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A UNIFIED WAY FOR OBTAINING DIVIDING FORMULAS $\mathbf{n|Q(n)}$

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Abstract. We show that interesting dividing formulas such as, Chinese theorem, Fermat's little theorem, and Euler's theorem can easily be derived from some well-known iterated maps. Other dividing formulas concerning Fibonacci numbers, generalized Fibonacci numbers of degree m, and numbers of other types can also be derived. The results show that iterated maps offer a systematic and unified way for obtaining nontrivial dividing formulas $n|Q(n)$, and we can thus understand the dividing formulas from the point of view of iterated maps.

1. Introduction

Interesting dividing formulas seemed to begin with the Chinese theorem more than two thousand years ago [8], which stated that $n|(2^n - 2)$, for n a prime. It took a long time to put this theorem in a more general form. The following are more examples of dividing formulas:

$n|$ $(a^n - a)$, for n a prime, and $a = 2, 3, 4, \dots$ (Fermat's little theorem).

$n|$ $(a^{\varphi(n)} - 1)$, where $\varphi(n)$ is Euler's totient function of n, and a and n are relatively prime (Euler's theorem).

$n|$ $(F_{n+1} + F_{n-1} - 1)$, for n a prime, where F_n is the nth Fibonacci number.

In this work, we use iterated maps to generate dividing formula $n|Q(n)$, where n is any positive integer and show that the above examples can all be re-derived from some simple iterated maps. It then seems that we have a systematic and unified way for obtaining dividing formulas.

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In Section 2, we give a general discussion on iterated maps. In Section 3, we derive the fundamental theorem $n|N(n)$ for each iterated map, where $N(n)$ is the number of period-n points for the iterated map. In Section 4, we show that what we obtain from an iterated map f is $N_\Sigma(n)$, the number of fixed points of $f^{[n]}$, and we give the relation between $N(n)$ and $N_\Sigma(n)$. In Section 5, we give several examples showing the applications of this fundamental theorem $n|N(n)$ for several maps to reproduce well-known results and obtain some new dividing formulas.

2. A General Discussion

For a general discussion [1], we consider a map $f(x)$ in some interval. we define $f(x)$ the first iterate of x for $f, f^{[2]}(x)$ the second iterate of x for f, and $f^{[n]}(x) = f(f^{[n-1]}(x))$ the nth iterate of x for f. Iterates of x form the sequence, $\{f^{[n]}(x)\}_{n=0}^{\infty}$, which is called the orbit of x.

We call x a fixed point if x is the point whose iterates are the same point. The fixed points of f are thus determined from the formula

$$f(x) = x. \tag{2.1}$$

Geometrically, the number of fixed points for f can be determined from the number of intersections of the curve $y = f(x)$ with the diagonal line $y = x$.

We call x a period-n point if $f^{[n]}(x) = x$, and in addition, $x, f(x), f^{[2]}(x), \ldots, f^{[n-1]}(x)$ are distinct, that is, n is the least number such that the nth iterate of x is again x. The period-n points of f are thus determined from the following two equations:

$$f^{[n]}(x) = x, \tag{2.2}$$

$$f^{[i]}(x) \neq x \quad \text{for } i = 1, 2, \ldots, n-1. \tag{2.3}$$

In what follows, we denote by $N(n)$ the number of period-n points for f. If x is a period-n point, the orbit of x is a periodic orbit, the orbit is denoted by $\{x, f(x), f^{[2]}(x), \ldots, f^{[n-1]}(x)\}$. This orbit is called an n-cycle.

3. The Fundamental Theorem: $n|N(n)$

Theorem 3.1. *Suppose f is an iterated map. Let $N(n)$ denote the number of period-n points for the map. Then*

$$n|N(n). \tag{3.1}$$

Proof. If $N(n) = 0$, formula (3.1) is obvious. If $N(n) \neq 0$, the orbit of a period-n point is an n-cycle containing n distinct period-n points. Two distinct n-cycles are disjoint. For if they contain a common element x_0, then all iterates of x_0 in each of these two cycles should be the same; these two n-cycles are then identical. Thus there are no common elements in any two distinct n-cycles. $N(n)$ can then be divided into disjoint n-cycles, and $N(n)/n$ is an integer representing the number of n-cycles for the map. ∎

As a consequence of this fundamental theorem, each iterated map offers a desired $Q(n)$ function such that $n|Q(n)$, where $Q(n) = N(n)$, the number of period-n points of the map. We have therefore an additional way to understand the dividing formula $n|Q(n)$ from the point of view of iterated maps. We also note that the result is quite general, as n can be any positive integer instead of being prime or relatively prime to some base number a.

