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MATRICES WITH MAXIMUM UPPER MULTIEXPONENTS IN THE CLASS OF PRIMITIVE, NEARLY REDUCIBLE MATRICES

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Abstract. B. Liu has recently obtained the maximum value for the kth upper multiexponents of primitive, nearly reducible matrices of order n with $1 \le k \le n$. In this paper primitive, nearly reducible matrices whose kth upper multiexponents attain the maximum value are completely characterized.

1. Introduction

A square Boolean matrix A is *primitive* if one of its powers, A^k, is the matrix J of all 1's for some integer $k \ge 1$. The smallest such k is called the *primitive exponent* of A. The matrix A is *reducible* if there is a permutation matrix P such that

$$P^t AP = \begin{bmatrix} A_1 & 0 \\ X & A_2 \end{bmatrix},$$

where A_1 and A_2 are square and nonvacuous; otherwise A is *irreducible*. The matrix A is *nearly reducible* if A is irreducible but each matrix obtained from A by replacing any nonzero entry by zero is reducible.

It is well known that there is an obvious one-to-one correspondence between the set B_n of n by n Boolean matrices and the set of digraphs on n vertices. Given $A = (a_{ij}) \in B_n$, the associated digraph $D(A)$ has vertex set $V(D(A)) = \{1, 2, \ldots, n\}$, and arc set $E(D(A)) = \{(i, j) : a_{ij} = 1\}$. A is primitive if and only if $D(A)$ is strongly connected and the greatest common divisor (gcd for short) of all the distinct cycle lengths of $D(A)$ is 1, and A is nearly reducible if and only if $D(A)$ is a minimally strongly connected digraph. We say a digraph

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is *primitive* with primitive exponent γ if it is the associated digraph of some primitive matrix with primitive exponent γ.

Now we give the definition of the upper multiexponent for a primitive digraph, which was introduced by R. A. Brualdi and B. Liu [1].

Let D be a primitive digraph on n vertices. The *exponent* of a subset $X \subseteq V(D)$ is the smallest integer p such that for each vertex i of D there exists a walk from at least one vertex in X to i of length p (and of course every length greater than p, since D is strongly connected). We denote it by $\exp_D(X)$. The number

$$F(D,k) = \max\{\exp_D(X) : X \subseteq V(D), |X| = k\}$$

is called the *kth upper multiexponent* of D.

Clearly $F(D,1)$ is the primitive exponent of D. Hence the kth upper multiexponent of a primitive digraph is a generalization of its primitive exponent.

Let A be an $n \times n$ primitive matrix, and let k be an integer with $1 \le k \le n$. The kth upper multiexponent of A is the kth upper multiexponent of $D(A)$, denoted by $F(A,k)$. Thus $F(A,k) = F(D(A),k)$. Clearly $F(A,k)$ is the smallest power of A for which no set of k rows has a column consisting of all zeros.

In [2] B. Liu obtained the maximum value for the kth upper multiexponents of primitive, nearly reducible matrices of order n with $1 \le k \le n$. In this paper, we provide a complete characterization of matrices in the class of $n \times n$ primitive, nearly reducible matrices whose kth upper multiexponents for $1 \le k \le n$ attain the maximum value.

Using the correspondence between matrices and digraphs, we express the results in the digraph version.

2. Main Results

We first give several lemmas that will be used.

Lemma 1. [3]. *Let D be a primitive digraph on n vertices, $1 \le k \le n-1$, and let h be the length of the shortest cycle of D. Then*

$$F(D,k) \le n + h(n-k-1).$$

Let

$$F(n,k) = \begin{cases} n^2 - 4n + 6, & k = 1; \\ (n-1)^2 - k(n-2), & 2 \le k \le n. \end{cases}$$

The following lemma has been proved in [2] for $n \ge 5$. For $n = 4$ it can be checked readily.

Lemma 2. [2]. *$F(D_{n-2}, k) = F(n, k), n \geq 4$, where D_{n-2} is the digraph given by Fig.* 1.

Lemma 3. [1]. *Let D be a primitive digraph with n vertices and let h and t be respectively the smallest and the largest cycle lengths of D. Then*

$$F(D, n-1) \leq \max\{n-h, t\}.$$

Let PMD_n be the set of all primitive, minimally strongly connected digraphs with n vertices. The following theorem has recently been proved by B. Liu.

Theorem 1. [2]. $\mathrm{Max}\{F(D,k) : D \in PMD_n\} = F(n,k), 1 \leq k \leq n.$

A problem that deserves investigation is to characterize the extreme digraphs, or the digraphs in PMD_n whose kth upper multiexponents assume the maximum value $F(n, k)$.

