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ON THE COMPACTNESS AND THE MINIMIZATION

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Abstract. In this article, we present various compactness and minimizations from very classical theorems such as Bolzano–Weierstrass theorem to modern theorems such as Palais–Smale conditions. Then apply them to assert the existence and multiplicity of solutions of partial differential equations.

1. Introduction

In this article, we present various compactness and minimizations from the very classical theorems such as Bolzana Weierstrass theorem to modern theorems such as Palais–Smale conditions. Then we find various conditions and apply them to assert the existence and multiplicity of solutions of partial differential equations.

In section 2, we present the compactness theorems in R^N, general Banach spaces, function spaces $C(K)$, L^p spaces, and Sobolev spaces $H_0^1(\Omega)$. In section 3, we describe the minimizers and Lagrange multiplier theorem with application to solve partial differential equations. In section 2, we need to investigate every sequence to see if it is nice. This is not economic. For example, to solve the equation (2) in Theorem 20. We associate the equation (2) with an energy functional J. The critical points of J are exactly the same as the solutions of equation (2). Thus, it suffices to find the critical points of J. Therefore, we need only to investigate the sequences related to the functional J. On the other hand, in section 3, we present the minimizers. In Theorem 16, we impose some coercive and weakly lower semicontinuous conditions on the function f to assure that there is a u such that

$$f(u) = \min_{v \in M} f(v);$$

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then apply the Lagrange multiplier theorem 18 to find such u. But it does not work if a function f satisfies neither the coercive condition nor weakly lower semicontinuous condition. Instead of finding the minimizer

$$f(u) = \min_{v \in M} f(v);$$

we try to find the saddle point u satisfying

$$f(u) = \inf_{°} \max_{t} f(°(t)):$$

Therefore, we need to consider the new compactness concepts and the new minimizers in section 4: for example let f be of C^1,

$$® = \min_{v \in M} f(v) \text{ or } ® = \inf_{°} \max_{t} f(°(t));$$

and for every minimizing sequence $\{u_n\}$ such that

$$f(u_n) \to ®;$$
$$f'(u_n) \to 0;$$

there is a subsequence, again denoted by $\{u_n\}$ and u such that

$$u_n \to u:$$

Then $f(u_n) \to ® = f(u)$. This is the Palais Smale conditions. In section 4, we present various Palais Smale conditions, and apply them to assert the existence and multiplicity of solutions of partial differential equations.

2. Compactness

Two main tools in analysis are the compactness and the minimization. The fundamental theorem of algebra was proved by the compactness in analysis. The fundamental and the most important compactness theorem is the following result in R^N:

Theorem 1. (*Bolzana-Weierstrass*) *Let* $\{u_n\}$ *be a bounded sequence in* R^N; *then there exist a subsequence* $\{u_{n_i}\}$ *and* $u \in R^N$ *such that* u_{n_i} *converges* (*strongly*) *to* u.

Proof. First, suppose that there are only finite different points in $\{u_n\}$, then we can find a v which occurs infinite many times in $\{u_n\}$, take such a subsequence $\{u_n\}$ by letting each $u_n = v$, then we are done. On the other hand, suppose that there are infinite many different point in $\{u_n\}$. Let J_1 be a closed cube with edge length l in R^N such that $\{u_n\} \subset J_1$. Bisecting each edge of J_1, we obtain from it 2^N closed subcubes of edge length $l/2$. One of the subcubes J_2 should be contain infinite u_n. If we continue bisecting, then we obtain a sequence of closed cubes $\{J_n\}$ such that $J_1 \supset J_2 \supset \cdots$ and

$$\lim_{n\to\infty} \operatorname{diam}(J_n) = 0. \tag{1}$$

By the finite intersection property and (1), there is a u such that

$$\{u\} = \bigcap_{n=1}^{\infty} J_n.$$

Take a subsequence $\{u_{n_i}\}$ such that $u_{n_i} \in J_i$ for each i. Then u_{n_i} converges to u. ■

In the following we note that the strongly convergence and the weakly convergence are equivalent in R^N.

Theorem 2. *Let* $\{u_n\}$ *and* u *be in* R^N, *then* $\|u_n - u\| \to 0$ *if and only if* $u_n \rightharpoonup u$ *weakly.*

Proof. It suffices to prove that if u_n converges weakly to u, then u_n converges strongly to u. Suppose that u_n converges weakly to u and $R^N = \langle e_1, \ldots, e_N \rangle$. Then for each i,

$$\langle u_n - u, e_i \rangle = o(1) \text{ as } n \to \infty.$$

Now for $\varphi = \sum_{i=1}^{N} a_i e_i$,

$$\begin{aligned} \|u_n - u\| &= \sup_{\|\varphi\| \le 1} \langle u_n - u, \varphi \rangle \\ &= \sup_{\|\varphi\| \le 1} \sum_{i=1}^{N} a_i \langle u_n - u, e_i \rangle \\ &\le \sum_{i=1}^{N} |\langle u_n - u, e_i \rangle| \\ &= o(1) \text{ as } n \to \infty. \end{aligned}$$

■

From the following Theorem 3, we have that the only normed linear space with compact balls is the Euclidean space R^N.

Theorem 3. (*Riesz*) *Let* X *be a normed linear space. If the unit sphere* $\{x \in X \mid \|x\| \le 1\}$ *is compact, then* $X = R^N$.

Proof. See Brézis [3, Theorem VI.5]. ∎

The compactness concepts can be extended to various infinite dimensional spaces.

Theorem 4. (*Banach – Alaoglu – Bourbaki*) *Let* X *be a separable Banach Space and* $\{u_n\}$ *be a bounded sequence in* X^*; *then there exist a subsequence* $\{u_n\}$ *and* $u \in X^*$ *such that* u_n * u *weak**.

Proof. See Brézis [3, Theorem III.15 and III.25] and Marsden-Hoffman [16, Theorem 3.1.3]. ∎

If the spaces are reflexive Banach Spaces, then Theorem 4 can be simplified.

Theorem 5. *Let* X *be a reflexive Banach Space and* $\{u_n\}$ *be a bounded sequence in* X; *then there exist a subsequence* $\{u_n\}$ *and* $u \in X$ *such that* u_n * u *weakly.*

There is a nice extension of the classical compactness theorem 1 to the continuous function space $C(K)$.

Theorem 6. (*Arzela-Ascoli*) *Let* K *be a compact metric space and* $C(K)$ *the set of all continuous functions in* K. *Let* $\{u_n\}$ *be a bounded and equicontinuous sequence in* $C(K)$; *then there exist a subsequence* $\{u_n\}$ *and* $u \in C(K)$ *such that* $u_n \to u$ *uniformly.*

Proof. See Yosida [24, p. 85]. ∎

There is an analogue result of Theorem 6 in $L^p(\Omega)$ spaces.

Theorem 7. (*Frechet – Kolmogorov*) *Let* Ω *be a domain in* R^N *and* $L^p(\Omega)$ *the set of all* L^p*-functions in* Ω; *where* $1 \le p < \infty$. *Let* $\{u_n\}$ *be a bounded sequence in* $L^p(\Omega)$. *Suppose that* $\{u_n\}$ *is an equicontinuous sequence in* $L^p(\Omega)$, *that is*

(i) *for* " > 0; ! ⊂⊂ Ω; *there exists* ± > 0; ± < dist(!; Ω^c) *such that*

$$\|(u_n)_h - u_n\|_{L^p(!)} < \text{"} \text{ for } h \in R^N;\ \|h\| < \pm;\ \text{ for } n \in N;$$

(ii) *for* " > 0; *there exists* ! ⊂⊂ Ω *such that* $\|u_n\|_{L^p(\Omega\setminus !)} <$ " *for* $n \in N$, *then there exist a subsequence* $\{u_n\}$ *and* $u \in L^p(\Omega)$ *such that* $u_n \to u$ *in* $L^p(\Omega)$.