4. Obtaining the $N(n)$ of an Iterated Map

As discussed in Section 2, $N(n)$, the number of period-n points of a map f, is determined from points of x satisfying (2.2) and (2.3). It is therefore not straightforward to determine $N(n)$ directly from a map. Instead, we denote by $N_\Sigma(n)$ the number of fixed points of $f^{[n]}$. $N_\Sigma(n)$ is then determined from points of x satisfying (2.2), i.e., $f^{[n]}(x) = x$. This enables the $N_\Sigma(n)$ to be determined directly from a map, that is, to count the number of intersections of the curve $y = f^{[n]}(x)$ with the diagonal line $y = x$. The relation of $N(n)$ and $N_\Sigma(n)$ can be seen from what follows. We note that a fixed point of $f^{[n]}$ is not necessarily a period-n point of f, for it could be a fixed point of f or in general a period-m point of f, where $m < n$ and $m|n$. Thus $N_\Sigma(n)$, the number of fixed points for $f^{[n]}$, includes the number of fixed points of f and the number of periodic points whose periods divide n. Accordingly, we have

$$N_\Sigma(n) = \sum_{d|n} N(d), \tag{4.1}$$

where the sum is over all the divisors of n (including 1 and n). We need a reverse formula expressing $N(n)$ in terms of $N_\Sigma(d)$. This has been done, as we know that there are already the following two formulas [3, 9]:

$$N_\Sigma(n) = \sum_{d|n} N(d), \tag{4.2}$$

and

$$(4.3)\qquad \begin{aligned} N(n) &= \sum_{d|n} \mu(n/d) N_\Sigma(d) \\ &= \sum_{d|n} \mu(d) N_\Sigma(n/d), \end{aligned}$$

where $\mu(d)$ is the Möbius function. $N_\Sigma(n)$ is called the Möbius transform of $N(n)$, and $N(n)$ the inverse Möbius transform of $N_\Sigma(n)$. Thus, after calculating $N_\Sigma(n)$ from an iterated map, we obtain a dividing formula $n|N(n)$ from (4.3).

Note that (4.2) and (4.3) are in fact quite general. They are not necessarily related to iterated maps, and hence the $N(n)$ and $N_\Sigma(n)$ in (4.2) and (4.3) need not have specific meanings like periods in iterated maps. We can arbitrarily specify an $N_\Sigma(n)$, and obtain $N(n)$ from (4.3). However, such $N(n)$ in general does not give a dividing formula $n|N(n)$. Thus, guessing an $N_\Sigma(n)$ to obtain an $N(n)$ such that $n|N(n)$ is difficult. Yet, if we start from an iterated map, it then naturally offers an $N_\Sigma(n)$, from which the $N(n)$ obtained is guaranteed to give a dividing formula $n|N(n)$. We should also note that, in general, it is not an easy task to obtain $N_\Sigma(n)$ from an arbitrary map, though in principle it exists. There are examples of iterated maps for which $N_\Sigma(n)$ can be calculated [2, 6].

We summarize all these in the following theorem.

Theorem 4.1. *Let $N(n)$ and $N_\Sigma(n)$ be defined as above. Then for an iterated map*

$$(4.4)\qquad n|N(n) \quad \text{with } N(n) = \sum_{d|n} \mu(n/d) N_\Sigma(d).$$

When n is a prime,

$$(4.5)\qquad n|(N_\Sigma(n) - N_\Sigma(1)).$$

In the following, we consider various maps for which $N_\Sigma(n)$ can be exactly calculated, and then obtain the accompanied dividing formulas.

