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ON THE SOLVABILITY OF A NONSELFADJOINT QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract. A solvability theorem is obtained for a quasilinear elliptic boundary value problem. The linear part of the problem is an elliptic operator of order $2m$ which has a nontrivial kernel not necessarily symmetric. The nonlinear part may grow sublinearly and contain derivatives of order up to $2m$. The proof is based on Borsuk's Theorem and the Nussbaum–Sodovskii degree.

1. Introduction

Let Ω be a bounded open set of R^N with sufficiently smooth boundary $\partial\Omega$ and let

$$Au = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha u$$

be a linear elliptic partial differential operator of order $2m$ with a nontrivial kernel acting on the Sobolev space $H^{2m}(\Omega)$. We consider the problem

$$\begin{aligned} Au(x) &= f(x; D^{\alpha_1}u(x); \cdots ; D^{\alpha_k}u(x)); \quad x \in \Omega \\ Bu(x) &= 0; \quad x \in \partial\Omega; \end{aligned} \tag{1}$$

where each α_i is a multiindex with $|\alpha_i| \leq 2m$, and B represents a system of m linear boundary operators of order less than $2m - 1$ defined on $\partial\Omega$. This type of problems was first studied by Landesman and Lazer ([7]) in 1970 and later extended by many authors in various directions. In this paper we consider the problem (1) with the nonlinearity which may grow sublinearly and contain derivatives of order up to $2m$. The solvability condition obtained below improves those in [1, 3, 6-9, 10, 12]. Our results is based on Borsuk's theorem and the Nussbaum-Sadovskii degree for condensing maps (see [2]).

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2. The Main Theorem

Let $H^{2m}(\Omega)$ be the usual Sobolev space with the norm

$$\|u\| = \left[\sum_{|®| \le 2m} \int_\Omega |D^{®} u(x)|^2 \, dx \right]^{1=2}$$

and let $X = \{u \in H^{2m}(\Omega) | Bu = 0\}$. We assume that

$$L : X \to L^2(\Omega); \qquad L(u) = A(u)$$

is a Fredholm operator with index zero and has a nontrivial kernel of finite dimension $l \ge 1$. Then there are orthonormal L^2-bases $(w_1; \cdots ; w_l)$ of $\mathrm{Ker}(L)$ and $(Á_1; \cdots ; Á_l)$ of $\mathrm{Im}(L)^\perp$. Moreover, we assume that L has the unique continuation property, that is, if $w \in \mathrm{Ker}(L)$ and $w \ne 0$, then $D^{®_i} w(x) = 0$ only on a set of measure zero in Ω for $®_i$, $1 \le i \le k$.

We assume further that $f : \Omega \times R^k \to R$ has the form

$$f(x; t_1; \cdots ; t_k) = f_1(x; t_1) + \cdots + f_k(x; t_k);$$

where $f_i : \Omega \times R \to R$ is a Caratheodory function for $1 \le i \le k$ satisfying the following conditions:

$$|f(x; t_1; \cdots ; t_k)| \le a(|t_1|^{¾_1} + \cdots + |t_k|^{¾_k}) + b(x) \tag{2}$$

for some $a > 0$, $b \in L^2(\Omega)$ and $0 \le ¾_i < 1$;

$$|f(x; t; y_1) - f(x; t; y_2)| \le \bar{}\,|y_1 - y_2| \tag{3}$$

for all $x \in \Omega$, $t \in R^{s(2m)-s(2m-1)}$ and $y_1, y_2 \in R^{s(2m)-s(2m-1)}$, where $s(j)$ denotes the number of those multiindices $®$ with $|®| \le j$ and $\bar{} > 0$ is a constant such that with $\bar{}\,\|L^{-1}\| < 1$, in which L^{-1} is the right inverse of L, and the $2k$ limits

$$\lim_{t_1 \to \pm\infty} \frac{f_1(x; t_1)}{|t_1|^{¾}} = f_1^{\pm}(x); \cdots ; \lim_{t_k \to \pm\infty} \frac{f_k(x; t_k)}{|t_k|^{¾}} = f_k^{\pm}(x) \tag{4}$$

exist, where $¾ = \max\{¾_i\}$.