Proof. See Brézis [3, Theorem IV.25].

The Lebesgue dominated convergence theorem also is a well-known compactness theorem.

Theorem 8. (*Lebesgue Dominated Convergence Theorem*) *Suppose* Ω *is a domain in* $\mathbb{R}^N$; $\{u_n\}_{n=1}^{\infty}$ *and* u *are measurable functions in* Ω *such that* $u_n \to u$ *a.e. in* Ω. *If there is a* $\varphi \in L^1(\Omega)$ *such that for each* $n \in \mathbb{N}$

$$|u_n| \leq \varphi \quad \text{a.e. in } \Omega,$$

then $u_n \to u$ *in* $L^1(\Omega)$.

Proof. See Wheeden-Zygmund [22, p. 173]. ■

In Example 9 it shows that the converse of the Lebesgue dominated convergence theorem fails.

Example 9. For $n = 1, 2, \ldots$, let $u_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$u_n(x) = \begin{cases} 0, & \text{for } x \leq n \\ 2, & \text{for } x = n + 1/2n \\ 0, & \text{for } x \geq n + 1/n \\ \text{linear,} & \text{otherwise.} \end{cases}$$

Then we have

$$\int_{\mathbb{R}} u_n(x)dx = \frac{1}{n} < \infty \text{ for each } n \in \mathbb{N}.$$

Then $u_n \to 0$ a.e. in $\mathbb{R}$ and strongly in $L^1(\mathbb{R})$. Suppose that there exists a $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying

$$|u_n| \leq \varphi \text{ a.e. in } \mathbb{R} \text{ for each } n \in \mathbb{N}.$$

Then $\infty = \sum \frac{1}{n} = \int_{\mathbb{R}} \sum u_n \leq \int_{\mathbb{R}} \varphi$. Consequently, $\varphi \notin L^1(\mathbb{R})$.

However the Vitali convergence theorem provides a necessary and sufficient result for L^1 convergence.

Theorem 10. (*Vitali Convergence Theorem for* $L^1(\Omega)$) *Suppose* Ω *is a domain in* $\mathbb{R}^N$; $\{u_n\}_{n=1}^{\infty} \subset L^1(\Omega)$; $u \in L^1(\Omega)$; *and* $u_n \to u$ *a.e. in* Ω. *Then* $\|u_n - u\|_{L^1} \to 0$ *if and only if the following two conditions hold* :

(UI) (*Uniformly integrable condition*) *For each* $\varepsilon > 0$; *there exists a measurable set* $E \subset \Omega$ *such that* $|E| < \infty$ *and*

$$\int_{E^c} |u_n| d\mu < \varepsilon \text{ for each } n \in N;$$

(UC) (*Uniformly continuous condition*) *For each* $\varepsilon > 0$; *there exists a* $\delta > 0$ *such that* $|E| < \delta$ *implies*

$$\int_{E} |u_n| d\mu < \varepsilon \text{ for each } n \in N:$$

Proof. See Gariepy-Ziemer [13, p. 150]. ■

Proposition 11. *In the* (UI) *condition of the Vitali Convergence Theorem* 10, *the condition* $|E| < \infty$ *can be replaced by "*E *is bounded".*

Proof. Let $E_n = E \cap B(0; n)$ for $n = 1; 2; \dots$, then $E_1 \subset E_2 \subset \cdots \nearrow E$. Thus, $|E_1| \le |E_2| \le \cdots \nearrow |E|$: For $\delta > 0$ as in the (UC) condition of Theorem 10, there is a E_N such that $|E \backslash E_N| < \delta$. Now $\int_{E_N^c} |u_n| dx = \int_{E^c} |u_n| dx + \int_{E \backslash E_N} |u_n| dx < 2\varepsilon$ for each $n \in N$. ■

If we replace $L^1(\Omega)$ in the Vitali Convergence Theorem by $H_0^1(\Omega)$, we can drop the (UC) condition.

Theorem 12. (i) (*Rellich Theorem*) *Let* Ω *be a domain in* R^N *of finite measure. Then the embedding of* $H_0^1(\Omega)$ *into* $L^p(\Omega)$ *is compact*; (ii) (*Vitali Convergence Theorem for* $H_0^1(\Omega)$) *Let* Ω *be a domain in* R^N. *Suppose that* $\{u_n\}_{n=1}^{\infty} \subset H_0^1(\Omega)$ *is a bounded sequence such that* $u_n \to u$ *a.e. in* Ω *for some* $u \in H_0^1(\Omega)$. *If for each* $\varepsilon > 0$; *there exists a measurable set* E *such that* $|E| < \infty$ *and* $\int_{E^c} |u_n|^p dx < \varepsilon$ *for each* $n \in N$; *then* $\|u_n - u\|_{L^p} \to 0$:

Proof. (i) Let $\mathcal{F}$ be the unit ball in $H_0^1(\Omega)$.

(a) For $\varepsilon > 0$, $w \subset\subset \Omega$, $u \in \mathcal{F}$, $|h| < \mathrm{dist}(w, \Omega^c)$. For $q \in [1, 2^*)$, write

$$\frac{1}{q} = \frac{\alpha}{1} + \frac{1-\alpha}{2^*} \text{ for some } \alpha; \quad \text{where } 0 < \alpha \le 1:$$

By the interpolation property,

$$\frac{1}{q} = \frac{1}{\frac{1}{\alpha}} + \frac{1}{\frac{2^*}{1-\alpha}};$$

for $f \in L^1(\Omega) \cap L^{2^*}(\Omega)$

$$\begin{aligned}\|f\|_{L^q} &\le \| |f|^{\alpha}\|_{L^{1/\alpha}} \| |f|^{1-\alpha}\|_{L^{2^*/(1-\alpha)}} \\ &\le \|f\|_{L^1}^{\alpha} \|f\|_{L^{2^*}}^{1-\alpha}.\end{aligned}$$

That is,

$$\begin{aligned}\|\tau_h u - u\|_{L^q(w)} &\le \|\tau_h u - u\|_{L^1(w)}^{\alpha} \|\tau_h u - u\|_{L^{2^*}(w)}^{(1-\alpha)} \\ &\le \|\tau_h u - u\|_{L^1(w)}^{\alpha} \left(2\|u\|_{L^{2^*}(w)} \right)^{(1-\alpha)} \\ &\le C\|\tau_h u - u\|_{L^1(w)}^{\alpha} 2^{(1-\alpha)} \|\nabla u\|_{L^2(\Omega)}^{(1-\alpha)};\end{aligned}$$

where $\tau_h u(z) = u(z-h)$. Now Ω is of finite measure, so $u \in H_0^1(\Omega) \subset W^{1,1}(\Omega)$ and

$$\begin{aligned}\|\tau_h u - u\|_{L^1(w)} &\le |h| \, \|\nabla u\|_{L^1(\Omega)} \\ &\le c|h| \, \|\nabla u\|_{L^2(\Omega)}.\end{aligned}$$