Obviously, for any $D \in PMD_n, F(D, n) = F(n, n) = 1$. We are going to consider the case $1 \leq k \leq n-1$.

Theorem 2. *Let $D \in PMD_n, 1 \leq k \leq n-1, n \geq 4$. Then for $1 \leq k \leq n-2, F(D,k) = F(n,k)$ if and only if $D \cong D_{n-2}$, where D_{n-2} is the digraph given by Fig.* 1*; $F(D, n-1) = F(n, n-1) = n-1$ if and only if $D \cong D_{n,s}$ with $1 \leq s \leq n-3$ and* $\gcd(n-1, s+1) = 1$*, where $D_{n,s}$ is the digraph given by Fig.* 2.

FIG. 1.

FIG. 2.

Remark. When $s = n-3$, $D_{n,s}$ is the digraph D_{n-2} for $n \geq 4$.

Proof. We begin the proof with the case $k = n-1$ first.

Suppose $D \cong D_{n,s}$. For $i = 2, 3, \ldots, s (s > 1)$, any walk from the vertex i to the vertex $n-1$ has a length of the form $n-1-i+a(n-1)+b(s+1)$, where a and b are non-negative integers. Consider the equation $n-1-i+a(n-1)+b(s+1) = n-2$, i.e., $a(n-1)+b(s+1) = i-1$. Since $i \leq s \leq -3$, we have $a = 0$, $b = 0$, which is impossssible. Hence there is no walk of length $n-2$ from the vertex i to the vertex $n-1$ for $i = 2, 3, \ldots, s$. For $i = s+1, \ldots, n-1 (s \geq 1)$, we have the same conclusion as above. Also it is easy to see that there is no walk of length $n-2$ from the vertex n to the vertex $n-1$. Now take $X_0 = V(D) \setminus \{1\}$. There does not exist any walk from a vertex in X_0 to the vertex $n-1$ of length $n-2$. Hence $\exp_{D_{n,s}}(X_0) \geq n-1$. By the definition of the $(n-1)$th upper multiexponent and Theorem 1 it follows that

$$F(D, n-1) = F(D_{n,s}, n-1) = n-1.$$

Conversely, suppose $F(D, n-1) = n-1$. Let h and t be respectively the smallest and the largest cycle lengths of D. D cannot have a cycle of length n, because, if so, the digraph is still strongly connected after the removal of any arc lying outside such cycle, contradicting the fact that D is minimally strongly connected. Similarly, we can show that D has no loops. So we have $2 \leq h \leq n-2, t \leq n-1$. By Lemma 3 we obtain

$$n-1 = F(D, n-1) \leq \max\{n-h, t\},$$

which implies $t = n-1$. Suppose D contains a cycle of length $n-1$ whose arcs are $(i, i+1)$ for $i = 1, 2, \ldots, n-2$, and $(n-1, 1)$. By the strong connectedness of D there exist u and v (u and v may be equal) in $\{1, 2, \ldots, n-1\}$ such that

(u, n) and (n, v) are arcs in D. Without loss of generality we assume that $v = 1$. Thus D contains a subdigraph $D_{n,u}$ with $1 \le u \le n-3$.

Since D is minimally strong, it is easy to see that D has no arcs other than those in $D_{n,u}$. It follows from the primitivity of D that $\gcd(n-1, u+1) = 1$.

Now we turn to the case $1 \le k \le n-2$. The case $k = 1$ is proved in [4]. Suppose $2 \le k \le n-2$. If $D \cong D_{n-2}$, by Lemma 2 we have $F(D, k) = F(n, k)$. Conversely, suppose $F(D, k) = F(n, k)$ and let h be the length of the shortest cycle in D. Since D is primitive, it has at least two different cycle lengths. In addition, D has no cycles of length n, being a minimally strong connected digraph of order n. It follows that $h \le n-2$.

If $h = n-2$, then the set of all distinct cycle lengths of D is $\{n-2, n-1\}$. By the minimally strong connectedness of D, it follows that $D \cong D_{n-2}$. We are going to show that it is impossible to have $h \le n-3$. We divide our argument into two cases.

Case 1: $2 \le k < n-2$.
If $h \le n-3$, applying Lemma 1 we have

$$\begin{aligned} F(D,k) &\le n + h(n-k-1) \\ &\le n + (n-3)(n-k-1) \\ &= (n-1)^2 - k(n-2) - (n-k-2) \\ &< (n-1)^2 - k(n-2) = F(n,k), \end{aligned}$$

a contradiction.