5. Applications of the Theorem $n|N(n)$ for Various Maps

5-1. Baker's map with base a

We consider first Baker's map with base a, or simply the B_a map, defined by:

$$
B_a(x) = \begin{array}{ll} ax & \text{for } 0 \le x \le 1/a, \\ ax - 1 & \text{for } 1/a < x \le 2/a, \\ ax - 2 & \text{for } 2/a < x \le 3/a, \\ \cdots & \\ ax - (a-1) & \text{for } (a-1)/a < x \le 1, \end{array} \tag{5.1.1}
$$

where a is any positive integer ≥ 2. If $a = 2$, it is Baker's map. The graph of $B_a(x)$ in the range $0 \le x \le 1$ contains a parallel line segments with slope a. Fig. 1 shows the graph of $B_a(x)$ with $a = 5$.

It is easy to see that the graph of $B_a^{[2]}(x)$ contains a^2 parallel line segments with slope a^2, and therefore, in general, the graph of $B_a^{[n]}(x)$ contains a^n parallel line segments with slope a^n. Each of these line segments intersects

FIG. 1. $B_a(x)$ map, with $a = 5$.

the diagonal line once. It follows that there are, in total, a^n intersections for these parallel line segments in $B^{[n]}(x)$ intersecting with the diagonal line. There are thus a^n fixed points for $B^{[n]}$, and we have

$$N_\Sigma(n) = a^n. \tag{5.1.2}$$

From (4.5), we have Fermat's little theorem:

$$n|(a^n - a) \quad \text{for } n \text{ a prime.} \tag{5.1.3}$$

We have now another way to understand Fermat's little theorem. The meaning of $N(n) = a^n - a$, when n is a prime, simply means that, in order to get the number of true period-n points, we need to subtract fixed points of B, whose number is a, from fixed points of $B_a^{[n]}$, whose number is $N_\Sigma(n) = a^n$. Fermat's little theorem is a natural consequence of the B_a map, and the Chinese theorem, with $a = 2$, is the consequence of Baker's map.

If n is a composite number, from (4.4), we have

$$n|\sum_{d|n} \mu(n/d)a^d. \tag{5.1.4}$$

Formula (5.1.4) is a known result, first stated by J. A. Serret as early as 1855 and proved by T. Szele in 1948 [10]. We have here simply reproduced this result from the iterated map. We show how to derive Euler's theorem from (5.1.4) in the Appendix. Formula (5.1.4) is more general than Euler's theorem, as we need not require the base a and n be relatively prime. We may call (5.1.4) the generalized Euler's theorem (in the sense that we are free to choose the base a for any n).

5-2. $B(\mu; x)$ map

The $B(\mu; x)$ map is defined as

$$B(\mu; x) = \begin{cases} \mu x & \text{for } 0 \le x \le 1/2, \\ \mu(x - 1/2) & \text{for } 1/2 < x \le 1, \end{cases} \tag{5.2.1}$$

where μ is a parameter whose value is restricted in the range $0 \le \mu \le 2$, so that x in the interval $[0, 1]$ is mapped to the same interval. Fig. 2 shows the graph of $B(\mu; x)$, with $\mu = (1 + \sqrt{5})/2$.

We discuss the following cases.

5-2-1. The case that $x = 1/2$ is a period-2 point

If $x = 1/2$ is a period-2 point of the $B(\mu)$ map, then $B^{[2]}(\mu; x) = x$, and $B(\mu; x) \ne x$. If $\mu \le 1$, the iterates of $1/2$ should be: $1/2 \mapsto \mu/2 \mapsto \mu^2/2 = 1/2$.

FIG. 2. $B(\mu; x)$ map, with $\mu = (1+\sqrt{5})/2$.

From this, we have $\mu = 1$; however, this simply means that $x = 1/2$ is a fixed point but not a period-2 point. If $\mu > 1$, the iterates of $1/2$ should be: $1/2 \mapsto \mu/2 \mapsto \mu(\mu/2 - 1/2) = 1/2$. It then follows that $\mu^2 - \mu - 1 = 0$. Solving this, we have $\mu = (1+\sqrt{5})/2 \approx 1.618$, which is the well-known golden mean. We denote this μ by $\sum_2$, indicating that for this parameter value $x = 1/2$ is a period-2 point. The detailed discussions of this map can be seen in [4, 5]. From counting the intersections of the function $y = B^{[n]}(\sum_2; x)$ with the diagonal line, we obtain $N_\Sigma(n)$. The calculation is not so straightforward but quite lengthy; we will discuss elsewhere how the results are obtained. Here, we only present the result:

$$N_\Sigma(n) = F_{n+1} + F_{n-1}, \tag{5.2.2}$$

where F_n is the nth Fibonacci number, for which $F_0 = 0,\ F_1 = 1$ and $F_2 = 1$. We have, from (4.4) and (4.5),

$$n|N(n) \quad \text{with } N(n) = \sum_{d|n} \mu(n/d)[F_{d+1} + F_{d-1}], \tag{5.2.3}$$

$$n|(F_{n+1} + F_{n-1} - 1) \quad \text{for } n \text{ a prime.} \tag{5.2.4}$$

Formula (5.2.4) is also a known result [7]; however, we see that it can be derived from the iterated maps. For n a composite number, (5.2.3) offers additional relations for the Fibonacci numbers. Let $N_\Sigma[n] = F_{n+1} + F_{n-1} \equiv L[n]$. Then $L[1] = 1,\ L[2] = 3$. It follows that $L[n]$ is in fact the Lucas sequence. For a simple example, consider $n = 15$. Then $N(15) = L[15] - L[3] - L[5] + L[1] = 1364 - 4 - 11 + 1 = 1350$. Indeed, we have $15|1350$.

5-2-2. The case that $x = 1/2$ is a period-m point

In general, there are many parameter values in the range $0 \le \mu \le 2$ such that $x = 1/2$ is a period-m point. We here choose the largest one among them. It follows that such a μ satisfies the equation $\mu^m - \sum\limits_{i=0}^{m-1} \mu^i = 0$. We denote this μ by $\sum_m$ indicating that for this parameter value $x = 1/2$ is a period-m point. We refer the details to [5]. From counting the intersections of the function $y = B^{[n]}(\sum_m; x)$ with the diagonal line, we have

$$N_\Sigma(n) = F_{n+1}^{(m)} + \sum_{k=1}^{m-1} kF_{n-k}^{(m)} \tag{5.2.5}$$

$$= \sum_{k=0}^{m-1} (k+1)F_{n-k}^{(m)}, \tag{5.2.6}$$

where $F_n^{(m)}$ is the Fibonacci number of degree m, i.e., $F_n^{(m)} = \sum\limits_{j=1}^{m} F_{n-j}^{(m)}$, and with the first m elements: $F_1^{(m)} = 1$, and $F_k^{(m)} = 2^{k-2}$ for $2 \le k \le m$.

In the following, we show some unfamiliar results for the purpose of interest showing how these results are obtained from various iterated maps with suitable parameter values chosen properly. Readers interested in this can take them as exercises.

5-3. $B_3(\mu; x)$ map

We consider the $B_3(\mu; x)$ map defined as

$$\begin{aligned} B_3(\mu; x) = \ & \mu x && \text{for } 0 \le x \le 1/3, \\ & \mu(x - 1/3) && \text{for } 1/3 < x \le 2/3, \\ & \mu(x - 2/3) && \text{for } 2/3 < x \le 1. \end{aligned} \tag{5.3.1}$$

In this map, the values of parameter μ are restricted in the range $0 \le \mu \le 3$. In what follows, we list several results:

5-3-1. The case that $x = 1/3$ is a period-2 point

The 2-cycle is $\{1/3, \mu/3\}$,

(1) For $1 < \mu < 2$:
The corresponding μ satisfies the equation: $\mu = 1 + \mu^{-1}$.
We have $\mu \approx 1.618$, the golden mean, and
$N_\Sigma(n) = F_{n+1} + F_{n-1}$. We don't get new results in this case.

(2) For $2 < \mu < 3$:
The corresponding μ satisfies the equation: $\mu = 1 + 2\mu^{-1}$.
We have $\mu = 1 + \sqrt{2} \approx 2.4142$. Then:
$N_\Sigma(n) = 2s(n) + 2s(n-1)$, where $s(n)$ is a sequence satisfying the relation

$$\begin{aligned} &s(n) = 2s(n-1) + s(n-2), \text{ with the first two elements :} \\ &s(1) = 1 \text{ and } s(2) = 2. \end{aligned} \tag{5.3.2}$$

5-3-2. The case that $x = 1/3$ is a period-3 point

The 3-cycle is denoted by $\{1/3, \mu/3, A\}$.