Hence

$$\begin{aligned}\|\tau_h u - u\|_{L^q(w)} &\le C|h|^{\alpha} \|\nabla u\|_{L^2(\Omega)}^{\alpha} c \, 2^{(1-\alpha)} \|\nabla u\|_{L^2(\Omega)}^{(1-\alpha)} \\ &\le C|h|^{\alpha} \|\nabla u\|_{L^2(\Omega)} \\ &\le c|h|^{\alpha} \text{ since } u \in \mathcal{F}.\end{aligned}$$

If we take $\delta > 0$, $\delta^{\alpha} = \frac{\varepsilon}{c}$, $\delta < \operatorname{dist}(w, \Omega^c)$, then $|h| < \delta$ implies

$$\|\tau_h u - u\|_{L^q(w)} < \varepsilon.$$

(b) For $\varepsilon > 0$, $u \in \mathcal{F}$, and $q \in [1, 2^*)$, we have

$$\begin{aligned}\|u\|_{L^q(\Omega \setminus w)} &= \left(\int_{\Omega \setminus w} |u|^q \right)^{\frac{1}{q}} \\ &\le \left(\int_{\Omega \setminus w} |u|^{qr} \right)^{\frac{1}{qr}} (|\Omega \setminus w|)^{\frac{1}{r'}}.\end{aligned}$$

Take $r = \frac{2^*}{q}$, then $r' = \frac{2^*}{2^*-q}$, so

$$\begin{aligned}\|u\|_{L^q(\Omega \setminus w)} &\le \|u\|_{L^{2^*}(\Omega \setminus w)} |\Omega \setminus w|^{1-\frac{q}{2^*}} \\ &\le \|\nabla u\|_{L^p(\Omega)} |\Omega \setminus w|^{1-\frac{q}{2^*}} \\ &\le |\Omega \setminus w|^{1-\frac{q}{2^*}} \text{ (since } u \in \mathcal{F}).\end{aligned}$$

If we choose

$$w = \{x \in \Omega \mid \text{dist}\,(x, \partial\Omega) > \delta\}$$

with δ small, then

$$\|u\|_{L^q(\Omega\backslash w)} < \varepsilon.$$

By Theorem 7, $\mathcal{F}$ is relatively compact in $L^p(\Omega)$.

(ii) By the Fatou lemma, $\int_{E^c} |u|^p dx \leq \varepsilon$. Since $|E| < \infty$, by part (i), we have

$$\int_E |u_n - u|^p dx = o(1).$$

Therefore,

$$\int_\Omega |u_n - u|^p dx = \int_E |u_n - u|^p dx + \int_{E^c} |u_n - u|^p dx = o(1). \qquad \blacksquare$$

Recall that a function u is radially symmetric if there is a function f such that $u(z) = f(|z|)$ for each z. Let $H_r^1(R^N) = \{u \in H^1(R^N) \mid u$ is radially symmetric$\}$. We state the following well-known result of Strauss [18].

Lemma 13. *For* $N \geq 2$, *every* $u \in H_r^1(R^N)$ *is equal to a continuous function* U *in* $R^N\backslash\{0\}$ *such that for* $z \neq 0$

$$|U(z)| \leq \left(\frac{2}{\omega_N}\right)^{1/2} |z|^{\frac{1-N}{2}} \left(\int_{|t|\geq|z|} |u(t)|^2 dt\right)^{1/4} \left(\int_{|t|\geq|z|} |\nabla u(t)|^2 dt\right)^{1/4}.$$

Let A be an annulus, say $A = \{z \in R^N \mid 1 < |z|\}$ with $N \geq 3$. Let

$$H_r^1(A) = \{u \in H_0^1(\Omega) \mid u \text{ is radially symmetric}\}.$$

Moreover, if we replace $L^1(A)$ in the Vitali convergence theorem for $H_r^1(A)$, we can drop both the (UI) and the (UC) conditions.

Theorem 14. (*Vitali Convergence Theorem for* $H_r^1(A)$). *The embedding of* $H_r^1(A)$ *into* $L^p(A)$ *is compact.*

Proof. By Lemma 13, see Berestycki-Lions [2, p. 341]. $\blacksquare$

3. MINIMIZERS

Since the maximum of the function $-f$ is the minimum of f, we only need to consider the minimum problems. First, let us see the classical minimizing problem.

Theorem 15. *If* $K \subset R^N$ *is compact and* $f : K \to R$ *is continuous, then there exists a* $u \in K$ *such that*

$$f(u) = \min_{v \in K} f(v):$$

Proof. Let $\text{®} = \inf_{v \in K} f(v)$ and $\{v_n\}$ a minimizing sequence in K such that $f(v_n) \to \text{®}$. Since K is compact, there are a subsequence $\{v_n\}$ and $u \in K$ such that

$$v_n \to u \text{ weakly}:$$

Since f is continuous, we have

$$f(v_n) \to f(u) = \text{®}:$$

Thus,

$$f(u) = \min_{v \in K} f(v): \qquad \blacksquare$$

For more applied analysis, we need the following minimizers in reflexive Banach spaces.

Theorem 16. *Let* X *be a reflexive Banach space*; $M \subset X$ *a weakly closed subset*; *and* $f : M \to R$ *satisfying*

(i) f *is coercive: if* $u \in M;\ \|u\| \to \infty;$ *then* $f(u) \to \infty$;

(ii) f *is weakly lower semicontinuous on* M*: if* $u_n \to u$ *weakly in* M; *then* $f(u) \le \liminf_{n \to \infty} f(u_n)$.

Then there exists a $u \in M$ *such that*

$$f(u) = \min_{v \in M} f(v):$$

Proof. Let $\text{®} = \inf_{v \in M} f(v)$ and $\{v_n\}$ a minimizing sequence in M such that $f(v_n) \to \text{®}$. Since $\text{®} < \infty$; by the coercion of f, $\{v_n\}$ is bounded. There exists a subsequence $\{v_n\}$ and $u \in X$ such that

$$v_n * u:$$

Since M is weakly closed, $u \in M$. From the assumption that f is weakly lower semicontinuous on M, we have

$$f(u) \le \liminf_{n \to \infty} f(v_n) = \text{®}:$$

Thus,

$$f(u) = \min_{v \in M} f(v): \qquad \blacksquare$$

Theorem 17. (*Implicit Function Theorem*) *Let* E, F, G *be Banach spaces, and let* $U \subset E, V \subset F$ *be open sets, and* $f : U \times V \to G$ *a* C^1 *map. Assume that at* $(a, b) \in U \times V$, $f(a, b) = 0$, *and the partial derivative* $D_2 f(a, b) \in GL(F, G)$, *where* $GL(F, G)$ *is the general linear group. Then there exist an open neighborhood A of a, an open neighborhood* B *of* b, *and a unique* C^1 *map* $g : A \to B$ *such that*

$$\begin{cases} g(a) = b \\ f(x, g(x)) = 0, \quad \text{for} \quad x \in A, \end{cases}$$

and $g'(x) = -[D_2 f(x, g(x))]^{-1} \circ D_1 f(x, g(x))$ *for all* $x \in A$.

Proof. See Dieudonne [11, p. 270]. ∎

In Theorem 16, we asserted that $f(u) = \min_{v \in M} f(v)$. In order to obtain such u, we need the following Lagrange multiplier theorem.

