P. R. China
Department of Mathematics, South China Normal University
Guangzhou 510631, P. R. China

Jian Shen
Department of Mathematics, Southwest Texas State University
San Marcos, TX 78666 USA