Case 2: $k = n-2$.
If $h \le n-4$, by Lemma 1,

$$\begin{aligned} F(D,k) &\le n + h(n-k-1) \\ &\le n + (n-4)(n-k-1) \\ &= 2n-4 < 2n-3 = F(n,k), \end{aligned}$$

a contradiction.

If $h = n-3$, observing that D cannot have loops, we have $h \ge 2$ and $n \ge 5$. If $n = 5$, then $h = 2$. Since D cannot have a cycle of length 5 and D is primitive, D must have a cycle of length 3. It follows from the fact that D is minimally strongly connected that D is isomorphic with D_1 or D_2 or D_3 as displayed in Fig. 3. In all such cases, it is easy to verify that we have $F(D, 1) \le 6$. Hence $F(D, n-2) = F(D, 3) \le F(D, 1) \le 6 < 7 = F(5, 3)$, which is a contradiction.

Now suppose $h = n-3$ and $n > 5$. Since D cannot have a cycle of length n, by the primitivity of D, D must contain a cycle of length of $n-2$ or $n-1$.

FIG. 3.

If there is a walk of length t from vertex j to vertex i, we say that j is a t-in vertex of i. And the set of all t-in vertices of i in D is denoted by $R_D(t,i)$.

Case 2.1: D has no cycles of length $n-1$. Then D must have a cycle of length $n-2$. Take a cycle C of D of length $n-2$. Then D has precisely two vertices, say, x, y, lying outside C. We divide this situation into the following two subcases.

(1) D contains one of the arcs (x,y) or (y,x). Say, D contains the arc (x,y). Then (y,x) cannot be an arc of D; otherwise, $n-3=h=2$ and so $n=5$, which is a contradiction. By the strong connectedness of D, there must exist vertices u, v of C (the cycle of length $n-2$) such that (u,x) and (y,v) are both arcs of D. If $u=v$, then $n-3=h=3$, so we have $n=6$ and D is the digraph D^1_{6-3}. If $u\neq v$, then since D has precisely two cycles, of lengths $n-2$ and $n-3$ respectively, it will follow that D is isomorphic with $D^1_{n-3}(n\geq 7)$. $D^1_{n-3}(n\geq 6)$ is given by Fig. 4. Suppose $D=D^1_{n-3}$.

For $n\geq 6$, we describe $R_D(2n-5,i)$ explicitly:

$$\begin{aligned}
&R_D(2n-5,1)=\{n,1,2\},\\
&R_D(2n-5,i)=\{i-1,i,i+1\}, i=2,3,\ldots,n-4,\\
&R_D(2n-5,n-3)=\{n-4,n-3,n-2,n-1\},\\
&R_D(2n-5,n-2)=\{n-2,n-1,n,1\},\\
&R_D(2n-5,n-1)=\{n-4,n-3,n-2,n-1\},\\
&R_D(2n-5,n)=\{n-2,n-1,n,1\}.
\end{aligned}$$

FIG. 4.

It is clear that each vertex has at least three $(2n-5)$-in vertices in D, and so $\exp_D(X) \leq 2n-5$ for any set of $n-2$ vertices. It follows from the definition of the $(n-2)$th upper multiexponent that $F(D, n-2) \leq 2n-5 < 2n-3 = F(D, n-2)$, which is a contradiction.

(2) Neither (x, y) nor (y, x) is an arc of D. By the strong connectedness of D, there must exist vertices u, v, u' and v' of C such that $(u, x), (x, v), (u', y)$ and (y, v') are arcs of D. We have $u \neq v$ and $u' \neq v'$; otherwise, $n-3 = h = 2$ and so $n = 5$, which is a contradiction. Also neither (u, v) nor (u', v') is an arc of C; otherwise D has a cycle of length $n-1$, which is a contradiction. Suppose that $uu_1u_2 \cdots u_r v$ and $u'v_1v_2 \cdots v_t v'$ are two paths of C, of lengths $r+1$ and $t+1$ respectively, where $r \geq 1$ and $t \geq 1$. If $r = t = 1$, then by the minimally strong connectedness of D, D has no cycles of length $h = n-3$, which is a contradiction. If $r \geq 3$ or $t \geq 3$, then there is a cycle with length less than $h = n-3$, which is also a contradiction. Hence we have $r = 2$ or $t = 2$. So D contains a subdigraph which is isomorphic with $D_{(n-1)-2}$ (see Fig. 1 for D_{n-2}). Assume $D_{(n-1)-2}$ is a subdigraph of D. Note that $V(D_{(n-1)-2}) = \{1, 2, \ldots, n-1\}$. By the strong connectedness of D, there exists a vertex $j \in \{1, 2, \ldots, n-1\}$ such that (j, n) is an arc of D.