Theorem 18. (*Lagrange Multiplier Theorem*) *Let* X *be a Banach space,* $f, g : X \to R$ *be of* C^1 *and* $M = \{x \in X | g(x) = 0\}$. *Let* $u \in M$ *such that* f *admits a minimum at* u *constrained on* M. *Suppose that* $Dg(u) \neq 0$, *then there exists a* $\lambda \in R$, *called the Lagrange multiplier, such that*

$$Df(u) = \lambda Dg(u).$$

Proof. Write

$$f(u + h) - f(u) = Df(u)h + r(h),$$
$$g(u + h) - g(u) = Dg(u)h + s(h),$$

where $r(h) = o(\|h\|)$, $s(h) = o(\|h\|)$ as $h \to 0$. We may assume $\dim X > 1$, and then take $v, w \in X$, $v \neq 0$, $w \neq 0$ such that $Dg(u)v = 1$ and $Dg(u)w = 0$. Consider the function $\varphi : R \times R \to R$ defined by

$$\varphi(t, \varepsilon) = g(u + tv + \varepsilon w) - c, \quad c = g(u).$$

Then $\varphi \in C^1$ satisfying $\varphi(0, 0) = 0$ and $D_1\varphi(0, 0) = Dg(u)v = 1$. By Implicit Function Theorem 17, there are open intervals I, J of 0, respectively, a unique C^1 map $t : J \to I$ such that

$$t(0) = 0,$$
$$t(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$
$$\varphi(t(\varepsilon), \varepsilon) = 0 \text{ for } \varepsilon \in J.$$

Now

$$\begin{aligned} 0 &= \varphi(t(\varepsilon);\varepsilon) \\ &= g(u + t(\varepsilon)v + \varepsilon w) - g(u) \\ &= Dg(u)[t(\varepsilon)v + \varepsilon w] + s(t(\varepsilon)v + \varepsilon w) \\ &= t(\varepsilon) + s(t(\varepsilon)v + \varepsilon w). \end{aligned}$$

For $0 < \delta < \|v\|$, there is a $\varepsilon_0 > 0$ such that $|\varepsilon| < \varepsilon_0$ implies $\frac{\|s(t(\varepsilon)v+\varepsilon w)\|}{\|t(\varepsilon)v+\varepsilon w\|} < \frac{\delta}{2\|v\|} < 1/2$. Therefore,

$$\frac{|t(\varepsilon)|}{|\varepsilon|} \le \frac{\|s(t(\varepsilon)v + \varepsilon w)\|}{\|t(\varepsilon)v + \varepsilon w\|}\,\frac{\|(t(\varepsilon)v + \varepsilon w\|}{|\varepsilon|} < \frac{1}{2}\frac{|t(\varepsilon)|}{|\varepsilon|} + \frac{\delta}{2}\frac{\|w\|}{\|v\|},$$

or $\frac{|t(\varepsilon)|}{|\varepsilon|} < c\delta$. We hence have $t(\varepsilon) = o(|\varepsilon|)$ as $\varepsilon \to 0$. Next we claim that $Df(u)w = 0$. In fact, since $g(u + t(\varepsilon)v + \varepsilon w) = c$ for $\varepsilon \in J$, we have that $\{u + t(\varepsilon)v + \varepsilon w \mid |\varepsilon| < \delta\}$, for some δ, represents a curve on M through u. Now

$$\begin{aligned} f(u + t(\varepsilon)v + \varepsilon w) - f(a) &= Df(u)(t(\varepsilon)v) + Df(u)(\varepsilon w) + o(t(\varepsilon)v + \varepsilon w) \\ &= t(\varepsilon)Df(u)v + \varepsilon(Df(u)w) + o(\varepsilon) \\ &= \varepsilon Df(u)w + o(\varepsilon). \end{aligned}$$

Note that

$$\frac{o(t(\varepsilon)v + \varepsilon w)}{\varepsilon} = \frac{o(t(\varepsilon)v + \varepsilon w)}{t(\varepsilon)v + \varepsilon w}\,\frac{t(\varepsilon)v + \varepsilon w}{\varepsilon} \to 0 \text{ as } \varepsilon \to 0,$$

since $t(\varepsilon) = o(\varepsilon)$, therefore $o(t(\varepsilon)v + \varepsilon w) = o(\varepsilon)$. Since f attains its extremum at u with respect to M,

$$Df(u)w = \lim_{\varepsilon\to 0} \frac{f(u + t(\varepsilon)v + \varepsilon w) - f(u)}{\varepsilon} = 0.$$

We conclude that, for $w \in X$

$$Dg(u)w = 0 \Rightarrow Df(u)w = 0.$$

Now $Dg(u)$, $Df(u) \in L(E, R)$ satisfy $\ker Dg(u) \subset \ker Df(u)$. Recall that if $T : X \to R$ is linear, then either $\ker T = X$ or $X = \langle \ker T; x_0\rangle$ for some $x_0 \in X \setminus \ker T$. Since $Dg(u) \neq 0$, there is a $x_0 \in \ker Dg(u)$ with $X = \ker Dg(u) \oplus Rx_0$. For any $x \in X$, $x = y + \beta x_0$ where $y \in \ker Dg(u)$. So, for $\lambda = \frac{Df(u)(x_0)}{Dg(u)(x_0)}$,

$$\begin{aligned} Df(u)(x) &= \beta Df(u)(x_0) = \beta\lambda Dg(u)(x_0) \\ &= \lambda Dg(u)(\beta x_0) = \lambda Dg(u)(x), \end{aligned}$$

or $Df(u) = \lambda Dg(u)$. ■

We give here an application of Theorems 16 and 18: consider the semilinear elliptic equation

$$(2)\qquad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N > 2$, and $2 < p < \frac{2N}{N-2}$. Let the potential operators $a, b : H_0^1(\Omega) \to \mathbb{R}$, and the energy functional $J : H_0^1(\Omega) \to \mathbb{R}$ be given by

$$a(u) = \int_\Omega \left(|\nabla u|^2 + \lambda u^2\right),$$

$$b(u) = \int_\Omega |u|^p,$$

$$J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u).$$

Theorem 19.

(i) J *is of* $C^{1,1}$,

(ii) J *satisfies the mountain pass hypothesis: There are* $r, \delta > 0$ *and* $e \in H_0^1(\Omega)$, *such that* $e \notin \overline{B(0;r)}$, $f(e) = 0$, $f(u) \geq \delta > 0$ *for* $u \in \partial B(0;r)$,

(iii) J *is weakly lower semicontinuous.*

Proof. See Rabinowitz [17, Theorem 2.15 and Proposition B.10]. ■

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ denote the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$.

Theorem 20. *For any* $\lambda > -\lambda_1$, *there exists a solution* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *of equation* (2).