For a fixed $w \in \mathrm{Ker}(L)$ which does not vanish, we define the $2k$ sets

$$\Omega_1^{\pm} = \{x \in \Omega | D^{®_1} w(x) \; ? \; 0\}; \cdots ; \Omega_k^{\pm} = \{x \in \Omega | D^{®_k} w(x) \; ? \; 0\};$$

so that Ω is divided into two parts in the following k ways

$$|\Omega| = |\Omega_1^+ \cup \Omega_1^-| = \cdots = |\Omega_k^+ \cup \Omega_k^-|;$$

where $|\cdot|$ denotes the Lebesgue measure in R^N. We denote by $< \cdot; \cdot >$ the inner product of $L^2(\Omega)$ and set

$$S(w) = \sum_{i=1}^{l} < w; w_i > Á_i$$

for $w \in \mathrm{Ker}(L)$.

Main Theorem. *Let all the assumptions stated above be satisfied. If*

$$\begin{aligned}(5)\qquad &\left[\int_{\Omega_1^+} f_1^+(x)|D^{®_1}w(x)|^{¾}S(w)(x)\,dx + \cdots \right.\\ &\left. + \int_{\Omega_k^i} f_k^-(x)|D^{®_k}w(x)|^{¾}S(w)(x)\,dx\right]\\ &\neq t\left[\int_{\Omega_1^+} f_1^-(x)|D^{®_1}w(x)|^{¾}S(w)(x)\,dx + \cdots \right.\\ &\left. + \int_{\Omega_k^i} f_k^+(x)|D^{®_k}w(x)|^{¾}S(w)(x)dx\right];\end{aligned}$$

for $w \in$ *Ker*$(L) \setminus \{0\}$ *and* $t \in [0; 1]$; *then the problem* (1) *has at least one solution in* X.

Proof. Define $F : X \to L^2(\Omega)$ by

$$F(u)(x) = f(x; D^{®_1}u(x); \cdots ; D^{®_k}u(x)):$$

Then the problem (1) is equivalent to the operator equation

$$(6)\qquad L(u) = F(u):$$

We define two projection operators $P : X \to X$, $Q : L^2(\Omega) \to L^2(\Omega)$ by $P(u) = \sum_{i=1}^{l} < u; w_i > w_i$, $Q(v) = \sum_{i=1}^{l} < v; Á_i > Á_i$. Therefore, $\mathrm{Im}(P) = \mathrm{Ker}(L)$, $\mathrm{Im}(L) = \mathrm{Ker}(Q)$ and $L^{-1} : \mathrm{Ker}(Q) \to \mathrm{Ker}(P)$ is bounded by the open mapping theorem. First, we claim there exists an integer $n \geq 1$ such that

$$(L - F)(u) \neq t(L - F)(-u); \quad u \in @B(n); \quad t \in [0; 1]$$

where $B(n) = \{u \in X \mid \|u\| < n\}$ and $@B(n) = \{u \in X \mid \|u\| = n\}$. If this is not the case, we can find out two sequences (u_n), $u_n \in @B(n)$, and (t_n), $t_n \in (0; 1)$, such that

$$(L - F)(u_n) = t_n(L - F)(-u_n)$$

or

$$(1+t_n)L(u_n) = F(u_n) - t_nF(-u_n):$$

From above we obtain

$$(1+t_n)L(u_n) = (I-Q)\big[F(u_n) - t_nF(-u_n)\big]; \tag{7}$$

$$JQ\big[F(u_n) - t_nF(-u_n)\big] = 0; \quad J = S^{-1}: \tag{8}$$

From (7) we know that

$$(I-P)(u_n) = \frac{1}{1+t_n}L^{-1}(I-Q)\big[F(u_n) - t_nF(-u_n)\big]:$$

Using (2) it follows after a tedious calculation that

$$\begin{aligned} ||(I-P)(u_n)|| &\le ||L^{-1}||\,||F(u_n) - t_nF(-u_n)||_{L^2}; \\ &\le ||L^{-1}||\big[ak^{1-\frac{¾}{2}}|\Omega|^{\frac{1\,¡\,¾}{2}}||u_n||^{¾} + ||b||_{L^2}\big]; \end{aligned}$$

which implies that

$$||(I-P)(u_n)||/||u_n|| \to 0 \text{ and } ||P(u_n)|| \to \infty$$

as $n \to \infty$. Let $w_n = P(u_n)/||P(u_n)||$. Then $||w_n|| = 1$. As $\dim \mathrm{Ker}(L) < \infty$, we may select a subsequence of (u_n) also denoted by (u_n) such that