(1) For $2/3 > \mu/3 > 1/3 > A$:
The corresponding μ satisfies the equation: $\mu = 1 + \mu^{-2}$.
We have $\mu \approx 1.4656$, and $A = \mu(\mu - 1)/3$. Then:
$N_\Sigma(n) = s(n) + 3s(n-2)$, where

$$\begin{aligned} &s(n) = s(n-1) + s(n-3), \\ &\text{with } s(1) = 1,\ s(2) = 1,\ s(3) = 1. \end{aligned} \tag{5.3.3}$$

(2) For $2/3 > \mu/3 > A > 1/3$:
The corresponding μ satisfies the equation: $\mu = 1 + \mu^{-1} + \mu^{-2}$.
We have $\mu \approx 1.839$, and $A = \mu(\mu - 1)/3$. Then:
$N_\Sigma(n) = F^{(3)}_{n+1} + F^{(3)}_{n-1} + 2F^{(3)}_{n-2}$, where $F^{(3)}_n$ is the nth Fibonacci number of degree 3.

This corresponds to the result in (5.2.5) with $m = 3$.

(3) For $\mu/3 > 2/3 > 1/3 > A$:
The corresponding μ satisfies the equation: $\mu = 2 + \mu^{-2}$.
We have $\mu \approx 2.2056$, and $A = \mu(\mu - 2)/3$. Then:
$N_\Sigma(n) = 2s(n) + 3s(n-2)$, where

$$\begin{aligned} &s(n) = 2s(n-1) + s(n-3), \\ &\text{with } s(1) = 1,\ s(2) = 2, \text{ and } s(3) = 4. \end{aligned} \tag{5.3.4}$$

(4) For $\mu/3 > 2/3 > A > 1/3$:
The corresponding μ satisfies the equation: $\mu = 2 + \mu^{-1} + \mu^{-2}$.
We have $\mu \approx 2.5468$, and $A = \mu(\mu - 2)/3$. Then:
$N_\Sigma(n) = 2s(n) + 2s(n-1) + 3s(n-2)$, where

$$\begin{aligned} &s(n) = 2s(n-1) + s(n-2) + s(n-3), \\ &\text{with } s(1) = 1,\ s(2) = 2,\ \text{and } s(3) = 5. \end{aligned} \tag{5.3.5}$$

(5) For $\mu/3 > A > 2/3 > 1/3$:
The corresponding μ satisfies the equation: $\mu = 2 + 2\mu^{-1} + \mu^{-2}$
We have $\mu \approx 2.8312$, and $A = \mu(\mu - 2)/3$. Then:
$N_\Sigma(n) = 2s(n) + 4s(n-1) + 3s(n-2)$, where

$$\begin{aligned} &s(n) = 2s(n-1) + 2s(n-2) + s(n-3), \\ &\text{with } s(1) = 1,\ s(2) = 2,\ \text{and } s(3) = 6. \end{aligned} \tag{5.3.6}$$

In general, we may consider $x = 1/3$ to be a point of higher period. Or we may consider more general maps, such as $B_m(\mu; x)$ maps, etc. We consider the last map for which the point we considered can be an eventually fixed point or an eventually periodic point.

5-4. $\boldsymbol{T}(\mu; x)$ map

We consider the triangular map with a parameter μ, i.e., the $T(\mu; x)$ map, which is defined by

$$T(\mu; x) = 1 - \mu|x| \quad \text{for } -1 \le x \le 1, \tag{5.4.1}$$

where μ is the parameter whose value is restricted in the range $0 \le \mu \le 2$. The x in the interval $[-1, 1]$ is then mapped to the same interval. Fig. 3 shows the graph of $T(\mu; x)$, with $\mu = (1 + \sqrt{5})/2$.

(1) Consider $\mu = \sqrt{2}$. Then $x = 0$ is an eventually fixed point. As the iterates of $x = 0$ is $0 \mapsto 1 \mapsto (1 - \sqrt{2}) \mapsto (\sqrt{2} - 1) \mapsto (\sqrt{2} - 1)$, we have the quite interesting result:

$$\begin{aligned} N_\Sigma(n) &= 1 && \text{if } n \text{ is odd,} \\ N_\Sigma(n) &= 2^{1+n/2} - 1 && \text{if } n \text{ is even.} \end{aligned} \tag{5.4.2}$$

We check that if p is a prime, then from $2p|N(2p)$, we have $2p|(N_\Sigma(2p) - N_\Sigma(2) - N_\Sigma(p) + N_\Sigma(1))$. That is, $2p|(2^{p+1} - 4)$ or $p|(2^p - 2)$, which is a well-known result.