Let $X \subseteq V(D)$ with $|X| = n-2$. For each vertex $1, 2, \ldots, n-1$, there is a walk to the vertex from a vertex in $X \setminus \{n\}$ of length $\exp_{D_{(n-1)-2}}(X \setminus \{n\})$ (and hence also every length greater). This is because, each such vertex belongs to the subgraph $D_{(n-1)-2}$. Note that

$$\exp_{D_{(n-1)-2}}(X \setminus \{n\}) \leq \begin{cases} F(D_{(n-1)-2}, n-2) = n-2, & n \notin X; \\ F(D_{(n-1)-2}, n-3) = 2n-5, & n \in X. \end{cases}$$

So $\exp_{D_{(n-1)-2}}(X \setminus \{n\}) \leq 2n-5$ whether $n \in X$ or $n \notin X$. Thus for every integer $t \geq 2n-5$, and for each vertex $1, 2, \ldots, n-1$, there is a walk to the vertex from a vertex in $X \setminus \{n\}$ of length t. Since $j \in \{1, 2, \ldots, n-1\}$ and (j, n) is an arc of D, it follows that there is a walk to the vertex n from a vertex in $X \setminus \{n\}$ of length $t+1$ for every integer $t \geq 2n-5$. So we have proved that there is a walk to each vertex of D from a vertex in $X \setminus \{n\}$ of length $t+1$ for every integer $t \geq 2n-5$. This implies that

$$\exp_D(X) \leq \exp_D(X \setminus \{n\}) \leq 2n-4 < 2n-3.$$

By the definition of the $(n-2)$th upper multiexponent, we have $F(D, n-2) < 2n-3 = F(n, n-2)$, which is a contradiction.

Case 2.2: D has a cycle of length of $n-1$. Since $h = n-3, D$ also has a cycle of length $n-3$. By the minimally strong connectedness of D, one can readily show that in this case D is composed of precisely two cycles, of lengths $n-1$ and $n-3$ respectively. But $\gcd\{n-1, n-3\} = 1$, so n is even, and D must be isomorphic with $D^2_{n-3}(n \geq 6)$, where D^2_{n-3} is given by Fig. 5.

Suppose $D = D^2_{n-3}$. We have

$$\begin{aligned}
&R_D(2n-4, 1) = \{n, 1, 3\},\\
&R_D(2n-4, 2) = \{2, 4, n-3, n-1\},\\
&R_D(2n-4, 3) = \{n, 1, 3, 5\},\\
&R_D(2n-4, i) = \{i-2, i, i+2\}, i = 4, \ldots, n-3,\\
&R_D(2n-4, n-2) = \{n-4, n-3, n-2, 1\},\\
&R_D(2n-4, n-1) = \{n-3, n-1, 2\},\\
&R_D(2n-4, n) = \{n-4, n, 1\}.
\end{aligned}$$

By similar arguments as for the case $D \cong D^1_{n-3}$, we get $F(D, n-2) \leq 2n-4 < 2n-3 = F(n, n-2)$, which is also a contradiction.

Now we have proved that it is impossible to have $h \leq n-3$. Thus the proof of the theorem is completed. ■

Theorem 2 gives complete characterizations of the extreme digraphs in the class of primitive, minimally strong digraphs of order n whose kth $(1 \leq k \leq n-1)$ upper multiexponents assume the maximum value.

Note that there is not any digraph D in PMD_n with $F(D, 1) = m$ if $n^2 - 5n + 9 < m < F(n, 1)$, or $n^2 - 6n + 12 < m < n^2 - 5n + 9$ for $n \geq 4$ (see [4]).

As a by-product of the proof of Theorem 2 we have a similar result.

Corollary 1. *Let k and n be integers. If $2 \leq k \leq n-3$, then for any integer m satisfying $n + (n-3)(n-k-1) < m < F(n, k)$, there is no digraph $D \in PMD_n$ such that $F(D, k) = m$.*

FIG. 5. D^2_{n-3} (n is even, $n \geq 6$).

This corollary tells us that there are gaps in the set of kth upper multiexponents of digraphs in $PMD_n (1 \leq k \leq n-3)$.

Corollary 2. *The number of non-isomorphic extreme digraphs in PMD_n with the $(n-1)$th upper multiexponent equal to $n-1 (n \geq 4)$ is $\phi(n-1)-1$, where ϕ is Euler's totient function.*

Finally, we point out that the maximum value for the k-exponents of primitive, nearly reducible matrices is also obtained in [2], and we have characterized the corresponding extreme matrices in another paper.

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