Proof. Let $X = H_0^1(\Omega)$, a Hilbert space and the manifold

$$M = \{u \in H_0^1(\Omega) \mid b(u) = 1\}.$$

By the Rellich compactness Theorem 12 (i), M is weakly closed. Moreover, a is coercive: for $\lambda > -\lambda_1$, from

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2},$$

we have

$$a(u) \geq \min\left\{1,\ \left(1+\frac{\lambda}{\lambda_1}\right)\right\}||u||_{H^1}.$$

a is weakly lower semilinear continuous since a is a norm. By Theorem 16, there exists a $u \in M$ such that

$$a(u) = \min_{v\in M} a(v).$$

By the Lagrange multiplier Theorem 18, there exists a $\mu \in R$ such that

$$a'(u)\varphi = \mu b'(u)\varphi,$$

for each $\varphi \in H_0^1(\Omega)$, or

$$\int_\Omega (\nabla u \nabla\varphi + \lambda u\varphi) = \mu \int_\Omega u|u|^{p-1}\varphi.$$

Let $\varphi = u$ to get

$$a(u) = \mu b(u) = \mu.$$

Thus, $\mu > 0$. Then $w = \mu^{1/(p-2)}u$ solves equation (2). ■

4. Palais Smale Conditions

In section 2, we presents the compactness. But in these compactness, we need to investigate every sequence. It is not economic. For example, to solve the equation (2) in Theorem 20, we associate the equation (2) with an energy functional J. The critical points of J and the solutions of equation (2) are exactly the same. Thus, it suffices to find the critical points of J. We conclude that we only need to consider the sequences related to the functional J. On the other hand, in section 3, we present the minimizers. In Theorem 16, we impose some coercive and weakly lower semicontinuous conditions on the function f such as that Ω is an unbounded domain to assure that there is a u such that

$$f(u) = \min_{v\in M} f(v),$$

then we apply the Lagrange multiplier Theorem 18 to find such u. There are some problems: if a function f satisfies neither the coercive condition nor weakly lower semicontinuous condition, or instead of to finding the minimizer

$$f(u) = \min_{v\in M} f(v),$$

we try to find the saddle point u satisfying

$$f(u) = \inf_{\gamma} \max_{t} f(\gamma(t)).$$

Therefore we need to consider new compactness concepts and new minimizers: for example, let f be of C^1 and

$$® = \min_{v\in M} f(v) \text{ or } ® = \inf_{°} \max_{t} f(°(t));$$

and $\{u_n\}$ a minimizing sequence such that

$$f(u_n) \to ®;$$
$$f'(u_n) \to 0:$$

If there is a subsequence $\{u_n\}$ and u such that

$$u_n \to u:$$

Then $f(u_n) \to ® = f(u)$. This is the so called the (PS)$-$conditions. In the following we define the Palais$-$Smale (simply by (PS)) sequences for J:

Definition 21.

(i) For $¯ \in R$; a sequence $\{u_n\}$ in $H_0^1(\Omega)$ is called a (PS)$_¯-$sequence for J if $J(u_n) \to ¯$ and $J'(u_n) \to 0$ strongly as $n \to \infty$;

(ii) $¯ \in R$ is called a (PS)$-$value for J if there is a (PS)$_¯-$sequence for J;

(iii) J satisfies the (PS)$_¯-$condition if every (PS)$_¯-$sequence for J contains a convergent subsequence;

(iv) J satisfies the (PS)$-$condition if, for every $¯ \in R$; every (PS)$_¯-$sequence for J contains a convergent subsequence.

We may produce the (PS)$_¯-$sequence for J by the Ekeland variational principle and the deformation lemma.

Theorem 22. (*Ekeland' Variational Principle*) *Let* M *be a complete metric space with metric* d*. Let* $f : M \to (-\infty; +\infty]$ *satisfying*

(i) f *is lower semi-continuous*;

(ii) f *is bounded from below:* $¯ = \inf_M f > -\infty$;

(iii) $f \not\equiv +\infty$.

Then, for each $²; ± > 0$ *and for each* $u \in M$ *such that* $f(u) \le ¯ + ²$, *there exists* $v \in M$ *such that*

(a) $f(v) \le f(u)$;

(b) $\mathrm{dist}(u; v) \le ±$;

(c) $f(v) < f(w) + \frac{"}{±}\mathrm{dist}(w; v)$ *for each* $w \in M$; $w \ne v$:

Proof. See Chabrowski [7, p. 4]. ■

Theorem 23. (*Deformation Lemma*) *Let* X *be a Banach space*; $f \in C^1(X)$; *and* $\beta \in R$. *Then for each* $\delta < \frac{1}{8}$; *there exists a continuous deformation* η : $X \times [0;1] \to X$ *such that*

(i) $\eta(u;0) = u$ *for* $u \in X$;

(ii) $\eta(u;t) = u$ *for* $t \in [0;1]$ *if* $\|f'(u)\| \leq \sqrt{\delta}$;

(iii) $\eta(u;t) = u$ *for* $t \in [0;1]$ *if* $|f(u) - c| \geq 2\delta$

(iv) $t \to f(\eta(u;t))$ *is increasing in* t;

(v) $0 \leq f(u) - f(\eta(u;t)) \leq 4\delta$ *for* $u \in X$; $t \in [0;1]$;

(vi) $\|\eta(u;t) - u\| \leq 16\sqrt{\delta}$ *for* $u \in X$; $t \in [0;1]$;

(vii) *If* $u \in f_{c+\delta}$ *then either* (a) $\eta(u;1) \in f_{c-\delta}$ *or* (b) *for some* $t_1 \in [0;1]$; *we have* $\|f'(\eta(u;t_1))\| < 2\sqrt{\delta}$, where $f_a = \{u \in X \,|\, f(u) \leq a\}$;

(viii) *More generally*; *let* $\tau \in [0;1]$ *and assume that for all* $t \in [0;\tau]$; $\eta(u;t)$ *belongs to the set*

$$\mathcal{N} = \{v \in X \,|\, |f(v) - c| \leq \delta \text{ and } \|f'(v)\| \geq 2\sqrt{\delta}\};$$

then $f(\eta(u;\tau)) \leq f(u) - \tau/4$;

(ix) $\eta(\eta(u;t);s) = \eta(u;t+s)$ (*which implies that*; *for fixed* t; $u \to \eta(u;t)$ *is a homeomorphism of* X *onto* X).

Proof. See Brézis-Nirerberg [4, p. 947]. ■

We may also produce the $(PS)_\beta-$sequence for J by methods other than the Ekeland variational principle and the deformation lemma, see Lien-Tzeng-Wang [15] and Chen-Wang [8].

The most important thing is to present various compactness: the (PS)$-$conditions. In order to present the (PS)$-$conditions, we need to define the index of a domain Ω of a functional J.

Consider the following four important positive values.

(i) The constrained value $\alpha_\theta = (\frac{1}{2} - \frac{1}{p})\theta^{\frac{2p}{2-p}}$; where

$$\theta = \sup\left\{\|u\|_{L^p(\Omega)} \mid u \in H_0^1(\Omega);\ a(u) = 1\right\}:$$

Clearly, α_θ is a positive value.

(ii) The Nehari value $\alpha_M = \inf_{u \in M(\Omega)} J(u)$, where

$$\mathsf{M}(\Omega) = {}^{©}\mathsf{u} \in \mathsf{H}_0^1(\Omega)\backslash\{0\} \mid \mathsf{a}(\mathsf{u}) = \mathsf{b}(\mathsf{u})^{\text{a}} :$$

As a consequence of the following lemma, $®_{\mathsf{M}}$ is a positive value.