$$w_n + (I-P)(u_n)/||P(u_n)|| \to w \text{ in } X$$

and

$$D^{®_i}(w_n + (I-P)(u_n)/||P(u_n)||) \to D^{®_i}w \text{ in } L^2(\Omega)$$

for any $1 \le i \le k$. Moreover

$$w_n(x) + (I-P)(u_n)(x)/||P(u_n)|| \to w(x) \text{ a. e. } x \in \Omega$$

and

$$D^{®_i}(w_n(x) + (I-P)(u_n)(x)/||P(u_n)||) \to D^{®_i}w(x) \text{ a. e. }$$
$$x \in \Omega \text{ for each } 1 \le i \le k:$$

Taking the inner product with w in (8) we have

$$\begin{aligned} 0 &= \,< JQ(F(u_n) - t_nF(-u_n)); w > \\ &= \,< Q(F(u_n) - t_nF(-u_n)); J^*(w) > \\ &= \,< Q(F(u_n) - t_nF(-u_n)); S(w) > \\ &= \,< F(u_n) - t_nF(-u_n); S(w) > : \end{aligned}$$

From the above we get

$$\begin{aligned}&\int_\Omega f(x; D^{®_1}u_n(x); \cdots ; D^{®_k}u_n(x))S(w)(x)\,dx\\&= t_n\int_\Omega f(x; D^{®_1}(-u_n(x)); \cdots ; D^{®_k}(-u_n(x)))S(w)(x)\,dx\end{aligned}$$

so that

$$\begin{aligned}&\int_\Omega (f_1(x; D^{®_1}u_n(x)) + \cdots + f_k(x; D^{®_k}u_n(x)))/||P(u_n)||^{¾}\cdot S(w)(x)\,dx\\&= t_n\int_\Omega (f_1(x; -D^{®_1}u_n(x)) + \cdots + f_k(x; -D^{®_k}u_n(x)))/||P(u_n)||^{¾}\\&\quad\cdot S(w)(x)\,dx:\end{aligned}\tag{9}$$

Now by (4) for $1 \le i \le k$

$$\begin{aligned}&\lim_{n\to\infty} f_i(x; D^{®_i}u_n(x))/||P(u_n)||^{¾}\\&= \lim_{n\to\infty}\frac{f_i(x; D^{®_i}(||P(u_n)||(w_n(x) + (I-P)(u_n)(x)/||P(u_n)||)))}{||P(u_n)||^{¾}|D^{®_i}(w_n(x) + (I-P)(u_n)(x)/||P(u_n)||)|^{¾}}\\&\quad\cdot||D^{®_i}(w_n(x) + (I-P)(u_n)(x)/||P(u_n)||)|^{¾}\\&= f_i^{\pm}(x)|D^{®_i}w(x)|^{¾}\ \text{a. e. in}\ \Omega_i^+(\Omega_i^-):\end{aligned}\tag{10}$$

Similarly we have

$$\lim_{n\to\infty} t_n f_i(x; -D^{®_i}u_n(x))/||P(u_n)||^{¾} = t f_i^{\mp}(x)|D^{®_i}w(x)|^{¾}\ \text{a. e. in}\ \Omega_i^+(\Omega_i^-):\tag{11}$$

Since the sequences

$$(f_i(x; D^{®_i}u_n)/||P(u_n)||^{¾})\ \text{and}\ (t_n f_i(x; -D^{®_i}u_n)/||P(u_n)||^{¾})$$

are bounded in $L^2(\Omega)$ and hence in $L^2(\Omega_i^{\pm})$, their weak limits are just the pointwise limits given in (10) and (11). Consequently, we get