(2) Consider μ satisfying $\mu^3 = 2\mu + 2$, i.e., $\mu \approx 1.76929$. Then $x = 0$ is also an eventually fixed point. As the iterates of $x = 0$ is $0 \mapsto 1 \mapsto (1 - \mu) \mapsto (1 + \mu - \mu^2) \mapsto (\mu^2 - \mu - 1) \mapsto (\mu^2 - \mu - 1)$, we then have

FIG. 3. Triangular map $T(\mu; x)$, with $\mu = (1+\sqrt{5})/2$.

$$
\begin{aligned}
N_\Sigma(n) &= s(n) - 1 \quad \text{if } n \text{ is even,} \\
N_\Sigma(n) &= s(n) + 1 \quad \text{if } n \text{ is odd, where}
\end{aligned} \tag{5.4.3}
$$

$$
\begin{aligned}
&s(n) = s(n-1) + 2s(n-2) - 2s(n-4), \text{ with} \\
&s(1) = 0,\ s(2) = 4,\ s(3) = 6,\ s(4) = 8.
\end{aligned} \tag{5.4.4}
$$

(3) Consider μ satisfying $\mu^3 = \mu^2 + 2$, i.e., $\mu \approx 1.69562$. Then $x = 0$ is an eventually period-2 point. As the iterates of $x = 0$ is $0 \mapsto 1 \mapsto (1-\mu) \mapsto (1+\mu-\mu^2) \mapsto (\mu - 1) \mapsto (1+\mu-\mu^2)$, we have

$$
N_\Sigma(n) = s(n) - 4\delta_{n,4k}, \tag{5.4.5}
$$

where $s(n)$ is a sequence satisfying the relation

$$
\begin{aligned}
&s(n) = s(n-1) + s(n-2) + s(n-3) - 2s(n-5), \text{ with} \\
&s(1) = 1,\ s(2) = 3,\ s(3) = 7,\ s(4) = 11,\ s(5) = 11.
\end{aligned} \tag{5.4.6}
$$

We see that many can be done in this way, and plenty of $N(n)$ such that $n|N(n)$ can be obtained. In principle, infinitely many $N(n)$ can be obtained, since each iterated map contributes an $N(n)$. We should check the $N(n)$ for every iterated map we have ever used, and it is hoped that some interesting, important and useful dividing formulas can be found. We conclude that the existence of dividing formulas is not so mysterious from the point of view of iterated maps.

Appendix

We show here how to derive Euler's theorem from (5.1.4), the generalized Euler's theorem. Let $n = \prod_{i=1}^{s} p_i^{t_i}$ be the prime factorization of a given number n. We let $n_i \equiv p_i^{t_i}$. Then $N(n_i) = N_\Sigma(n_i) - N_\Sigma(n_i/p_i)$. With $N_\Sigma(n) = a^n$, we have

$$n_i|(a^{n_i} - a^{n_i/p_i})$$

or

$$n_i|(a^{n_i/p_i}[a^{n_i(1-1/p_i)} - 1]). \tag{A.1}$$

We may replace the term $n_i(1-1/p_i)$ in (A.1) by $\varphi(n)$, since $\varphi(n)$ is a multiple of it. Therefore,

$$n_i|(a^{n_i/p_i}[a^{\varphi(n)} - 1]).$$

If n and a are relatively prime, we then have

$$n_i|(a^{\varphi(n)} - 1). \tag{A.2}$$

Thus, $a^{\varphi(n)} - 1$ is divisible by each n_i. As these n_i are relatively by prime to each other, therefore $a^{\varphi(n)} - 1$ is divisible by the product of these n_i, which is n. We therefore reproduce Euler's theorem:

$$n|(a^{\varphi(n)} - 1) \text{ for } n \text{ and } a \text{ relatively prime.} \tag{A.3}$$

Accordingly, Euler's theorem is a natural consequence of the B_a map.

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