Lemma 24. *Let* $\mathsf{S}(\Omega) = \{\mathsf{u} \in \mathsf{H}_0^1(\Omega) \mid \|\mathsf{u}\|_{\mathsf{H}^1} = 1\}$ *be the unit sphere. Then there is a bijective* $\mathsf{C}^{1;1}$ *map* m *from* $\mathsf{S}(\Omega)$ *to* $\mathsf{M}(\Omega)$. *Moreover*; $\mathsf{M}(\Omega)$ *is path-connected and there exists a constant* $\mathsf{c} > 0$ *such that for* $\mathsf{u} \in \mathsf{M}(\Omega)$; $\|\mathsf{u}\|_{\mathsf{H}^1} \geq \mathsf{c}$ *and* $\mathsf{J}(\mathsf{u}) \geq \mathsf{c}$.

Proof. See Chen-Wang [8, Lemma 2.2]. ■

(iii) The minimax value $®_{\Gamma} = \inf_{\mathsf{v}\in\Gamma} \max_{\mathsf{t}\in[0;1]} \mathsf{J}(\mathsf{v}(\mathsf{t}))$; where

$$\Gamma = \{\mathsf{v} \in \mathsf{C}([0;1];\mathsf{H}_0^1(\Omega)) \mid \mathsf{v}(0) = 0; \mathsf{v}(1) = \mathsf{e}\} \text{ and } \mathsf{J}(\mathsf{e}) = 0:$$

Since J satisfies the mountain pass hypothesis, $®_{\Gamma}$ is a positive value.

(iv) The minimal value $®_{\mathsf{P}} = \inf_{¯\in\mathsf{P}(\Omega)} ¯$, where $\mathsf{P}(\Omega)$ is the set of all positive (PS)$-$values for J in Ω: As a consequence of the following lemma, $®_{\mathsf{P}}$ is a positive value.

Lemma 25. *There is a* $¯_0 > 0$ *such that* $¯ \geq ¯_0$ *for every positive* $(\mathsf{PS})-$*value* $¯$:

Proof. See Chen-Lin-Wang [10, Lemma 11]. ■

We state the following useful lemma.

Lemma 26. *Let* $\{\mathsf{u}_\mathsf{n}\} \subset \mathsf{H}_0^1(\Omega)$ *be a* $(\mathsf{PS})_{¯}-$*sequence for* J *with* $¯ > 0$: *Then there is a sequence* $\{\mathsf{s}_\mathsf{n}\}$ *in* R^+ *such that* $\{\mathsf{s}_\mathsf{n}\mathsf{u}_\mathsf{n}\} \subset \mathsf{M}(\Omega)$ *and* $\mathsf{J}(\mathsf{s}_\mathsf{n}\mathsf{u}_\mathsf{n}) = ¯ + o(1)$:

Proof. See Wang [20, Lemma 8]. ■

Now we now study several important (PS)$-$values.

Lemma 27. $®_{\mu}$; $®_{\mathsf{M}}$; $®_{\Gamma}$ *and* $®_{\mathsf{P}}$ *are positive* $(\mathsf{PS})-$*values for* J: *Moreover, every minimizing sequence for* $®_{\mathsf{M}}$ *is a* $(\mathsf{PS})_{®_{\mathsf{M}}}-$*sequence for* J.

Proof.

(i) By Lien-Tzeng-Wang [15, Theorem 2.1], $®_{\mu}$ is a positive (PS)–value for J:

(ii) By Stuart [19, Lemma 3.4], $®_{\mathsf{M}}$ is a positive (PS)–value for J:

(iii) By Brézis-Nirenberg [4], $®_{\Gamma}$ is a positive (PS)$-$value for J:

(iv) For each $\mathsf{n} \in \mathsf{N}$; take $\mathsf{u}_\mathsf{n} \in \mathsf{H}_0^1(\Omega)$ and $¯_\mathsf{n} \in \mathsf{P}(\Omega)$ such that

$$|¯_n - ®_P| < \frac{1}{2n}; \quad |J(u_n) - ¯_n| < \frac{1}{2n}; \quad \|J'(u_n)\| < \frac{1}{2n}:$$

Then $J(u_n) = ®_P + o(1)$ and $J'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$. Thus, $®_P \in P(\Omega)$. ■

In the following, we present a comparison lemma.

Lemma 28. *Let* $\{u_n\} \subset H_0^1(\Omega)$ *be a* $(PS)_¯$*–sequence for* J *with* $¯ > 0$: *Then* $¯ \geq ®_\mu; ¯ \geq ®_M; ¯ \geq ®_\Gamma$ *and* $¯ \geq ®_P$:

Proof. By Wang [20, Lemma 9], $¯ \geq ®_\mu; ¯ \geq ®_M$ and $¯ \geq ®_\Gamma$. Clearly, $¯ \geq ®_P$: ■

By Lemma 27 and 28, we have the following interesting result.

Theorem 29. *Four important* (PS)*–values are equal:* $®_\mu = ®_M = ®_\Gamma = ®_P$:

Remark 1. *For the equalities* $®_\mu = ®_M = ®_\Gamma$; *see also Willem* [23].

Definition 30. By Theorem 29, the positive (PS)–values $®_\mu; ®_\Gamma; ®_M$ and $®_P$ for J are the same. Any one of them is called the index of J in Ω and denoted by $®(\Omega)$ (simply by $®$). By the definition of $®_M$; if u is a nonzero solution of equation (2), then $J(u) \geq ®$. Follows from Berestycki-Lions [2], we call that a solution u of equation (2) is a ground state solution if $J(u) = ®$ and u is a higher energy solution if $J(u) > ®$:

Let Ω^1 and Ω^2 are two domains in R^N; $\Omega^1 \ \$ \ \Omega^2$ and $®_i = ®(\Omega^i)$ for $i = 1; 2$, then clearly $®_2 \leq ®_1$: If $®_2 = ®_1$, then we have the following useful results.

Lemma 31. *Let* $\Omega^1 \ \$ \ \Omega^2$ *and* $J : H_0^1(\Omega^2) \to R$ *be the energy functional. Suppose that* $®_2 = ®_1$: *Then*

(i) J *does not satisfy the* $(PS)_{®_1}$*–condition in* Ω^1;

(ii) $®^1$ *does not admit any ground state solution*;

(iii) J *does not satisfy the* $(PS)_{®_2}$*–condition in* Ω^2:

Proof. See Chen-Lin-Wang [10, Lemma 20] and Wang-Wu [21, Lemma 22]. ■

Let Ω be an unbounded domain in R^N and Ω^i a proper subdomain of Ω for $i = 1; 2; \cdots; k$ such that $\Omega^i \cap \Omega^j$ is bounded if $i \neq j$ and $\Omega = \Omega^1 \cup \cdots \cup \Omega^k$. Let $® = ®(\Omega)$, $®_i = ®(\Omega^i)$, $M(\Omega) = \{u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b(u)\}$ and $M(\Omega^i) = \{u \in H_0^1(\Omega^i) \setminus \{0\} \mid a(u) = b(u)\}$ for $i = 1; 2; \cdots$, k. Since $H_0^1(\Omega^i) \subset H_0^1(\Omega)$ and

$M(\Omega^i) \subset M(\Omega)$, for $i = 1, 2, \cdots, k$, we have $\alpha \le \min\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$. If one of Ω^i is bounded, say Ω^1 is bounded, then J satisfies the $(PS)_{\alpha_1}-$condition. Let

$$\widetilde{\Omega}_n = \Omega \setminus \overline{B^N(0; n)} \subsetneq \Omega;$$
$$\widetilde{M}_n = \left\{ u \in H_0^1(\widetilde{\Omega}_n)\setminus\{0\} \mid a(u) = b(u) \right\};$$
$$\widetilde{\alpha}_n = \alpha(\widetilde{\Omega}_n) = \inf_{u \in \widetilde{M}_n} J(u).$$

If $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_\alpha-$sequence for J, then $\{u_n\}$ is bounded. There are a subsequence $\{u_n\}$ and $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, a.e. in Ω and strongly in $L^p_{loc}(\Omega)$. Define $\alpha_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\Omega\cap\{|x|>R\}} |u_n|^p$. For the quantity α_∞, which measures a loss of mass at infinity of a weakly convergent sequence, see Chabrowski [5], Ben-Naoum-Troestler-Willem [1], and Willem [23]. A lot of information on α_∞ and its significance for weak convergence methods can be found in the book of Chabrowski [6].