$$\begin{aligned}&\lim_{n\to\infty}\left[\int_{\Omega_1^+} f_1(x; D^{®_1}u_n(x))/||P(u_n)||^{¾}S(w)(x)\,dx + \cdots\right.\\&\quad\left.+\int_{\Omega_k^i} f_k(x; D^{®_k}u_n(x))/||P(u_n)||^{¾}S(w)(x)\,dx\right]\\&= \left[\int_{\Omega_1^+} f_1^+(x)|D^{®_1}w(x)|^{¾}S(w)(x)\,dx + \cdots\right.\\&\quad\left.+\int_{\Omega_k^i} f_k^-(x)|D^{®_k}w(x)|^{¾}S(w)(x)\,dx\right]\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} t_n \Bigg[\int_{\Omega_1^+} f_1(x; -D^{®_1}u_n(x))/||P(u_n)||^{¾} S(w)(x)\,dx + \cdots \\
&\quad + \int_{\Omega_k^i} f_k(x; -D^{®_k}u_n(x))/||P(u_n)||^{¾} S(w)(x)\,dx\Bigg] \\
&= t\Bigg[\int_{\Omega_1^+} f_1^-(x)|D^{®_1}w(x)|^{¾} S(w)(x)\,dx + \cdots \\
&\quad + \int_{\Omega_k^i} f_k^+(x)|D^{®_k}w(x)|^{¾} S(w)(x)\,dx\Bigg]:
\end{aligned}$$

Using the unique continuation property and taking the limits in both sides of (9), we have

$$\begin{aligned}
&\Bigg[\int_{\Omega_1^+} f_1^+(x)|D^{®_1}w(x)|^{¾} S(w)(x)\,dx + \cdots \\
&\quad + \int_{\Omega_k^i} f_k^-(x)|D^{®_k}w(x)|^{¾} S(w)(x)dx\Bigg] \\
&= t\Bigg[\int_{\Omega_1^+} f_1^-(x)|D^{®_1}w(x)|^{¾} S(w)(x)\,dx + \cdots \\
&\quad + \int_{\Omega_k^i} f_k^+(x)|D^{®_k}w(x)|^{¾} S(w)(x)\,dx\Bigg];
\end{aligned}$$

which leads to a contradiction to (5).

To complete the proof, let n be the number chosen above. We define the map $T : X \to X$ by

$$T(u) = P(u) + JQF(u) + L^{-1}(I - Q)F(u):$$

Then T is a condensing map. This can be proved in the following way. We define

$$\begin{aligned}
&S : H^{2m}(\Omega) \to H^{2m}(\Omega); \\
&S(u; v) = AF(x; D^{´}u; D^{³}v);
\end{aligned}$$

where $A = L^{-1}(I - Q) : L^2(\Omega) \to X$, $D^{´}u = \{(D^{®}u) : 0 \le |®| \le 2n - 1\}$ and $D^{³}v = \{(D^{®}v) : |®| = 2m\}$. Then for each fixed $v \in H^{2m}(\Omega)$ the map $S(\cdot; v) : H^{2m}(\Omega) \to H^{2m}(\Omega)$ is compact as $H^{2m}(\Omega)$ is compactly embedded in $H^{2m-1}(\Omega)$ (see [4]), and for each fixed $u \in H^{2m}(\Omega)$ the map $S(u; \cdot) : H^{2m}(\Omega) \to H^{2m}(\Omega)$

is a contraction, since

$$
\begin{aligned}
&\|S(u;v_1)-S(u;v_2)\| = \|AF(x;D^{\eta}u;D^{\zeta}v_1)-AF(x;D^{\eta}u;D^{\zeta}v_2)\| \\
&\quad\le \|A\|\Big(\int |f(x;D^{\alpha_1}u(x);\cdots;D^{\alpha_W}u(x);D^{\alpha_{W+1}}v_1(x);\cdots;D^{\alpha_k}v_1(x)\big) \\
&\quad -f(x;D^{\alpha_1}u(x);\cdots;D^{\alpha_W}u(x);D^{\alpha_{W+1}}v_2(x);\cdots;D^{\alpha_k}v_2(x))|^2\,dx\Big)^{\frac12} \\
&\quad\le \|A\|\Big(\int \beta^2|(D^{\alpha_{W+1}}(v_1(x)-v_2(x)\big);\cdots;D^{\alpha_k}(v_1(x)-v_2(x)))|^2dx\Big)^{\frac12} \\
&\quad\le \beta\|L^{-1}(I-Q)\|\,\|v_1-v_2\| < \|v_1-v_2\|;
\end{aligned}
$$

where $W = s(2m)-s(2m-1)$. Set $L^{-1}(I-Q)F(u) = S(u;u)$. This proves that T is a condensing map (see [11], Theorem 1). Thus we can use the Nussbaum-Sodovskii degree to solve the following operator equation

$$u - T(u) = 0$$

which is equivalent to the operator equation (6).