We have the following characterization of compactness.

Theorem 32. *The following properties are equivalent*:

(i) J *satisfies the* $(PS)_\alpha-$*condition*;

(ii) *For every* $(PS)_\alpha-$*sequence* $\{u_n\} \subset H_0^1(\Omega)$ *for* J, *there exists a subsequence* $\{u_n\}$ *and* $u \ne 0$ *in* $H_0^1(\Omega)$ *such that* $u_n \rightharpoonup u$ *weakly in* $H_0^1(\Omega)$;

(iii) *For every* $(PS)_\alpha-$*sequence* $\{u_n\} \subset H_0^1(\Omega)$ *for* J, *there are a* $c > 0$, *a subsequence* $\{v_m\}$ *of* $\{u_n\}$, *a positive integer* $K > 0$ *such that for each* $k \ge K$ *there is a positive integer* $N(k)$ *such that for* $m \ge N(k)$, *we have*
$$\int_{\Omega_k} |v_m|^p \ge c;$$

(iv) *For every* $(PS)_\alpha-$*sequence* $\{u_n\} \subset H_0^1(\Omega)$ *for* J, *there is a subsequence* $\{u_n\}$ *such that for* $\varepsilon > 0$ *there is a measurable set* E *such that* $|E| < \infty$ *and* $\int_{E^c} |u_n|^p dx < \varepsilon$ *for each* $n \in \mathbb{N}$;

(v) $\alpha < \widetilde{\alpha}_n$ *for each* $n \in \mathbb{N}$;

(vi) $\alpha < \min\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$;

(vii) $\alpha_\infty < \frac{2p}{p-2}\alpha$.

Proof. See Chen-Lin-Wang [10, Theorem 23]. ∎

In the following we give one application of Theorem 32. Let $z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and Ω is a domain in $\mathbb{R}^N$. Define the ball $B^N(z_0; s)$ in the Euclidean space R^N; the infinite strip A^r, the upper semi-strip A_0^r; the finite strip $A_{s;t}^r$ for $s; t \in R$; and the interior flask domain F_s^r for $s > 0$ as follows:

$$\begin{aligned}
&B^N(z_0; s) = \{z \in R^N \mid |z - z_0| < s\};\\
&A^r = \{(x; y) \in R^N \mid |x| < r\};\\
&A_0^r = \{(x; y) \in A^r \mid 0 < y\};\\
&A_{s;t}^r = \{(x; y) \in A^r \mid s < y < t\};\\
&F_s^r = A_0^r \cup B^N(0; s):
\end{aligned}$$

Esteban-Lions [12, Theorem I.1]es-li proved the following:

Theorem 33. *Equation* (2) *in* A_0^r *does not admit any nontrivial solution.*

However, in the following Theorem 34 we apply Theorem 32 to assert that Equation (2) in a perturbation of the upper semi-strip A_0^r admits a ground state solution.

Theorem 34. *There exists* $s_0 > 0$ *such that equation* (2) *has a ground state solution in* F_s^r *if* $s > s_0$; *but does not have any ground state solution if* $s < s_0$:

Proof. Let $\Omega = F_s^r$; $\Omega_1 = A_0^r$, and $\Omega_2 = B(0; s)$: By Lien-Tzeng-Wang [15], equation (2) has a ground state solution in A^r. By Lemma 31, we have $®(A^r) > ®(R^N)$: Note that $®(A^r) = ®(A_0^r)$ and $\lim_{s\to\infty} ®(B^N(0; s)) = ®(R^N)$: Take s large enough such that

$$®(B(0; s)) < ®(A^r) = ®(A_0^r):$$

It is well-known that there is a ground state solution of equation (2) in $B^N(0; s)$. By Lemma 31, we have

$$®(\Omega) = ®(F_s^r) < ®(B^N(0; s)):$$

We conclude that

$$®(\Omega) = ®(F_s^r) < ®(B^N(0; s)) = ®(\Omega_2) < ®(A_0^r) = ®(\Omega_1):$$

By Theorem 32, equation (2) has a ground state solution in F_s^r for large s. If equation (2) has a ground state solution in $F_{s_1}^r$ and $s_1 < s_2$, then $F_{s_2}^r = F_{s_1}^r \cup B^N(0; s_2)$. By Lemma 31, $®(F_{s_2}^r) < ®(B^N(0; s_2))$ and $®(F_{s_2}^r) < ®(F_{s_1}^r)$. By Lemma 31, equation (2) has a ground state solution in $F_{s_2}^r$: Let

$$s_0 = \inf\{s > r \mid \text{equation (2) has a ground state solution in } F_s^r\}:$$

We then conclude that equation (2) has a ground state solution in F^r_s if $s > s_0$; and equation (2) has no ground state solution in F^r_s if $s < s_0$: ■

Let $z = (x; y) \in R^{N-1} \times R$ and Ω be a domain in R^N.

Definition 35.

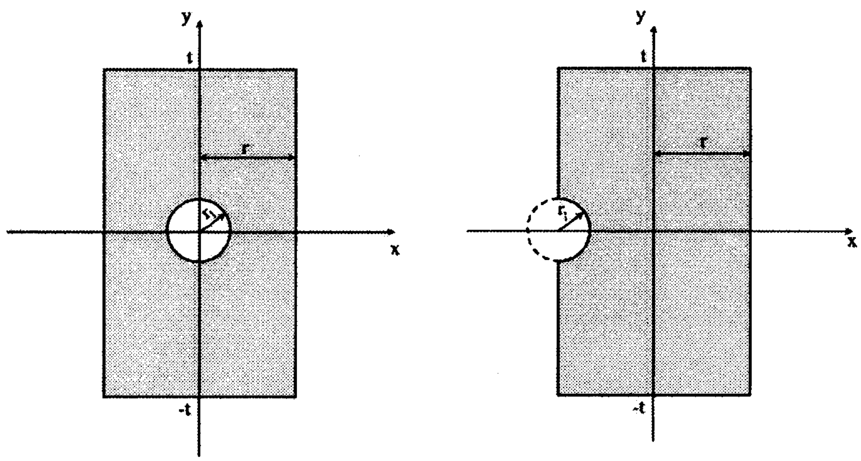
(i) Ω is $y-$symmetric provided $z = (x; y) \in \Omega$ if and only if $(x; |y|) \in \Omega$;

(ii) A domain Ω in A^r is large if for any $m > 0$ there exists $s < t$ such that $t - s = m$ and $A^r_{s;t} \subset \Omega$;

(iii) Let Ω be an $y-$symmetric domain in R^N. A function $u : \Omega \to R$ is $y-$symmetric (axially symmetric) if there is a function $f : R^{N-1} \times [0; \infty) \to R$ such that $u(x; y) = f(x; |y|)$ for $(x; y) \in \Omega$:

For $0 < r_1 < r$ and $t > 0$; consider the finite strip with a hole,

$$\Omega_t = A^r_{-t;t} \setminus \overline{B^N((x; 0); r_1)};$$



and the exterior domain $A^r \backslash \overline{B^N((X; 0); r_1)}$. Then A^r and Ω_t are $y-$symmetric domains and the exterior domain $A^r \backslash \overline{B^N((x; 0); r_1)}$ is a large $y-$symmetric domain.

Throughout this article, we let $H_s(\Omega)$ be the H^1-closure of the $\{u \in C^\infty_0(\Omega) \mid u$ is y–symmetric$\}$. Then $H_s(\Omega)$ is a closed linear subspace of $H^1_0(\Omega)$: In Definition 21, we may replace $H^1_0(\Omega)$ by $H_s(\Omega)$ to get

Definition 36. We define

(i) For $^- \in R$; a sequence $\{u_n\}$ in $H_s(\Omega)$ is a symmetric (PS)$_-$–sequence for J if $J(u_n) \to {}^-$ and $J'(u_n) \to 0$ strongly as $n \to \infty$;

(ii) ¯ $\in$ R is a symmetric (PS)$-$value for J if there is a symmetric (PS)¯$-$sequence for J;

(iii) J satisfies the symmetric (PS)¯$-$condition if every symmetric (PS)¯$-$sequence for J contains a convergent subsequence;

(iv) J satisfies the symmetric (PS)$-$condition if, for every ¯ $\in$ R; every symmetric (PS)¯$-$sequence for J contains a convergent subsequence.

We may replace $H_0^1(\Omega)$ in the definitions of the four (PS)$-$values ®$_\mu$, ®$_M$, ®$_\Gamma$, ®$_P$ by $H_s(\Omega)$ to get the four symmetric (PS)$-$values ®$^s_\mu$, ®^s_M, ®$^s_\Gamma$, and ®^s_P. By Wang-Wu [21, Theorem 16], we have

Theorem 37. *Four symmetric* $(PS)-$*values are equal:* ®$^s_\mu =$ ®$^s_M =$ ®$^s_\Gamma =$ ®^s_P:

Definition 38. By Theorem 37; the positive symmetric $(PS)-$values ®$^s_\mu$; ®^s_M; ®$^s_\Gamma$; and ®^s_P for J in $H_s(\Omega)$ are the same. Any one of them is called the symmetric index of J in Ω and denoted by ®$^s(\Omega)$ (simply by ®^{s}). By the definition of ®^s_M, if u is a nonzero y$-$symmetric solution of equation (1), then $J(u) \geq$ ®^{s}. Follows from Berestycki-Lions [2], we call that a nonzero y$-$symmetric solution u of equation (1) is a symmetric ground state solution if $J(u) =$ ®^{s} and is a symmetric higher energy solution if $J(u) >$ ®^{s}:

Let $X(\Omega)$ be either $H_0^1(\Omega)$ or $H_s(\Omega)$ with the index ®$_X(\Omega)$. Then we have the following lemma.

Lemma 39. *Let* $u \in X(\Omega)$ *be a change sign solution of equation* (2). *Then* $J(u) > 2$®$_X(\Omega)$.

Proof. See Wang-Wu [21, Lemma 20]. ■

The following Lemma 40 asserts that a bounded domain in R^N has nice property for PS–condition.

Lemma 40. *Let* Ω *be a bounded domain in* R^N. *Then the* $(PS)_{®_X(\Omega)}-$*condition holds in* $X(\Omega)$ *for* J.

Proof. See Wang-Wu [21, Lemma 25]. ■

Lemma 41. *If* Ω *is a large domain in* A^r; *then* ®$(\Omega) =$ ®(A^r):

Proof. See Lien-Tzeng-Wang [15, Lemma 2.5]. ■

Proposition 42. *We have that* ®$_s(A^r_{-t;t}) =$ ®$(A^r_{-t;t})$ and ®$_s(A^r) =$ ®(A^r):

Proof. By Gidas-Ni-Nirenberg [14] and Chen-Chen-Wang [9], we have that every positive solution of equation (2) in a finite strip $A^r_{-t;t}$ and in an infinite strip A^r is $y-$symmetric. ■

Corollary 43. *If* Ω *is a proper* $y-$*symmetric large domain in* A^r; *then* $®(A^r) < ®_s(\Omega)$.

Proof. See Wang-Wu [21, Corollary 33]. ■

Finally, we apply Corollary 43 to prove the existence of three solutions of (2).

Theorem 44. *There exists* $t_0 > 0$ *such that for* $t \geq t_0$; *the equation* (2) *on* Ω_t *has three positive solutions in which one is* $y-$*symmetric and the other two are nonaxially symmetric.*

Proof. Let $\Omega = A^r \setminus \overline{B^N((x;0);r_1)}$, then Ω is an $y-$symmetric large domain in A^r. By Corollary 43, we have $®(A^r) < ®_s(\Omega)$: By Lemma 40 and Lien-Tzeng-Wang [15], equation (2) admits a ground state solution in $A^r_{0;t}$ and in A^r, we have that $®(A^r_{0;t})$ is strictly decreasing as t is strictly increasing and

$$®(A^r_{0;t}) \searrow ®(A^r) \text{ as } t \to \infty:$$

Take $t_1 > 0$ such that for $t \geq t_1$;

$$®(A^r) < ®(A^r_{0;t}) < ®_s(\Omega): \tag{3}$$

Note that $A^r_{r_1;t_1+r_1} \$ \Omega_t \$ A^r$ for $t \geq t_0 = t_1 + r_1$. By Lemma 31, we conclude that

$$®(A^r) < ®(\Omega_t) < ®(A^r_{r_1;t_1+r_1}): \tag{4}$$

By Lien-Tzeng-Wang [15], if Ω is a domain of R^N, then $®(\Omega)$ is invariant by rigid motions. Thus,

$$®(A^r_{r_1;t_1+r_1}) = ®(A^r_{0;t_1}): \tag{5}$$

Therefore, by (3)-(5)

$$®(A^r) < ®(\Omega_t) < ®(A^r_{0;t_1}) < ®_s(\Omega): \tag{6}$$

Since $\Omega_t \subset \Omega$, we have

$$®_s(\Omega) \leq ®_s(\Omega_t): \tag{7}$$

By (6) and (7), we obtain

$$®(\Omega_t) < ®_s(\Omega_t): \tag{8}$$

By Lemma 41, there exist an $y-$symmetry solution u_1 and a solution u_2 of equation (2) in domain Ω_t such that

$$J(u_1) = ®_s(\Omega_t);$$
$$J(u_2) = ®(\Omega_t):$$

By Lemma 39, we may take u_1 and u_2 to be positive. Let

$$u_3(x; y) = u_2(x; -y);$$

then u_3 is the third solution. By (8), u_1, u_2 and u_3 are different. Moreover, u_1 is an $y-$symmetric solution while both u_2 and u_3 are nonaxially symmetric solutions of equation (2) in domain Ω_t. ■

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