For each $u \in X$, $t \in [0;1]$ we define the operator equations

$$H(t;u) = u - P(u) - (JQ + L^{-1}(I-Q))\left[\frac{1}{1+t}(F(u)-tF(-u))\right]:$$

Then we have $H(t;u) \neq 0$ for all $t \in [0;1]$, $u \in \partial B(n)$. By the homotopy invariance of degree theory and the Borsuk odd mapping theorem we have

$$\deg[H(0;\cdot);B(n);0] = \deg[H(1;\cdot);B(n);0] \neq 0:$$

Since $H(0;u) = 0$ is equivalent to our original equation (6), this implies the existence of a solution of (1) and the theorem is proved.

3. Examples

We give two examples as application of the main theorem.

Example 1. Let us consider the boundary value problem

$$\Delta u + 2u = u^{\frac{q}{p}}; \quad u\big|_{\partial\Omega} = 0;$$

where $\Omega = (0;\pi)\times(0;\pi)$ and p, q are odd integers, $1 \le q < p$. Then the kernel of L is spanned by the function $w(x;y) = \sin x \sin y$. We have $X = H^2(\Omega)\cap H^1_0(\Omega)$, and $\Omega^+ = \Omega$, $\Omega^- = \emptyset$, $f^+(x;y) = 1$, $f^-(x;y) = -1$. The inequality (5) becomes

$$\begin{aligned}&\int_0^{¼}\int_0^{¼} |k\sin x\sin y|^{\frac{q}{p}} k\sin x\sin y\,dxdy\\ &\neq t\int_0^{¼}\int_0^{¼}(-1)|k\sin x\sin y|^{\frac{q}{p}} k\sin x\sin y\,dxdy;\end{aligned}$$

where $k \neq 0$, and since

$$\int_0^{¼}\int_0^{¼} |\sin x\sin y|^{1+\frac{q}{p}}\,dxdy \neq t(-1)\int_0^{¼}\int_0^{¼} |\sin x\sin y|^{1+\frac{q}{p}}\,dxdy;$$

it follows that the boundary value problem (12) has at least one solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Example 2. Let us consider the equation

$$\Delta u + 2u = (u_x)^{\frac{q}{p}} \tag{13}$$

under the boundary condition

$$\begin{cases} u(0;y) = u_x\left(\frac{¼}{2};y\right) = 0; & \text{if } 0 < y < \frac{¼}{2}\\ u_y(x;0) = u|\left(x;\frac{¼}{2}\right) = 0; & \text{if } 0 < x < \frac{¼}{2}; \end{cases} \tag{14}$$

where $\Omega = (0;\frac{¼}{2}) \times (0;\frac{¼}{2})$ and p, q are odd integers, $1 \leq q < p$ as above. Then the kernel of L is spanned by the function $w(x;y) = \sin x\cos y$. We have $X = \{u \in H^2(\Omega) : u \text{ satisfies (14)}\}$, and $\Omega^+ = \{(x;y) \in \Omega : D^{(1;0)}w > 0\} = \{(x;y) \in \Omega : \cos x\cos y > 0\} = \Omega$, $\Omega^- = \emptyset$, $f^+(x;y) = 1$, $f^-(x;y) = -1$. The inequality (5) becomes

$$\begin{aligned}&\int_0^{\frac{¼}{2}}\int_0^{\frac{¼}{2}} |k\cos x\cos y|^{\frac{q}{p}} k\sin x\cos y\,dxdy\\ &\neq t\int_0^{\frac{¼}{2}}\int_0^{\frac{¼}{2}}(-1)|k\cos x\cos y|^{\frac{q}{p}} k\sin x\cos y\,dxdy;\end{aligned}$$

where $k \neq 0$, and since

$$\begin{aligned}&\int_0^{\frac{¼}{2}}\int_0^{\frac{¼}{2}} |\cos x\cos y|^{\frac{q}{p}} \sin x\cos y\,dxdy\\ &\neq t(-1)\int_0^{\frac{¼}{2}}\int_0^{\frac{¼}{2}} |\cos x\cos y|^{\frac{q}{p}} \sin x\cos y\,dxdy;\end{aligned}$$

it follows that the boundary value problem (13)-(14) has at least one solution $u \in X$.

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